THE COVERING OF SPACE BY SPHERES

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1. Introduction. Bambah **(1)** has recently determined the most economical covering of three dimensional space by equal spheres whose centres form a lattice, the density of this covering being

$$\vartheta_3 = \frac{5\sqrt{5}}{24}\pi. \tag{1.1}$$

As is well known, this problem may be interpreted in terms of the inhomogeneous minimum of a positive definite quadratic form. If $f(x) = f(x_1, x_2, \ldots, x_n)$ $(n \geqslant 2)$ is a positive quadratic form of determinant D, then, for any real $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define $\mathfrak{m}(f; \alpha)$ to be the minimum of $f(x+\alpha)$ for integral x. The inhomogeneous minimum of $f(x)$ is then defined as

$$\mathfrak{m}(f) = \max_{\alpha} \mathfrak{m}(f; \alpha).$$

If now ϑ_n is the density of the most economical covering of n-dimensional space by lattice-ordered spheres, we have

$$\left(\frac{\vartheta_n}{J_n}\right)^{2/n} = \min_f \frac{\mathfrak{m}(f)}{D^{1/n}},$$

where J_n is the volume of the unit sphere:

$$x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant 1.$$

Thus (1.1) is equivalent to the assertion that

$$\mathfrak{m}(f) \geqslant \left(\frac{125}{1024} D\right)^{\frac{1}{3}} \tag{1.2}$$

for all $f(x_1, x_2, x_3)$, and that the equality sign holds for some form f.

It is natural to introduce here the notion of an extreme form, by analogy with the corresponding homogeneous problem. We shall say that $f(x)$ is *extreme* if the ratio $\mathfrak{m}(f)/D^{1/n}$ is a (local) minimum, i.e. is not increased by any sufficiently small variation of the coefficients of f. Forms for which $\mathfrak{m}(f)/D^{1/n}$ is an absolute minimum may be called *absolutely extreme.* Since $\mathfrak{m}(f)$ and D are invariant under equivalence transformations (integral unimodular transformations of $x_1, \ldots, x_n$), while $\mathfrak{m}(f)/D^{1/n}$ is unaltered by multiplying f by an arbitrary positive constant, the property of being extreme is shared by the class of forms consisting of all forms equivalent to a multiple of some one form of the class.

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I prove here:

THEOREM 1. *If $n = 3$, there is just one class of extreme forms represented by*

$$f_0(x_1, x_2, x_3) = 3x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3; \tag{1.3}$$

and for this class

$$\mathfrak{m}(f) = \left(\frac{125}{1024} D\right)^{\frac{1}{3}}. \tag{1.4}$$

This theorem clearly includes the results of Bambah **(1)** (where the question of the existence of other classes of extreme forms is left open).

The object of this paper is, however, not so much to establish the above refinement of Bambah's results as to give a much simpler proof, which also suggests a method of attacking the problem when $n \geqslant 4$.

The starting point of the proof is Voronoï's method of reduction of a positive form f and the construction of the polyhedron Π associated with f. These are discussed in §2. Theorem 1 is proved in §3, while §4 contains some remarks on the method and the possibility of extending it to higher dimensions.

2. Reduced forms and their polyhedra. Voronoï **(3**, p. 150**)** has shown that every class of equivalent positive forms in 3 variables contains a form expressible as

$$f(x_1, x_2, x) = \rho_{01}x_1^2 + \rho_{02}x_2^2 + \rho_{03}x_3^2 + \rho_{12}(x_1 - x_2)^2 + \rho_{13}(x_1 - x_3)^2 + \rho_{23}(x_2 - x_3)^2 \tag{2.1}$$

where

$$\rho_{ij} \geqslant 0 \qquad (i, j = 0, \ldots, 3);$$

and clearly the ρ_{ij} are uniquely determined by f. We call such a form *reduced* (in the sense of Voronoï).

The ρ_{ij} are not in general determined by the class of f. We have in fact, defining for convenience

$$\rho_{ij} = \rho_{ji}, \qquad i > j,$$

LEMMA 2.1. *If p, q, r, s is an arbitrary permutation of* 0, 1, 2, 3, *then the form*

$$\rho_{pq}x_1^2 + \rho_{pr}x_2^2 + \rho_{ps}x_3^2 + \rho_{qr}(x_1 - x_2)^2 + \rho_{qs}(x_1 - x_3)^2 + \rho_{rs}(x_2 - x_3)^2 \tag{2.2}$$

is equivalent to the form (2.1).

