ON AUTOMORPHISMS OF A KÄHLERIAN STRUCTURE

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Is every isometry, or more generally, every affine transformation of a Kählerian manifold a complex analytic transformation? The answer is certainly negative in the case of a complex Euclidean space. This question has been recently studied by Lichnerowicz [8] and Schouten-Yano [11] from the infinitesimal point of view; they have found some conditions in order that every infinitesimal motion of a Kählerian manifold preserve the complex structure. (As a matter of fact, [11] has dealt with the case of a pseudo-Kählerian manifold, which does not differ essentially from a Kählerian manifold as far as the question at hand is concerned.)

In the present paper, we generalize their results by a different approach. In order to explain our main idea, we shall first give a few definitions (**1** and **2**) and state our main results (**3**). The proofs are given in the subsequent sections.

1. Kählerian structures

Let M be a complex analytic manifold of complex dimension n. Its complex structure is defined by a real analytic tensor field I of type $(1, 1)$ with $I^2 = -\mathbf{1}$[1] on the underlying $2n$-dimensional real analytic manifold which satisfies the condition of integrability $I[X, Y] - [IX, Y] - [X, IY] - I[IX, IY] = 0$ for all real vector fields X and Y (for example, [1]). A differentiable transformation f of M is said to preserve the complex structure I if $\delta f \circ I = I \circ \delta f$, where δf denotes the differential of f. This is equivalent to saying that f is a complex analytic transformation. If $\delta f \circ I = -I \circ \delta f$, we say that f maps I into the conjugate complex structure $-I$; f is then a conjugate analytic transformation.

A real analytic Riemannian metric g on a complex analytic manifold M is called Kählerian if it is hermitian, that is, $g(IX, IY) = g(X, Y)$ for all real vector fields X and Y, and if I is a parallel tensor field with respect to the

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[1] Throughout the present note, **1** denotes the identity transformation.

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Riemannian connection arising from g. Likewise, given a real analytic manifold M with Riemannian metric g, we shall say that a complex structure I on M is Kählerian if g is Kählerian with respect to I in the above sense. Such a pair (I, g) defines a Kählerian structure on M. By an isometry (resp. affine transformation) of a Kählerian manifold M, we understand of course an isometry (resp. affine transformation) of the underlying Riemannian manifold. By an automorphism of M, we mean an isometry which preserves the complex structure.

A Kählerian manifold M (with complex structure I and Riemannian metric g) will be called *non-degenerate* if the restricted homogeneous holonomy group σ_x of the underlying Riemannian manifold contains the endomorphism I_x of the tangent space T_x at $x \in M$, where x is an arbitrary reference point for the holonomy group and I_x is the value at x of the tensor field I. Note that the condition $I_x \in \sigma_x$ is independent of the choice of a reference point x. For any point y, let τ denote the parallel displacement: $T_x \to T_y$ along an arbitrary curve from x to y. Since I is a parallel tensor field, we have $I_y = \tau \cdot I_x \cdot \tau^{-1}$. On the other hand, we have $\sigma_y = \tau \cdot \sigma_x \cdot \tau^{-1}$.

Finally, we shall say that a Riemannian manifold M of dimension > 1 is irreducible if the restricted homogeneous holonomy group is irreducible, that is, if it does not admit any non-trivial invariant subspace as a group of linear transformations on the (real) tangent vector space.

2. Complex and quaternionian structures on a real vector space

In this section, we shall indicate an intrinsic way of defining real representations of $GL(n, C)$, $U(n)$, $SU(n)$ or $GL(n, Q)$, $Sp(l)$, etc. such as described in C. Chevalley: Theory of Lie Groups I, Chapter I. The exposition is quite elementary but important for our purpose.

A complex structure I on a real m-dimensional vector space T is, by definition, an endomorphism of T such that $I^2 = -1$. It allows us to define the set T as an n-dimensional vector space over the field of complex numbers C, where $m = 2n$. More precisely, we define

$$(a + bi)X = aX + bIX$$

for $a, b \in R$ (field of real numbers) and $X \in T$. We denote by T^* the vector space over C thus obtained.

If g is a positive definite inner product on T such that $g(IX, IY) = g(X, Y)$

for all $X, Y \in T$, then we may define a positive definite hermitian inner product g^* on T^* by

$$g^*(X, Y) = g(X, Y) - ig(IX, Y)$$

for $X, Y \in T^*$, where X and Y are considered as elements of T on the right hand side of the above equation.

