MODELS FOR OFFICIAL ENTAILMENT

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Abstract. This paper shows how to set up Fine's "theory-application" type semantics so as to model the use-unrestricted "Official" consequence relation for a range of relevant logics. The frame condition matching the axiom $(((A \to A) \wedge (B \to B)) \to C) \to C$—the characteristic axiom of the very first axiomatization of the relevant logic **E**—is shown forth. It is also shown how to model propositional constants within the semantic framework. Whereas the related Routley–Meyer type frame semantics fails to be strongly complete with regards to certain contractionless logics such as **B**, the current paper shows that Fine's weak soundness and completeness result can be extended to a strong one also for logics like **B**.

§1. Introduction. This paper shows that there is a natural way to define semantical consequence using the "theory-application" type semantics first set forth in [17][1] which for a range of logics yields a strong soundness and completeness result relative to what Anderson and Belnap, tongue-in-cheek, called the "Official" relation of deducibility (cf. [1, sec. 22.2.1]).[2]

One of the striking features of Fine's presentation is that it doesn't present a frame condition matching to the axiom

$$(((A \to A) \wedge (B \to B)) \to C) \to C.$$

This is the characteristic axiom of the first system for the logic **E** ever presented [2].[3] One of the merely admissible rules of **E** is the

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1 The content of Fine's paper is also to be found as Section 51 of [3]. For a historical and conceptual account of how this type of semantics relates to the ternary Routley–Meyer semantics, see Dunn and Restall [16], Bimbó and Dunn [10], and Tedder [41].

2 Although the term is meant to carry a significant flavor of disapproval, I argued in Øgaard [26] that Anderson and Belnap, in fact, acknowledge it as one of the correct accounts of consequence.

3 Later Anderson and Belnap came to prefer an axiomatization in which this axiom was replaced by two axioms—$((A \to A) \to B) \to B$ and $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, where $\Box A := (A \to A) \to A$. The latter of these, however, fails to be a logical theorem of weaker logics than **E** even though the mentioned characteristic axiom (A13) is. For instance, the logic obtained from the system B (as set forth in this paper) augmented by A13, fails to have $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ as a logical theorem.

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following:[4]

$$\text{(assertion rule)} \quad \{A\} \Vdash (A \to B) \to B.$$

The system named 'E' in [17], however, has the assertion rule rather than the above axiom as its characteristic feature. Furthermore, the frame condition which is specified as adequate for the assertion rule, is as claimed, but fails, then, to adequately model the E-axiom $(((A \to A) \wedge (B \to B)) \to C) \to C$. Although Fine's set-up does suffice for a *weak* completeness result—every semantically valid formula is a logical theorem of **E**—it doesn't yield a *strong* completeness result in any straight forward sense.[5] This paper shows how to model the characteristic axiom of E, and how to tweak Fine's semantics so as to yield strong completeness. It is also shown how to model propositional constants—such as the *Ackermann constant*—within Fine's semantics.

Fine's semantics is a type of frame semantics, the evaluation points of which are best thought of as *theories*—sets of sentences closed under conjunction and what Anderson and Belnap called *entailment*, so that the consequent of a provable conditional belongs to the theory if the antecedent does. The ground theory, relative to a set of premises Θ, of any frame can in general be thought of as the theory generated by the Official consequence relation—the set of Official consequences of Θ. In Fine's set-up, such a set need not be *prime*. For instance, $p \vee {\sim}p$ is a logical theorem of **E**, but neither p nor ${\sim}p$ need to belong to the ground theory. The related ternary Routley–Meyer semantics, on the other hand, dispenses with all such nonsaturated evaluation points.

The Routley–Meyer semantics can also model Official consequence for a range of logics, including **E**.[6] It cannot, however, model this relation for all logics. Examples of this failure include the contractionless relevant logics **B**, **DW**, **TW**, and **EW**. The set of logical theorems of each of these logics is prime (cf. [38, 39]). This, however, does not extend in the requisite way to Official consequence in that the mentioned contractionless logics all fail to have derivable the disjunctive version of all of their primitive rules. In the case of the first three, $\{A \vee C, (A \to B) \vee C\} \Vdash B \vee C$ fails to be a derivable rule, whereas in the case of the latter, the disjunctive version of

A *system* is here to be understood as an axiom system—a set of axioms and rules—together with a definition of derivability. A *logic* is the consequence relation generated by such a system, relative to a language. To be precise, then, a logic is to be identified as a pair $\langle \mathcal{L}, \vdash_L \rangle$, where $\mathcal{L}$ is a language, and $\vdash_L$ is the consequence relation generated using the system L over the language $\mathcal{L}$. The languages considered in this paper are set forth in Definition 2.2. To avoid confusion, I'll use bold-faced capital letters to refer to logics, and italicized capital letters to refer to systems. The system E defined in Definitions 2.3 and 2.5, then, axiomatize the logic **E** as understood in this paper.