Proof. The result is obvious if $p = 0$, since then (2.2) arises from (2.1) by the transformation $x_q \to x_1$, $x_r \to x_2$, $x_s \to x_3$. It therefore suffices to prove the result for $p, q, r, s = 1, 0, 2, 3$; this however corresponds to transforming (2.1) by

$$x_1 \to x_1, \; x_2 \to x_1 - x_2, \; x_3 \to x_1 - x_3.$$

This Lemma is the genesis of the suffix notation in (2.1), and provides an "argument by symmetry" which will be frequently used in what follows.

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The set of points of space which are at least as near to the origin as to any integral point l (with the metric defined by f) forms a closed bounded convex polyhedron Π, the intersection of the half-spaces

$$f(x) \leqslant f(x - l),$$

where l runs through all integral points. Π may in fact be defined by a finite number $2\sigma \leqslant 2(2^3 - 1)$ of these inequalities of the type

$$f(x) \leqslant f(x \pm l_k) \qquad (k = 1, \ldots, \sigma).$$

The planes $f(x) = f(x \pm l_k)$ are then the faces of Π.

Perhaps the simplest method of obtaining $l_1, \ldots, l_\sigma$ is to use the criterion established by Voronoï (**4**, p. 277): a point $l(\neq 0)$ appears in the set $\pm l_1, \ldots, \pm l_\sigma$ if and only if the minimum of $f(x)$ over $x \equiv l \pmod 2$ is attained only for $x = \pm l$.

It is clear that, for the form (2.1), the minimum of $f(x)$ for prescribed parities of x_1, x_2, x_3 is attained when the even x_i are zero and the odd x_i are all 1 or all -1; and in general (e.g., if all $\rho_{ij} > 0$) only for these two sets. Thus, in general, Π has 7 pairs of parallel faces, for which we can find a symmetrical notation as follows:

Define $x_0 = 0$, so that

$$f = \sum_{0}^{3} \rho_{ij}(x_i - x_j)^2,$$

and set

$$y_i = \tfrac{1}{2}\frac{\partial f}{\partial x_i} = \sum_{j=0}^{3} \rho_{ij}(x_i - x_j) \qquad (i = 0, \ldots, 3);$$

then the 14 faces of Π are given by

$$(2.3)\qquad \begin{aligned} F_i:&\qquad 2y_i = \sum_{l\neq i} \rho_{il}, \\ F_{ij}:&\qquad 2(y_i + y_j) = \sum_{l\neq i,j} (\rho_{il} + \rho_{jl}), \\ F_{ijk}:&\qquad 2(y_i + y_j + y_k) = \sum_{l\neq i,j,k} (\rho_{il} + \rho_{jl} + \rho_{kl}), \end{aligned}$$

where all indices and summations run from 0 to 3. Since clearly $\sum y_i = 0$, the faces F_i, F_{jkl} and the faces F_{ij}, F_{kl} are parallel, where i, j, k, l is any permutation of 0, 1, 2, 3.

It is easy to verify the faces

$$(2.4)\qquad \begin{aligned} F_1:&\ 2y_1 = 2\rho_{01}x_1 + 2\rho_{12}(x_1 - x_2) + 2\rho_{13}(x_1 - x_3) = \rho_{01} + \rho_{12} + \rho_{13}, \\ F_{12}:&\ 2(y_1 + y_2) = 2\rho_{01}x_1 + 2\rho_{02}x_2 + 2\rho_{13}(x_1 - x_3) + 2\rho_{23}(x_2 - x_3) \\ &\qquad = \rho_{01} + \rho_{02} + \rho_{13} + \rho_{23}, \\ F_{123}:&\ 2(y_1 + y_2 + y_3) = 2\rho_{01}x_1 + 2\rho_{02}x_2 + 2\rho_{03}x_3 = \rho_{01} + \rho_{02} + \rho_{03}, \end{aligned}$$

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determine a vertex v_{123} of Π; thus for example we have

$$|2(y_0 + y_1 + y_3)| = |2y_2| = |\rho_{02} - \rho_{12} + \rho_{23}| \leqslant \rho_{02} + \rho_{12} + \rho_{23}.$$

Applying all 4! permutations of the suffixes 0, 1, 2, 3, we obtain 4! distinct sets (F_i, F_{ij}, F_{ijk}) of faces determining 4! vertices v_{ijk}. Since Π has at most 4! vertices (**4**, p. 205), we have therefore determined all vertices of Π.