If τ is an endomorphism of T which commutes with I, then it may be considered as an endomorphism τ^* of T^*, as is clear from $\tau(iX) = \tau(IX) = I(\tau X) = i\tau X$. If furthermore τ leaves g invariant, then τ^* leaves g^* invariant.

We shall say that a group of linear transformations of a real vector space T of dimension $m = 2n$ is *contained in a real representation of* $U(n)$ if T admits a complex structure I which commutes with every $\tau \in G$ and a positive definite inner product g which is invariant by I and every $\tau \in G$. In this case, G is isomorphic with the subgroup $G^* = \{\tau^*;\ \tau \in G\}$ (in the above notation) of the unitary group on T^* with respect to g^*. If furthermore $\det \tau^* = 1$ for every $\tau \in G$, then we say that G *is contained in a real representation of* $SU(n)$.

By a *quaternionian structure* on a complex vector space T^* of dimension n, we shall mean a conjugate linear transformation J of V with $J^2 = -1$, that is, a $1-1$ map of T^* onto itself such that

$$J(X+Y) = JX + JY, \quad J(aX) = \bar{a}JX \quad \text{and} \quad J^2X = -X$$

for all $X, Y \in T^*$ and $a \in C$, where $\bar{a}$ denotes the complex conjugate of a. It allows us to consider T^* as a vector space $\tilde{T}$ over the field of quaternions Q in the following fashion. We represent every quaternion q in the form $q = a + bj$ (j being an element of Q such that $j^2 = -1$ and $ij = -ji$) and define the scalar multiplication by

$$q \cdot X = aX + bJX$$

for all $X \in T^*$. As a vector space over Q, $\tilde{T}$ is of dimension l where $n = 2l$.

If g^* is a positive definite hermitian inner product on T^* such that $g^*(JX, JY) = g^*(Y, X) (= \overline{g^*(X, Y)})$, then we can define a positive definite symplectic inner product $\tilde{g}$ in the vector space $\tilde{T}$ over Q in the following fashion:

$$\tilde{g}(X, Y) = g^*(X, Y) + g^*(X, JY)j$$

for all $X, Y \in \tilde{T}$. Namely, $\tilde{g}$ is Q-valued and satisfies the following conditions:

1) $\tilde{g}(Y, X)$ is the symplectic conjugate of $\tilde{g}(X, Y)$;

2) $\tilde{g}(X+X', Y) = \tilde{g}(X, Y) + \tilde{g}(X', Y)$;

3) $g(X, X) \geqq 0$ for every X and it is 0 if and only if $X=0$.

If τ^* is an endomorphism of T^* over C which commutes with J, then it may be regarded as an endomorphism $\tilde{\tau}$ of $\tilde{T}$ over Q. If furthermore τ^* leaves g^* invariant, then $\tilde{\tau}$ leaves $\tilde{g}$ invariant.

We say that a group G^* of linear transformations of a complex vector space T^* of dimension $n=2l$ is *contained in a complex representation* of $S_p(l)$ if T^* admits a quaternionian structure J and a positive definite hermitian inner product g^* such that

$$g^*(JX, JY) = g^*(Y, X), \quad \tau^* \cdot J = J \cdot \tau^* \quad \text{and} \quad g^*(\tau^* X, \tau^* Y) = g^*(X, Y)$$

for all $X, Y \in T^*$ and $\tau^* \in G^*$. In this case, G^* is isomorphic with a subgroup of $S_p(l)$ on the vector space $\tilde{T}$ over Q with respect to the symplectic inner product $\tilde{g}$.

Finally, we define a *quaternionian structure on a real vector space* T. It is a pair of endomorphisms I and J of T such that $I^2 = J^2 = -1$ and $IJ = -JI$. Such a structure makes it possible to regard T as a vector space $\tilde{T}$ over Q, the scalar multiplication being defined by

$$(a+bi+cj+dk)X = aX + bIX + cJX + d(IJ)X$$

for $a, b, c, d \in R$ and $X \in T$. Another way of seeing this is to consider, first, T with a complex structure I as a vector space T^* over C and then consider the given endomorphism J as a quaternionian structure on T^*, which is obviously possible.

This being said, we are able to use the following expression. We say that a group of linear transformations G on a real m-dimensional vector space T is *contained in a real representation of* $S_p(l)$, with $m=4l$, if T admits a quaternionian structure (I, J) and a positive definite inner product g which are both invariant by I, J and every element of G. It is now easy to see that, in this case, G is isomorphic with a subgroup of $S_p(l)$ on the l-dimensional vector space $\tilde{T}$ over Q with a suitable symplectic inner product.