4 *Assertion* is a name for the axiom $A \to ((A \to B) \to B)$; *restricted assertion* a name for $(A \to C) \to (((A \to C) \to B) \to B)$ (cf. [1, p. 26]).

5 I should hasten to note that Official consequence is representable as enthymematical consequence in the case of logics like **E**—B is Officially derivable from a set of formulas Δ if and only if $A \wedge Tm \to B$ is a logical truth of **E**, where A is a conjunction of a subset of Δ, and Tm is a logical truth of **E**. No such result, however, has been shown to hold in case of weaker logics like **B**. Hence the need for the more direct approach to be provided in this paper. Furthermore, the enthymematical deduction theorem does not hold using Fine's axiomatization E, seeing as for any propositional variable p, there are no logical theorem T of **E** such that $(p \wedge T) \to ((p \to p) \to p)$ is a logical theorem. That this is so is easily checked using `MaGIC` [37].

6 Cf. [3] and [28].

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its characteristic rule—the already mentioned assertion rule—fails to be derivable. This is indeed the case not only for **EW**, but even for the logic obtained from Fine's axiomatization E mentioned above.[7] The semantical consequence relation used in the Routley–Meyer semantics, however, counts a rule as truth-preserving just in case it is truth-preserving over—depending on the set-up—a single or collection of ground points. Since each point validates a disjunction if and only if it validates at least one disjunct, it follows that if a rule $\{A_1, \dots, A_n\} \Vdash B$ is truth-preserving, then so is its *disjunctive* version $\{A_1 \vee C, \dots, A_n \vee C\} \Vdash B \vee C$. It follows, then, that such a set-up cannot yield a strong completeness result for the mentioned contractionless logics. In allowing for nonsaturated theories, then, Fine's semantics allows one to model Official consequence for a range of logics outside the reach of, if not all, then at least the standard ways of setting up the Routley–Meyer semantics.

Some systems for relevant logics come equipped with disjunctive rules.[8] This paper also shows that there is but a single frame condition which must be added so as to obtain a semantics fit for the disjunctive extension of a given logic. All in all, then, this makes Fine's semantics both very flexible as well as more encompassing than the Routley–Meyer semantics.

The Official consequence relation for relevant logics fails to satisfy the standard notion of relevance—variable sharing from premises to conclusion—as it allows every logical theorem to follow from any set of formulas. The question arises, therefore, as to the very motivation for studying Official consequence for these logics. Here are two reasons for doings so. First of all, relevant logicians often do *apply* the Official consequence relation.[9] A useful formal semantics ought to be able to model such applications. The other reason that I want to highlight is the following: In order to have an informed debate within the philosophy of logic as to how best to understand the core ideas of logical consequence—such as the very notion of relevance—it is arguably preferable not to have formal results restricted by the particular aims of a specific school of logic. To show that this latter point also pertains to the current case, note that some logicians have argued that the Official consequence is the correct notion of logical consequence whilst claiming that the Official consequence relation of any acceptable logic must satisfy certain properties of relevance. For instance, Avron [4] sets forth the property called the *basic relevance criterion (BRC)* and argued it to be the "most basic criterion for relevance" [4, p. 28].[10] Avron [5] showed that **R** satisfies BRC using a weak completeness theorem for the Routley–Meyer semantics. Avron's proof relies on the enthymematical deduction theorem mentioned in fn. 5. Avron's proof can, however, be tweaked so as not to rely on this feature if a strong completeness theorem is available (cf. [29]). It is easily shown that a strong completeness result using

7 See [28] for a discussion of this.

8 According to Brady [13], the use of such rules trace back to Meyer. The use of disjunctive rules as in this paper can be found in for instance [12], [31], and [32].

9 Here are but a few examples: For the case of relevant arithmetic, see for instance [14], [35], [40], [25], and [34]. McKubre–Jordens and Weber [23] explicitly acknowledges the use of Official consequence in investigating relevant analysis, whereas the naïve set theories investigated in [35], [13], [30], [42] and [18] are all closed under Official consequence.

10 See [29] for a discussion of Avron's suggestion. A consequence relation $\vdash$ satisfies BRC just in case $\Gamma \vdash A$ if $\Gamma \cup \Delta \vdash A$, where Δ shares no propositional variable with $\Gamma \cup \{A\}$.

Fine's semantics would do as well.[11] This paper shows for the first time that a range of familiar relevant logics—logics like **B**—which fail to have the disjunctive version of all of their rules derivable, are strongly sound and complete with regards to a frame semantics. The result, then, can be used to prove that such weak logics satisfy BRC.

The plan for the paper is as follows: The next section sets forth some initial definitions, including how axiomatizations of logics are to be pieced together and how Official consequence is to be understood. Then follows a section setting forth the proof-theoretic machinery, then one dedicated to the semantics. The trailing two sections show soundness and completeness, respectively, before the final section gives a short summary. There is also an appendix which goes into a slight trouble with Fine's semantic clause for the propositional constant expressing "maximum necessity." The appendix explains a different fix suggested by one of the referees than what is found in the main part of the paper.

§2. Initial definitions.