Our next task is to determine $\mathfrak{m}(f)$. From the definition of Π it is clear that

$$\mathfrak{m}(f) = \max_{x \in \Pi} f(x);$$

and by the convexity of Π and of the ellipsoid $f(x) \leqslant \mathfrak{m}(f)$, it follows that

$$\mathfrak{m}(f) = \max f(v)$$

over all vertices v of Π.

To calculate the values of $f(v)$, it suffices to evaluate $f(v_{123})$ and then to apply all permutations of suffixes in the ρ_{ij}; and the evaluation of $f(v_{123})$ may be simplified by observing that

$$f(x) = x_1y_1 + x_2y_2 + x_3y_3.$$

A direct calculation gives

$$(2.5) \quad 4Df(v_{123}) = D(\rho_{01} + \rho_{02} + \rho_{03} + \rho_{12} + \rho_{13} + \rho_{23}) - K - 4\rho_{01}\rho_{03}\rho_{12}\rho_{23}$$

where D is the determinant of f (and of the equations (2.9)) and[1]

$$K = \sum \rho_{01}\rho_{02}\rho_{03}(\rho_{12} + \rho_{13} + \rho_{23}).$$

Since D, $\sum\rho_{ij}$ and K are invariant under permutation of suffixes of the ρ_{ij}, it follows from (2.5) that $f(v)$ has at most 3 distinct values for vertices v of Π. Denoting these by f_1, f_2, f_3 and setting

$$(2.6) \qquad \lambda_1 = \rho_{01}\rho_{23}, \; \lambda_2 = \rho_{02}\rho_{13}, \; \lambda_3 = \rho_{03}\rho_{12},$$

we have

$$(2.7) \qquad \begin{aligned} 4Df_1 &= D(\textstyle\sum\rho_{ij}) - K - 4\lambda_2\lambda_3 \\ 4Df_2 &= D(\textstyle\sum\rho_{ij}) - K - 4\lambda_1\lambda_3 \\ 4Df_3 &= D(\textstyle\sum\rho_{ij}) - K - 4\lambda_1\lambda_2. \end{aligned}$$

Since

$$D(f_i - f_j) = \lambda_k(\lambda_i - \lambda_j)$$

for i, j, k a permutation of 1, 2, 3, the value of

$$(2.8) \qquad \mathfrak{m}(f) = \max(f_1, f_2, f_3)$$

is easily decided from the relative magnitudes of $\lambda_1, \lambda_2, \lambda_3$.

The above analysis has been carried out on the assumption that Π has 14

[1]We use here the usual summation convention for symmetric functions, so that K is the sum of the four distinct terms obtainable by cyclic permutations of 0, 1, 2, 3.

faces. If some of the ρ_{ij} vanish, some of the planes (2.4) are linearly dependent on the others and may be discarded. The effect of this is that certain of the 24 vertices coincide; thus if $\rho_{12} = \rho_{13} = \rho_{23} = 0$, Π degenerates to a parallelepiped, and $f(v)$ is the same for each of its 8 vertices. Such degeneration, however, does not affect the validity of our final results (2.7), (2.8).

It is convenient to note here, before proceeding to the proof of Theorem 1, some formulae concerning D, K and their derivatives.

We have

$$D = \begin{vmatrix} \rho_{01}+\rho_{12}+\rho_{13} & -\rho_{12} & -\rho_{13} \\ -\rho_{12} & \rho_{02}+\rho_{12}+\rho_{23} & -\rho_{23} \\ -\rho_{13} & -\rho_{23} & \rho_{03}+\rho_{13}+\rho_{23} \end{vmatrix}$$

$$= \sum \rho_{01}\rho_{02}\rho_{03} + \sum \rho_{01}\rho_{23}(\rho_{02}+\rho_{03}+\rho_{12}+\rho_{13}) \tag{2.9}$$

and, writing for convenience

$$\sigma_i = \rho_{jk}\rho_{jl} + \rho_{jk}\rho_{kl} + \rho_{jl}\rho_{kl} \tag{2.10}$$

(where i, j, k, l is any permutation of 0, 1, 2, 3),

$$\frac{\partial D}{\partial \rho_{01}} = \sigma_0 + \sigma_1 + \lambda_2 + \lambda_3. \tag{2.11}$$