By using the fact that $S_p(l)$ is connected, we can prove that if G is contained in a real representation of $S_p(l)$, then it is contained in a real representation of $SU(n)$, where $m=2n$ and $n=2l$. We omit the detail of the proof.

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3. Main results

THEOREM 1. *Every simply connected and complete Kählerian manifold M is a direct product $M_0 \times M_1 \times \ldots \times M_k$, where M_0 is a complex Euclidean space of dimension $\geqq 0$ and $M_1, \ldots, M_k$ are irreducible Kählerian manifolds. If M is non-degenerate, M_0 does not appear and $M_1, \ldots, M_k$ are all non-degenerate.*

THEOREM 2.[2)] *Let M be an irreducible Kählerian manifold whose restricted homogeneous holonomy group is not contained in a real representation of $S_p(l)$, where $\dim M = 4l$. Then every affine transformation of M preserves the complex structure I or maps I into the conjugate complex structure. The largest connected group of affine transformations $A^0(M)$ preserves the complex structure.*

THEOREM 3. *If M is a complete non-degenerate Kählerian manifold, then the largest connected group of affine transformations $A^0(M)$ consists of automorphisms.*

If M is a pseudo-Kählerian manifold, we can still define the notion of non-degeneracy. If we replace "Kählerian" by "pseudo-Kählerian" and "complex structure" by "almost complex structure" respectively, then all the results stated above remain true.

It is of some interest to compare our problem with the following: is every affine transformation of a Riemannian manifold an isometry? This question has been settled by Hano and one of the authors as follows. Every simply connected and complete Riemannian manifold M is a direct product of a Euclidean space M_0 and irreducible Riemannian manifolds $M_1, \ldots, M_k$ (the so-called de Rham decomposition) [10]. The largest connected group of affine transformations $A^0(M)$ is naturally isomorphic with $A^0(M_0) \times A^0(M_1) \times \ldots \times A^0(M_k)$ [2]. On the other hand, every affine transformation of a complete irreducible Riemannian manifold is an isometry [5]. It follows that, *if M is a complete Riemannian manifold whose restricted homogeneous holonomy group does not leave any non-zero vector invariant, then $A^0(M)$ consists of isometries.*

Our Theorem 1 corresponds to the de Rham decomposition of a Riemannian manifold. By using the above result of Hano, our problem is reduced to the case of an irreducible Kählerian manifold, to which Theorem 2 is an answer. Here we do not need the condition of completeness but require an assumption

2) A similar result has been obtained also by M. Obata.

on the holonomy group. Now, what is a condition which assures that every component of the de Rham decomposition of a given Kählerian manifold satisfies the assumption of Theorem 2? The non-degeneracy is such a condition. The following theorem shows the relationship of this notion to the known facts on Ricci curvature, thus giving a heuristic interpretation of the results of Lichnerowicz [7], [8].

THEOREM 4. *Let M be a Kählerian manifold of complex dimension n.*

1) *If M is irreducible and the Ricci curvature is not zero, then M is non-degenerate.*

2) *If M is non-degenerate and n is not divisible by 4, then the Ricci curvature of M is not zero.*

3) *If the Ricci curvature of M is non-singular at some point of M, then M is non-degenerate.*

Finally we add

COROLLARY. *Let M be a $2n$-dimensional simply connected real analytic Riemannian manifold which is irreducible. Then the following three cases are possible:*

1) *If the restricted homogeneous holonomy group σ is not contained in a real representation of $U(n)$, there exists no Kählerian structure at all on M.*

2) *If σ is contained in a real representation of $U(n)$ but not of $S_p(l)$, $n=2l$, then there exist exactly two Kählerian structures on M which are mutually conjugate.*

3) *If σ is contained in a real representation of $S_p(l)$, $n=2l$, then there exist continuously may distinct Kählerian structures on M.*

4. Proof of Theorem 1

The underlying Riemannian manifold of M admits the de Rham decomposition $M=M_0\times M_1\times \ldots \times M_k$. It is not difficult (see [3]) to see that every component M_i has a Kählerian structure induced from that of M and that M is the direct product of $M_0, M_1, \ldots, M_k$ as Kählerian manifolds. The homogeneous holonomy group $\sigma(M)$ of M is decomposed into the direct product of the homogeneous holonomy groups $\sigma(M_i)$ of M_i, $i=0, 1, \ldots, k$, where $\sigma(M_0)$ consits of the identity only. It follows that if M is non-degenerate, the Euclidean

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part M_0 does not exist and the irreducible components M_i, $i=1, 2, \ldots, k$, are all non-degenerate.

