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A SEMILINEAR DIRICHLET PROBLEM

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Introduction and notations. Let Ω be a bounded region in $\mathbf{R}^n$. In this note we discuss the existence of weak solutions (see [**4**, Section 2]) of the Dirichlet problem

$$\text{(I)}\quad \begin{aligned} \triangle u(x) + g(x, u(x)) + f(x, u(x), \nabla u(x)) &= 0 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega \end{aligned}$$

where $\triangle$ is the Laplacian operator, $g : \Omega \times \mathbf{R} \to \mathbf{R}$ and $f : \Omega \times \mathbf{R}^{n+1} \to \mathbf{R}$ are functions satisfying the Caratheodory condition (see [**2**, Section 3]), and ∇ is the gradient operator.

We let $\lambda_1 < \lambda_2 \leqq \ldots \leqq \lambda_m \leqq \ldots$ denote the sequence of numbers for which the problem

$$\text{(II)}\quad \begin{aligned} \triangle u(x) + \lambda u(x) &= 0 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \Omega \end{aligned}$$

has nontrivial weak solutions.

The main result of this paper is:

Suppose the following two hypotheses hold. (1.1) *The function* $g(x, u)$ *admits a derivative with respect to* u, $\partial g/\partial u : \Omega \times \mathbf{R} \to \mathbf{R}$, *which satisfies the Caratheodory condition; furthermore, there exist* $\alpha, \alpha_1 \in \mathbf{R}$ *and a positive integer* N *such that*

$$\lambda_N < \alpha \leqq \partial g/\partial u(x, u) \leqq \alpha_1 < \lambda_{N+1} \quad \textit{for all } (x, u) \in \Omega \times \mathbf{R}$$

(1.2) *There exist a constant* $\beta > 0$ *and a function* $c(x) \in L_2(\Omega)$ *such that*

$$|f(x, u, y)|^2 \leqq c(x) + \beta^2 ||y||^2$$

for all $(x, u, y) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$, *where* $||\ ||$ *denotes the usual norm in* $\mathbf{R}^n$.
If

$$\text{(1.3)}\quad \beta < (\min\{1 - \alpha_1/\lambda_{N+1}, \alpha/\lambda_N - 1\})/\sqrt{\lambda_1}$$

then (I) *has a weak solution.*

As a corollary of our main result we obtain bounds for the eigenvalues on $(\lambda_N, \lambda_{N+1})$ of a class of non-selfadjoint problems of the form:

$$\text{(III)}\quad \begin{aligned} \triangle u(x) + \langle (a_1(x), \ldots, a_n(x)), \nabla u(x) \rangle + \lambda u(x) &= 0 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega, \end{aligned}$$

where $\langle\, ,\, \rangle$ denotes the usual inner product in $\mathbf{R}^n$ and $a_1, \ldots, a_n \in L_\infty(\Omega)$.

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In [**2**, Theorem 1] and [**4**, Theorem 3.1] the problem (I) is considered and the existence of weak solutions is proved when $f(x, u, y) = o(||y||)$ as $||y|| \to +\infty$. In [**3**, Theorem 3.4] the problem (I) is studied when $\Omega \subset \mathbf{R}$ and f and g are permitted to depend on the second order derivatives. The results of [**3**] yield inequalities of the form (1.3) when $\alpha_1 < \lambda_1$. We denote in this paper by H^1 the Sobolev space $H_0^{1,2}(\Omega)$ (see [**1**, p. 45]). We take as inner product in H^1 the bilinear form defined by

$$\langle u, v\rangle_1 = \int_\Omega \langle \nabla u(\xi), \nabla v(\xi)\rangle d\xi.$$

We denote by $|| \; ||$ the norm on H^1 and by $|| \; ||_0$ the norm on $L_2(\Omega)$. We let X denote the closed subspace of H^1 spanned by the eigenfunctions of (II) corresponding to eigenvalues λ_k with $\lambda_k \leqq \lambda_N$. We use the symbol $\int$ to mean *integral over* Ω.