Using symmetry, we obtain

$$\frac{\partial D}{\partial \rho_{01}} - \frac{\partial D}{\partial \rho_{13}} = \sigma_0 - \sigma_3 - \lambda_1 + \lambda_2, \tag{2.12}$$

$$\frac{\partial D}{\partial \rho_{01}} + \frac{\partial D}{\partial \rho_{23}} - \frac{\partial D}{\partial \rho_{02}} - \frac{\partial D}{\partial \rho_{13}} = -2(\lambda_1 - \lambda_2). \tag{2.13}$$

Similarly we find

$$\frac{\partial K}{\partial \rho_{01}} = \rho_{02}\rho_{03}(\rho_{12}+\rho_{13}+\rho_{23}) + \rho_{12}\rho_{13}(\rho_{02}+\rho_{03}+\rho_{23}) + \rho_{02}\rho_{12}\rho_{23} + \rho_{03}\rho_{13}\rho_{23} \tag{2.14}$$

$$\frac{\partial K}{\partial \rho_{01}} - \frac{\partial K}{\partial \rho_{13}} = (\lambda_2 - \lambda_1)(\rho_{03}+\rho_{12}) - \lambda_3(\rho_{01}-\rho_{02}+\rho_{23}-\rho_{13}) - (\rho_{03}+\rho_{12})(\rho_{01}\rho_{02}-\rho_{13}\rho_{23}); \tag{2.15}$$

interchanging 1 and 2 and subtracting gives

$$\frac{\partial K}{\partial \rho_{01}} + \frac{\partial K}{\partial \rho_{23}} - \frac{\partial K}{\partial \rho_{02}} - \frac{\partial K}{\partial \rho_{13}} = 2(\lambda_2 - \lambda_1)(\rho_{03}+\rho_{12}) - 2\lambda_3(\rho_{01}-\rho_{02}+\rho_{23}-\rho_{13}). \tag{2.16}$$

3. Proof of Theorem 1. We take f in the form (2.1), and suppose that f is extreme. We prove successively: (i) the two greater of $\lambda_1, \lambda_2, \lambda_3$ must be equal; (ii) $\lambda_1 = \lambda_2 = \lambda_3$; (iii) all ρ_{ij} are equal. In each case the proof proceeds by exhibiting a variation of the coefficients ρ_{ij} which, if the stated conditions

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are not satisfied, contradicts our supposition that f is extreme. It will always suffice to work to the first order of small quantities; we denote generally by δR the first order variation in a function R of the ρ_{ij} resulting from small variations $\delta\rho_{ij}$.

In order to apply the analysis of §2 to both f and the neighbouring form $f' = \sum(\rho_{ij} + \delta\rho_{ij})(x_i - x_j)^2$, we must of course ensure that $\rho_{ij} + \delta\rho_{ij} \geqslant 0$ for all i, j. If all $\rho_{ij} > 0$, this will obviously hold for sufficiently small $\delta\rho_{ij}$ of either sign. If our hypotheses do not allow us to infer that $\delta\rho_{ij} \neq 0$ for some i, j we shall always choose the corresponding $\delta\rho_{ij} \geqslant 0$.

LEMMA 3.1. *If f is extreme, it is impossible that*

$$\lambda_1 > \lambda_2 \geqslant \lambda_3. \tag{3.1}$$

Proof. If (3.1) holds, we have $\mathfrak{m}(f) = f_1$, by (2.7), (2.8); and we shall have $\mathfrak{m}(f') = f'_1$ for any sufficiently near form f'.

We choose

$$\delta\rho_{01} = \delta\rho_{23} = -\epsilon, \ \delta\rho_{02} = \delta\rho_{13} = \epsilon \qquad (\epsilon > 0),$$

noting that (3.1) implies that $\rho_{01} > 0$, $\rho_{23} > 0$. Then, by (2.13),

$$\delta D = -\epsilon\left(\frac{\partial D}{\partial\rho_{01}} + \frac{\partial D}{\partial\rho_{23}} - \frac{\partial D}{\partial\rho_{02}} - \frac{\partial D}{\partial\rho_{13}}\right) = 2\epsilon(\lambda_1 - \lambda_2) > 0. \tag{3.2}$$