5. Proof of Theorem 2

Let f be an affine transformation of M and δf its differential. Then $I^f = \delta f^{-1} \cdot I \cdot \delta f$ is a tensor field of type (1, 1) which clearly satisfies the condition $I^f \cdot I^f = -\mathbf{1}$. We show that it is a parallel tensor field. Let c be an arbitrary curve from x to y and let τ be the linear mapping of T_x onto T_y defined by parallel displacement along c. Let c^* be the image curve of c by f and let τ^* be the linear mapping of $T_{f(x)}$ onto $T_{f(y)}$ defined by parallel displacement along c^*. Since f is an affine transformation, we have $\delta f \cdot \tau = \tau^* \cdot \delta f$ on T_x [9]. On the other hand, we have $I_{f(y)} \cdot \tau^* = \tau^* \cdot I_{f(x)}$ since I is a parallel tensor field. Therefore we get

$$\begin{aligned} I^f_y \cdot \tau &= \delta f^{-1} \cdot I_{f(y)} \cdot \delta f \cdot \tau = \delta f^{-1} \cdot I_{f(y)} \cdot \tau^* \cdot \delta f \\ &= \delta f^{-1} \cdot \tau^* \cdot I_{f(x)} \cdot \delta f = \tau \cdot \delta f^{-1} \cdot I_{f(x)} \cdot \delta f = \tau \cdot I^f_x, \end{aligned}$$

which proves our assertion. In particular, I^f_x commutes with every element of σ_x.

Now let A be the algebra (over the field of real numbers R) formed by all endomorphisms of T_x which commute with every element of σ_x. Since σ_x is irreducible, every non-zero element of A has an inverse, that is, A is a division algebra. By a well known theorem in algebra, A is isomorphic either with the field of real numbers R, or the field of complex numbers C, or else the field of quaternions Q. Since A contains an element I_x with $I_x^2 = -\mathbf{1}$, it cannot be isomorphic with R. If A were isomorphic with Q, it would follow that σ_x is contained in a real representation of $S_p(l)$, with $n=2l$; indeed, again by the irreducibility of σ_x, we see that the inner product g_x of T_x induced from the Kählerian metric of M is invariant by the elements I and J of A which correspond to the units i and j of Q. Hence A is isomorphic with C. The only complex numbers whose square are -1 are i and $-i$. Since I^f_x is in A and $I^f_x \cdot I^f_x = -\mathbf{1}$, we have either $I^f_x = I_x$ or $I^f_x = -I_x$. Since I and I^f are parallel tensor fields, we have $I^f = I$ or $I^f = -I$. This concludes the proof of the first part of Theorem 2.

In order to prove the second part, let f be an element of $A^0(M)$. We take a continuous 1-parameter family f_t of affine transformations such that $f_0 =$ identity transformation and $f_1 = f$. If we form $I^t = \delta f_t^{-1} \cdot I \cdot \delta f_t$, then we have $I^t_x = I_x$ or $-I_x$ from what we have seen. Since I^t_x is a continuous 1-parameter family of endomorphisms of T_x such that $I^t_x = I_x$ for $t = 0$, I^t_x must coincide with I_x for every t. In particular, we have $I^f_x = I_x$, that is, $I^f = I$. This proves that f preserves the complex structure I.

6. Proof of Theorem 3

Let $\tilde{M}$ be the universal covering manifold of M provided with a naturally induced Kählerian structure. It is easy to see that $\tilde{M}$ is also complete and non-degenerate. By the argument in [2], it is sufficient to prove Theorem 3 for $\tilde{M}$. By Theorem 1, $\tilde{M} = M_1 \times \ldots \times M_k$ where each M_i is irreducible and non-degenerate. The homogeneous holonomy group $\sigma(M_i)$ is not contained in a real representation of $S_p(l)$, $n = 2l$. In fact, we show that the division algebra A considered in the proof of Theorem 1 cannot be isomorphic with Q. If it were so, the elements I, J and K of A corresponding to i, j and $k \in Q$ must commute with the element I_0, $I_0^2 = -\mathbf{1}$, of A determined by the given complex structure of M_i, which is contained in A since M_i is non-degenerate. This is a contradiction. Hence Theorem 2 shows that $A^0(M_i)$ preserves the complex structure of M_i. Since $A^0(\tilde{M})$ is the direct product of $A^0(M_i)$, $i = 1, 2, \ldots, k$, we see that $A^0(\tilde{M})$ preserves the complex structure of $\tilde{M}$. On the other hand, we already know ([2] and [5]) that $A^0(\tilde{M})$ consists of isometries. Hence $A^0(\tilde{M})$ consists of automorphisms of $\tilde{M}$.