Proofs. From now on we assume that (1.1) and (1.2) hold. Let $J: H^1 \times H^1 \to \mathbf{R}$ be defined by

$$J(y, u) = \int \{||\nabla u(\xi)||^2/2 - G(\xi, u(\xi)) - f(\xi, y(\xi), \nabla y(\xi))u(\xi)\}d\xi,$$

where $G : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function such that $\partial G/\partial u(x, u) = g(x, u)$ and $G(x, 0) = 0$. It is not difficult to see that for $y, u, v \in H^1$

$$\text{(2.1)} \quad \lim_{t\to\infty} (J(y, u + tv) - J(y, u))/t = \int \{\langle \nabla u(\xi), \nabla v(\xi)\rangle - g(\xi, u(\xi))v(\xi) - f(\xi, y(\xi), \nabla y(\xi))v(\xi)\}d\xi.$$

Therefore, by Vainberg's lemma (see [**6**, p. 63]), if (1.1) holds, the right hand side of (2.1) defines a continuous linear functional on $v \in H^1$. Hence, for each $(y, u) \in H^1 \times H^1$ there exist $S(y, u) \in H^1$ such that

$$\text{(2.2)} \quad \lim_{t\to 0} J(y, u + tv) - J(y, u)/t = \langle u, v\rangle_1 + \langle S(y, u), v\rangle_1.$$

By (1.1), (1, 2) and Vainberg's Lemma (see [**2**, Proposition 4]) the functions $u(\xi) \to g(\xi, u(\xi))$ and $y(\xi) \to f(\xi, y(\xi), \nabla y(\xi))$ are continuous functions from H^1 into $L_2(\Omega)$. Since, by Rellich's principle, the inclusion of $L_2(\Omega)$ into the dual space of H^1 is compact, $S(y, u)$ is a compact function.

From (1.1) and the results of [**5**, Section 7] it follows that

$$\begin{aligned} \Delta u(x) + g(x, u(x)) + f(x, y(x), \nabla y(x)) &= 0 \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega \end{aligned}$$

has a unique weak solution for each $y \in H^1$. Therefore, for each $y \in H^1$ there exists a unique $\varphi(y) \in H^1$ such that

$$\text{(2.3)} \quad \langle \varphi(y) + S(y, \varphi(y)), v\rangle_1 = 0 \quad \text{for all } v \in H^1.$$

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LEMMA 1. *The function* $\varphi : H^1 \to H^1$ *defined by* (2.3) *is compact.*

Proof. First we show that φ is continuous. From the discussion in [**5**, Section 7] we see that if $J_y : H^1 \to \mathbf{R}$ is defined by $J_y(u) = J(y, u)$ then J_y is of class C^2. Let $DJ_y(u)$ be the Hessian of J_y at u. An elementary computation show that

$$\langle DJ_y(u)v, v\rangle_1 = \int \{||\nabla v||^2 - \partial g/\partial u\ (\xi, u(\xi))\, v^2(\xi)\}d\xi.$$

Following the arguments of [**4**, Section 7] we see that $DJ_y(u)$ is a nonsingular Fredholm operator.

Let $T : H^1 \times H^1 \to H^1$ be defined by $T(y, u) = u + S(y, u)$. Hence T is continuously differentiable with respect to u and $\partial_u T(y, u) = DJ_Y(u)$. Thus, by (2.3), for any $y_0 \in H^1$ $T(y_0, \varphi(y_0)) = 0$. By the foregoing argument $\partial_u T(y_0, \varphi(y_0))$ is nonsingular. Therefore, by the implicit function theorem there exist a neighborhood V of y_0 and a continuous function $\psi : V \to H^1$ such that $T(y, \psi(y)) = 0$ for all $y \in V$. Consequently, by the uniqueness of $\varphi(y)$, we have $\varphi(y) = \psi(y)$ on V, and this proves that φ is continuous.

Now we prove that φ is bounded on bounded sets. For $y \in H^1$, let $\varphi_1(y)$ be the orthogonal projection of $\varphi(y)$ on X, and let $\varphi_2(y)$ be $\varphi(y) - \varphi_1(y)$. By (2.3) we have

$$0 = \langle \varphi(y) + S(y, \varphi(y)), \varphi_2(y) - \varphi_1(y)\rangle_1.$$