We set

$$L = D(\rho_{03} + \rho_{12}) - K - 4\lambda_2\lambda_3, \tag{3.3}$$

so that, by (2.7),

$$4f_1 = \rho_{01} + \rho_{02} + \rho_{13} + \rho_{23} + L/D. \tag{3.4}$$

Using (3.2) and (2.16) we find easily that

$$\begin{aligned} \delta L &= (\rho_{03} + \rho_{12})\,\delta D - \delta K - 4\delta(\lambda_2\lambda_3) \\ &= -2\epsilon\lambda_3(\rho_{01} + \rho_{02} + \rho_{23} + \rho_{13}), \end{aligned}$$

whence

$$\delta L \leqslant 0. \tag{3.5}$$

We have also

$$L \geqslant 0. \tag{3.6}$$

This may be verified by direct computation, using (3.3) and (3.1). We may argue more simply as follows:

Since $f(x) \geqslant f(1,1,0) = \rho_{01} + \rho_{02} + \rho_{13} + \rho_{23}$ for $x_1, x_2, x_3 \equiv 1, 1, 0 \pmod 2$, we have $f(x) \geqslant \frac{1}{4}(\rho_{01} + \rho_{02} + \rho_{13} + \rho_{23})$ for $x_1, x_2, x_3 \equiv \frac{1}{2}, \frac{1}{2}, 0 \pmod 1$; hence $\mathfrak{m}(f) \geqslant \frac{1}{4}(\rho_{01} + \rho_{02} + \rho_{13} + \rho_{23})$. Since $f_1 = \mathfrak{m}(f)$, (3.6) follows at once from (3.4).

We have thus shown that

$$\delta D > 0, \quad \delta f_1 \leqslant 0,$$

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whence, for all sufficiently small $\epsilon > 0$,

$$\mathfrak{m}(f')D'^{-\frac{1}{3}} = f'_1 D'^{-\frac{1}{3}} < f_1 D^{-\frac{1}{3}} = \mathfrak{m}(f)D^{-\frac{1}{3}}.$$

This contradicts our assumption that f is extreme.

LEMMA 3.2. *If f is extreme, it is impossible that*

$$\lambda_1 = \lambda_2 > \lambda_3. \tag{3.7}$$

Proof. If (3.7) holds, we have $\mathfrak{m}(f) = f_1 = f_2 > f_3$; and, for any sufficiently near form f', $\mathfrak{m}(f') = \max(f'_1, f'_2)$. We choose

$$\delta\rho_{01} = \delta\rho_{23} = -\epsilon_1, \quad \delta\rho_{02} = \delta\rho_{13} = -\epsilon_2,$$

$$\delta\rho_{03} = \delta\rho_{12} = \epsilon_1 + \epsilon_2,$$

where $\epsilon_1 > 0$, $\epsilon_2 > 0$ (noting that (3.7) implies that $\rho_{01}, \rho_{23}, \rho_{02}, \rho_{13}$ are all positive). By restricting ϵ_1, ϵ_2 to satisfy

$$\epsilon_1(\rho_{01} + \rho_{23}) = \epsilon_2(\rho_{02} + \rho_{23}),$$

we ensure that

$$\delta(\lambda_1 - \lambda_2) = 0.$$

By (2.13) and (3.7), and writing for convenience

$$\lambda = \lambda_1 = \lambda_2,$$

we have

$$\begin{aligned}\delta D &= \epsilon_1\left(\frac{\partial D}{\partial\rho_{03}} + \frac{\partial D}{\partial\rho_{12}} - \frac{\partial D}{\partial\rho_{01}} - \frac{\partial D}{\partial\rho_{23}}\right) + \epsilon_2\left(\frac{\partial D}{\partial\rho_{03}} + \frac{\partial D}{\partial\rho_{12}} - \frac{\partial D}{\partial\rho_{02}} - \frac{\partial D}{\partial\rho_{13}}\right)\\ &= 2\epsilon_1(\lambda_1 - \lambda_3) + 2\epsilon_2(\lambda_2 - \lambda_3)\\ &= 2(\epsilon_1 + \epsilon_2)(\lambda - \lambda_3) > 0.\end{aligned}$$

We set

$$M = (\rho_{12} + \rho_{13} + \rho_{23})D - K - 4\lambda_2\lambda_3,$$

so that

$$4f_1 = (\rho_{01} + \rho_{02} + \rho_{03}) + M/D.$$

Arguing as in Lemma 3.1, we have

$$f_1 = \mathfrak{m}(f) \geqslant f(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{4}(\rho_{01} + \rho_{02} + \rho_{03}),$$

whence

$$M \geqslant 0.$$

Also, using (2.16) (with suitable permutations of the suffixes), we obtain

$$\begin{aligned}\delta M &= (\rho_{12} + \rho_{13} + \rho_{23})\delta D - \delta K - 4\delta(\lambda_2\lambda_3)\\ &= -(\epsilon_1 + \epsilon_2)[(\lambda - \lambda_3)(\rho_{01} + \rho_{02}) + \lambda\rho_{03} + \lambda_3\rho_{12}]\\ &< 0,\end{aligned}$$

since $\lambda > \lambda_3$, $\rho_{01} + \rho_{02} > 0$.