7. Proof of Theorem 4

1) The complex structure I_x of the tangent space T_x is invariant by the restricted homogeneous holonomy group σ_x operating on T_x, we may consider σ_x as a subgroup of $U(n)$ as indicated in **2**. Since the Ricci curvature is not zero, σ_x is not a subgroup of $SU(n)$ ([7], see also [4] and [6]) which means that σ_x has a non-discrete center. σ_x being irreducible, the center must be of dimension 1 by Schur's lemma. Hence σ_x contains the 1-parameter subgroup $\{e^{2\pi i r} \cdot \mathbf{1};\ r \text{ reals}\}$ of $U(n)$, in particular, the transformation $i\,\mathbf{1}$. In real representation, this means that σ_x contains the endomorphism I_x, that is, M is non-degenerate.

2) If we consider σ_x as a subgroup of $U(n)$, then σ_x contains the transformation $i\,\mathbf{1}$ whose determinant, the n-th power of i, is not equal to 1 since n is

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not divisible by 4. Hence σ_x is not a subgroup of $SU(n)$ and the Ricci curvature is not zero.

3) Let x be a point of M where the Ricci curvature is non-singular. Let U be a properly chosen neighborhood of x which is isometric to the direct product $U_0 \times U_1 \times \ldots \times U_k$, where U_0 is locally a complex Euclidean space and $U_1, \ldots, U_k$ are irreducible Kählerian manifolds (the argument is similar to that in the proof of Theorem 1). We may consider $U_0, U_1, \ldots, U_r$ as submanifolds of U passing through x. Then the Ricci curvature of U at x is the direct sum of the Ricci curvatures of $U_0, U_1, \ldots, U_k$. Therefore the Ricci curvature of each U_i is non-singular at x. It follows that there does not exist U_0 and that $U_1, \ldots, U_k$ are non-degenerate by 1) of Theorem 4. Since $\sigma_x(M)$ contains $\sigma_x(U) = \sigma_x(U_1) \times \ldots \times \sigma_x(U_k)$, we see that $\sigma_x(M)$ contains the endomorphism I_x.

8. Proof of Corollary

At any reference point x, we consider the division algebra A formed by all endomorphisms of T_x commuting with every element of σ_x. If A is isomorphic with R, then there is no element $I \in A$ with $I^2 = -\mathbf{1}$; there is no Kählerian structure on M. If A is isomorphic with C, then let I_x be an element of A with $I_x^2 = -\mathbf{1}$. By parallel displacement of I_x, we get a parallel tensor field I of type $(1, 1)$ such that $I^2 = -\mathbf{1}$ and $g(IX, IY) = g(X, Y)$. It is known that an almost complex structure which is parallel with respect to a Riemannian connection (or, more generally, an affine connection without torsion) is integrable [1]. Hence I is a Kählerian structure. It is clear that I and its complex conjugate structure $-I$ are the only Kählerian structures on M.

If A is isomorphic with Q, then it contains continuously many elements S with $S^2 = -\mathbf{1}$. Namely, if I, J and K are the elements of A which correspond to i, j and k of Q respectively, then we may take $S = bI + cJ + dK$, where b, c and d are real numbers such that $b^2 + c^2 + d^2 = 1$. For any such element S of A, we get a Kählerian structure on M by the same argument as before. Hence M has continuously many distinct Kählerian structures.

9. Remarks

In the case of a compact Kählerian manifold M, the largest connected group of affine transformations $A^0(M)$ consists of isometries (theorem of Yano, which has been generalized in [7]) and preserves the complex structure of M, as is remarked in [8]. Indeed, the form F associated to the Kählerian structure of $M: F(X, Y) = g(IX, Y)$ is harmonic and invariant by every 1-parameter group of isometries. It follows that I is also invariant by the 1-parameter group.

The above statement is no longer true for the total group $A(M)$ of affine transformations. For example, in a complex projective space P_n with usual Fubini-Study metric, the transformation defined by $(z^0, z^1, \ldots, z^n) \to (\bar{z}^0, \bar{z}^1, \ldots, \bar{z}^n)$ in terms of homogeneous coordinates $z^0, \ldots, z^n$ is isometric but not complex analytic.

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