Hence,

$$\begin{aligned}(2.4)\quad 0 = {} & ||\varphi_2(y)||_1^2 - ||\varphi_1(y)||_1^2 - \int g(\xi, \varphi(y)(\xi))(\varphi_2(y)(\xi) - \varphi_1(y)(\xi))d\xi \\ & - \int f(\xi, y(\xi), \nabla y(\xi))(\varphi_2(y)(\xi) - \varphi_1(y)(\xi))d\xi \\ \geqq {} & ||\varphi_2(y)||_1^2 - ||\varphi_1(y)||_1^2 - \sqrt{\lambda_1}||g(\xi, 0)||_0 \cdot ||\varphi(y)||_1 - \alpha_1||\varphi_2(y)||_0^2 \\ & + \alpha||\varphi_1(y)||_0^2 - \left(\int f^2(\xi, y(\xi), \nabla y(\xi))d\xi\right)^{1/2} \cdot \sqrt{\lambda_1} \cdot ||\varphi(y)||_1 \\ \geqq {} & (1 - \alpha_1/\lambda_{N+1})||\varphi_2(y)||_1^2 + (\alpha/\lambda_N - 1)||\varphi_1(y)||_1^2 \\ & - \sqrt{\lambda_1}\, ||g(\xi, 0)||_0||\varphi(y)||_1 - \left(\int f^2(\xi, y(\xi), \nabla y(\xi))d\xi\right)^{1/2} \cdot \sqrt{\lambda_1} \cdot ||\varphi(y)||_1.\end{aligned}$$

Thus, if $m = \min\{1 - \alpha_1/\lambda_{N+1}, \alpha/\lambda_N - 1\}$ then we have

$$(2.5)\quad \sqrt{\lambda_1}||g(\xi, 0)||_0 + \sqrt{\lambda_1}\left(\int f^2(\xi, y(\xi), \nabla y(\xi))d\xi\right)^{1/2} \geqq m||\varphi(y)||_1.$$

Since, by (1.2), the Nemytski operator $y(\xi) \to f(\xi, y(\xi), \nabla y(\xi))$ maps bounded sets of H^1 into bounded sets of $L_2(\Omega)$ we infer from (2.5) that φ is bounded on bounded sets.

Suppose $\{y_n\}$ is a bounded sequence in H^1. Hence $\{S(y_n, \varphi(y_n))\}$ contains a convergent subsequence $\{S(y_{n_j}, \varphi(y_{n_j}))\}$. By (2.3), $-\varphi(y_{n_j}) = S(y_{n_j}, \varphi(y_{n_j}))$. Therefore, $\{\varphi(y_{n_j})\}$ is a convergent sequence. Consequently, φ is compact and the lemma is proved.

THEOREM 2. *If* (1.1), (1.2) *and* (1.3) *hold then the problem* (I) *has a weak solution.*

Proof. By (1.2), there exists $K \in \mathbf{R}$ such that

$$(2.6)\quad \left(\int f^2(\xi, y(\xi), \nabla y(\xi))d\xi\right) \leqq K^2\|c(x)\|_0 + \beta^2\|y\|_1^2$$

for all $y \in H^1$. Combining (2.5) and (2.6) we have

$$(2.7)\quad m\|\varphi(x)\|_1 \leqq \sqrt{\lambda_1}\|g(\xi, 0)\|_0 + \sqrt{\lambda_1}K\|c(x)\|_0^{1/2} + \beta\sqrt{\lambda_1}\|y\|_1.$$

Therefore, by (1.3), if $R > 0$ is big enough then the function φ maps the ball of center 0 and radius R into itself. Consequently, by Schauder's fixed point theorem, φ must have a fixed point. Since any fixed point of φ is a weak solution of (I) the theorem is proved.

COROLLARY 3. *If* $(\int(a_1^2(\xi) + \ldots + a_n^2(\xi))d\xi)^{1/2} \leqq \beta$ *then the problem* (III) *does not have eigenvalues in the open interval*

$$(\lambda_N(1 + \beta\sqrt{\lambda_1}), \lambda_{N+1}(1 - \beta\sqrt{\lambda_1})) = D.$$

Proof. If $\lambda \in D$, then following the proof of Theorem 2 we see that for any $c(x) \in L_2(\Omega)$ the problem

$$\begin{aligned} \triangle u(x) + \langle(a_1(x), \ldots, a_n(x)), \nabla u(x)\rangle + \lambda u(x) &= c(x) \quad x \in \Omega \\ u(x) &= 0 \quad x \in \partial\Omega \end{aligned}$$

has a weak solution. Therefore by the Fredholm alternative (see [**2**, Proposition 1]) λ cannot be an eigenvalue of (III).

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