Since $\delta D > 0$, $\delta(\rho_{01} + \rho_{02} + \rho_{03}) = 0$ and $\delta M < 0$, we see that $\delta f_1 < 0$; and by symmetry $\delta f_2 < 0$. Hence for all sufficiently small ϵ_1, ϵ_2 we have

$$D' > D, \ \mathfrak{m}(f') < \mathfrak{m}(f),$$

contradicting our assumption that f is extreme.

LEMMA 3.3. *If f is extreme, then*

$$\lambda_1 = \lambda_2 = \lambda_3. \tag{3.8}$$

Proof. By a suitable permutation of suffixes we can ensure that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$; the result now follows from Lemmas 3.1, 3.2.

LEMMA 3.4. *If f is extreme, it is impossible that*

$$\rho_{01} > \rho_{13}, \ \rho_{02} > \rho_{23}. \tag{3.9}$$

Proof. Suppose that (3.9) holds. By Lemma 3.3, (3.8) holds and

$$\mathfrak{m}(f) = f_1 = f_2 = f_3.$$

We make the variation

$$\begin{aligned} -\delta\rho_{01} = \delta\rho_{13} &= \epsilon_1 = \epsilon(\rho_{01} + \rho_{13}), \\ -\delta\rho_{02} = \delta\rho_{23} &= \epsilon_2 = \epsilon(\rho_{02} + \rho_{23}), \\ \delta\rho_{03} = -\epsilon_3 &= -\epsilon(\rho_{01} + \rho_{02} - \rho_{13} - \rho_{23}), \\ \delta\rho_{12} &= 0, \end{aligned}$$

where $\epsilon > 0$. To justify this, we have to show that $\rho_{01} > 0$, $\rho_{02} > 0$, $\rho_{03} > 0$. Clearly $\rho_{01} > 0$, $\rho_{02} > 0$, by (3.9). If now $\rho_{03} = 0$, then $\lambda_3 = 0$, whence $\lambda_1 = \lambda_2 = 0$ by (3.8); this gives $\rho_{23} = 0$, $\rho_{13} = 0$, since $\rho_{01} \neq 0$, $\rho_{02} \neq 0$. But now $f = \rho_{01}x_1^2 + \rho_{02}x_2^2 + \rho_{12}(x_1 - x_2)^2$ and is clearly not positive definite.

It is easy to see that, for all sufficiently small ϵ, we have $\lambda'_1 = \lambda'_2 > \lambda'_3$, so that the neighbouring form f' has

$$\mathfrak{m}(f') = f'_1 \ (= f'_2).$$

For

$$\begin{aligned} \lambda'_1 - \lambda'_2 &= (\rho_{01} - \epsilon_1)(\rho_{23} + \epsilon_2) - (\rho_{02} - \epsilon_2)(\rho_{13} + \epsilon_1) \\ &= \lambda_1 - \lambda_2 - \epsilon_1(\rho_{02} + \rho_{23}) + \epsilon_2(\rho_{01} + \rho_{13}) = 0; \end{aligned}$$

and

$$\begin{aligned} \delta\lambda_1 &= \rho_{01}\epsilon_2 - \rho_{23}\epsilon_1 = \epsilon(\rho_{01}\rho_{02} - \rho_{13}\rho_{23}) > 0, \\ \delta\lambda_3 &= -\ \epsilon_3\rho_{12} \leqslant 0, \end{aligned}$$

so that $\lambda'_1 > \lambda'_3$.

We now obtain a contradiction to the fact that f is extreme by showing that

$$\delta D = 0, \quad \delta f_1 < 0. \tag{3.10}$$

By (2.12) and (2.11) we have

$$\frac{\partial D}{\partial \rho_{01}} - \frac{\partial D}{\partial \rho_{13}} = \sigma_0 - \sigma_3 - \lambda_1 + \lambda_2$$
$$= (\rho_{02} + \rho_{12} + \rho_{23})(\rho_{13} + \rho_{23} - \rho_{01} - \rho_{02}) + \rho_{02}^2 - \rho_{23}^2,$$

and, by symmetry,

$$\frac{\partial D}{\partial \rho_{02}} - \frac{\partial D}{\partial \rho_{23}} = (\rho_{01} + \rho_{12} + \rho_{13})(\rho_{13} + \rho_{23} - \rho_{01} - \rho_{02}) + \rho_{01}^2 - \rho_{13}^2.$$

Hence

$$- \epsilon_1\left(\frac{\partial D}{\partial \rho_{01}} - \frac{\partial D}{\partial \rho_{13}}\right) - \epsilon_2\left(\frac{\partial D}{\partial \rho_{02}} - \frac{\partial D}{\partial \rho_{23}}\right)$$
$$= \epsilon(\rho_{01} + \rho_{02} - \rho_{13} - \rho_{23})[(\rho_{01} + \rho_{13})(\rho_{02} + \rho_{12} + \rho_{23})$$
$$+ (\rho_{02} + \rho_{23})(\rho_{01} + \rho_{12} + \rho_{13})]$$
$$- \epsilon(\rho_{01} + \rho_{13})(\rho_{02}^2 - \rho_{23}^2) - \epsilon(\rho_{02} + \rho_{23})(\rho_{01}^2 - \rho_{13}^2)$$
$$= \epsilon_3[(\rho_{01} + \rho_{13})(\rho_{02} + \rho_{12} + \rho_{23}) + (\rho_{02} + \rho_{23})(\rho_{01} + \rho_{12} + \rho_{13})$$
$$- (\rho_{01} + \rho_{13})(\rho_{02} + \rho_{23})]$$
$$= \epsilon_3(\sigma_0 + \sigma_3 + \lambda_1 + \lambda_3)$$
$$= \epsilon_3 \frac{D}{\rho_{03}},$$

from which it follows immediately that $\delta D = 0$.

Writing, as in Lemma 3.1,

$$L = (\rho_{03} + \rho_{12})D - K - 4\lambda_2\lambda_3$$

we have, using $\delta D = 0$,

$$\delta L = D\delta\rho_{03} - \delta K - 4\delta(\lambda_2\lambda_3);$$

and a calculation similar to the above, using (2.14), (2.15) and (3.8), gives

$$\delta L = - 2\epsilon_3\rho_{03}[(\rho_{01} + \rho_{12} + \rho_{13})(\rho_{02} + \rho_{12} + \rho_{23}) - \rho_{12}^2] < 0,$$

since $\epsilon_3 > 0$, $\rho_{03} > 0$. As in Lemma 3.1 we deduce that $\delta f_1 < 0$.

This establishes (3.10), and the Lemma is proved.

LEMMA 3.5. *If f is extreme, then*

(3.11) $$\lambda_1 = \lambda_2 = \lambda_3 > 0.$$

Proof. By Lemma 3.3, it suffices to prove the impossibility of

(3.12) $$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Now if (3.12) holds, at least three ρ_{ij} are zero. Since in any three ρ_{ij} some suffix occurs at least twice, we may assume by symmetry that

(3.13) $$\rho_{13} = \rho_{23} = 0.$$

Since $\lambda_3 = \rho_{03}\rho_{12} = 0$ and $\rho_{03} \neq 0$ (else f does not involve x_3 and so is not definite) we have $\rho_{12} = 0$. Thus

$$f = \rho_{01}x_1^2 + \rho_{02}x_2^2 + \rho_{03}x_3^2,$$

and, since f is definite, we have

$$\rho_{01} > 0, \quad \rho_{02} > 0. \tag{3.14}$$

Now (3.13) and (3.14) contradict Lemma 3.4.

LEMMA 3.6. *If f is extreme, then*

$$\rho_{01} = \rho_{02} = \rho_{03} = \rho_{12} = \rho_{13} = \rho_{23}. \tag{3.15}$$

Proof. We first show that $\rho_{01} = \rho_{13}$.

If $\rho_{01} \neq \rho_{13}$, then, after interchanging 0 and 3 if necessary, we have

$$\rho_{01} > \rho_{13}.$$

Since by Lemma 3.5

$$\lambda_1 = \rho_{01}\rho_{23} = \rho_{02}\rho_{13} = \lambda_2 > 0,$$

we have also

$$\rho_{02} > \rho_{23}.$$

By Lemma 3.4, these inequalities cannot hold.

Thus $\rho_{01} = \rho_{13}$. By symmetry we have

$$\rho_{ij} = \rho_{jk}$$

for any distinct suffixes i, j, k; from this (3.15) follows immediately.

Lemma 3.6 shows that the only possible class of extreme forms is that represented by

$$f_0(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2;$$

and (1.4) of Theorem 1 is simply verified for $f = f_0$ by substituting $\rho_{ij} = 1$ in the formulae of §2.

Hence to complete the proof of Theorem 1 we have only to show that f_0 is in fact extreme. A direct proof of this is not difficult, but is rather tedious. It is simpler to appeal to a general theorem of Hlawka (2) which asserts the existence of a most economical lattice-covering of space, and hence the existence of a class of absolutely extreme forms (which can only be the class of f_0).

4. Remarks on the method. Voronoï (3; 4; 5) has given two distinct methods of reduction of positive quadratic forms. The first is based on the concept of perfect forms, and leads to a finite number of regions $R_0, R_1, \ldots, R_\tau$ in the $\frac{1}{2}n(n + 1)$-dimensional coefficient space, with the properties: (i) any form is equivalent to a form lying in one of the regions R; (ii) no two forms lying in the interior of different regions are equivalent.

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The second is based on the consideration of types of space-filling polytopes (which may be derived from positive forms, as we derived Π from f in §2), and leads to regions $R'_0, R'_1, \ldots, R'_\tau$ having the same two properties.

The "principal regions" R_0, R'_0 are derived respectively from the perfect form

$$\phi_0 = \sum_1^n x_i^2 + \sum_{i<j} x_i x_j$$

and its adjoint, a multiple of

$$f_0 = n\sum_1^n x_i^2 - 2\sum_{i<j} x_i x_j;$$

and in fact $R_0 \equiv R'_0$.

For $n = 2$ and $n = 3$, $R_0 = R'_0$ is the only region, and we obtain for $n = 3$ the definition of reduction used in §2. For general $n \geqslant 2$, R_0 is the set of forms expressible as

$$f(x) = \sum_{i<j} \rho_{ij}(x_i - x_j)^2,\ \rho_{ij} \geqslant 0\ (i, j = 0, 1, \ldots, n),\ x_0 = 0. \tag{4.1}$$

It is to be noted that the regions R or R' do *not* possess the property that no two forms interior to the same region are equivalent; for example, the result of Lemma 2.1 generalizes in the obvious way for the form (4.1). This fact, which (as Voronoï remarks) is normally a disadvantage in a method of reduction, is clearly seen from the analysis of §§2 and 3 to be of considerable advantage in the problem we have been investigating. What Voronoï's second method of reduction achieves is the specification of the broadest type of forms whose polytopes Π (when not degenerate) are defined by the same set of integral points l; there is therefore little doubt that this method of reduction is best suited to the covering problem for each $n \geqslant 2$.

In conclusion, it is perhaps worth noting that the case $n = 2$ (for which there is just one region $R_0 = R'_0$) is very simply settled by these methods, and leads to

THEOREM 2. *If $n = 2$, there is just one class of extreme forms, represented by*

$$f_0(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2,$$

and for this class

$$\mathfrak{m}(f) = \left(\frac{4}{27} D\right)^{\frac{1}{2}}.$$

We take f in R_0, i.e.

$$f(x_1, x_2) = \rho_{01}x_1^2 + \rho_{02}x_2^2 + \rho_{12}(x_1 - x_2)^2, \quad \rho_{ij} \geqslant 0,$$

for which

$$D = \rho_{01}\rho_{02} + \rho_{01}\rho_{12} + \rho_{02}\rho_{12},$$

$$4D\mathfrak{m}(f) = 4Df(v) = D(\rho_{01} + \rho_{02} + \rho_{12}) - \zeta_{01}\zeta_{02}\zeta_{12},$$

(the value of $f(v)$ being the same for all vertices v of Π).

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If $\rho_{01} > \rho_{02}$, we take $\delta\rho_{01} = -\epsilon$, $\delta\rho_{02} = \epsilon$, $\epsilon > 0$, whence trivially D is increased and

$$4f(v) = (\rho_{01} + \rho_{02}) + \rho_{12}^2(\rho_{01} + \rho_{02})/D$$

is not increased; thus f cannot be extreme.

By symmetry it now follows that, for extreme f, we require $\rho_{01} = \rho_{02}$, and so $\rho_{01} = \rho_{02} = \rho_{12}$. Theorem 2 follows at once.

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