DEFINITION 2.1 (Parenthesis conventions). *$\vee$ and $\wedge$ are to bind tighter than $\to$, and so I'll usually drop parenthesis enclosing conjunctions and disjunctions whenever possible. Association is otherwise to the left and so $\sim A \wedge B \wedge C \to D \vee E$ is simply shorthand for $((\sim A \wedge B) \wedge C) \to (D \vee E)$.*

DEFINITION 2.2. *The languages considered in this paper are all to be built up from the set of propositional variables $Var := \{p_0, p_1, \ldots\}$ together with any subset of the set of propositional constants $Con := \{\mathfrak{c}_0, \mathfrak{c}_1, \ldots\}$, using the connectives $\sim, \wedge, \vee$, and $\to$.*

For presentational purposes, it will suffice to restrict the attention to a single propositional constant—$\mathfrak{c}$—and so the axioms and rules to be considered for propositional constants will all be stated using '$\mathfrak{c}$.'[12]

DEFINITION 2.3 (Official consequence/entailment[13]). *An* OFFICIAL PROOF *of a formula A from a set of formulas Γ using the axiom system L is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of L, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of L. The existential claim that there is such a proof is written $\Gamma \vdash^{\mathbb{O}}_{L} A$.*

Although the only derivability relation which will be *used* in this paper is the Official one, it is worth stating clearly the one used in Fine's paper (cf. [17, p. 351]).

DEFINITION 2.4 (L-deducibility). *For any system L, B is L-DEDUCIBLE from Δ if there is a sequence of formulas $A_0, \ldots, A_n$ such that $B = A_n$ and $\forall i \leq n$: $A_i \in \Delta$ or $\exists j, k < i$ such that $A_i = A_j \wedge A_k$ or $\exists j < i$ such that $\varnothing \vdash^{\mathbb{O}}_{L} A_j \to A_i$.*

[11] The main ingredient of the BRC-proof is that the class of frames for a logic is closed under products, something the Fine-type frames set forth in this paper are easily verified to be.

[12] See [27] for a more general inquiry into propositional constants for relevant logics in the context of the so-called *simplified* Routley–Meyer semantics.

[13] This notion of a proof yields what is called an *E-theory* in [20]. See [9] for a detailed investigation into Maksimova's contribution to relevant logics.

DEFINITION 2.5 (Systems).

B	$A1–A7$; $R1–R5$
DW	$A1–A8$; $R1–R4$
TW	$A1–A10$; $R1–R2$
EW	$TW[R6]$
T	$TW[A11, A12]$
E	$T[A13]$
Π'	$E[R7]$
R	$T[A14]$.

DEFINITION 2.6 (Logic). *If 'L' is a name for a system over a language $\mathcal{L}$, then '**L**' is a name for the* LOGIC *obtained from L, i.e., the consequence relation* $\{\langle\Theta, A\rangle \mid \Theta \vdash^{\mathbb{O}}_{L} A \;\&\; \Theta \cup \{A\} \subseteq wff_{\mathcal{L}}\}$.

DEFINITION 2.7. *A logic* $\mathbf{L}_2$ EXTENDS *a logic* $\mathbf{L}_1$ *just in case every well-formed formula (wff) of* $\mathbf{L}_1$ *is a wff of* $\mathbf{L}_2$ *and that for every set of* $\mathbf{L}_1$*-wffs* $\Gamma \cup \{A\}$, *if* $\Gamma \vdash^{\mathbb{O}}_{L_1} A$, *then also* $\Gamma \vdash^{\mathbb{O}}_{L_2} A$.

Every logic considered in this paper will be extensions of the relevant logic **B**. The system for **B** used in this paper, along with some of the more common systems for relevant logics, is set forth in Definition 2.5. The axioms and rules of any system to be considered in this paper are all found in Table 1. Universal claims on the form "every system ..." are, then, to be understood as tacitly restricted to this rather restricted class of systems.

DEFINITION 2.8. *For any system L and axioms/rules* $\theta_1, \dots, \theta_n$: $L[\theta_1, \dots, \theta_n]$ *is the system obtained by adding* $\theta_1, \dots, \theta_n$ *as axioms/rules and expanding the language to include any connective occurring in* $\theta_1, \dots, \theta_n$ *which is not already present in L.*

DEFINITION 2.9 (Derivable versus admissible rules).

- *A rule* $\Delta \Vdash A$ *is said to be* DERIVABLE *in a logic* **L** *just in case* $\Delta^{\sigma} \vdash^{\mathbb{O}}_{L} A^{\sigma}$ *for every uniform substitution* σ.
- *A rule* $\Delta \Vdash A$ *is said to be* ADMISSIBLE *in a logic* **L** *just in case for every uniform substitution* σ, *if* ($\varnothing \vdash^{\mathbb{O}}_{L} \delta^{\sigma}$ *for every* $\delta \in \Delta$), *then* $\varnothing \vdash^{\mathbb{O}}_{L} A^{\sigma}$.

As we'll see, Fine's semantics can easily be set up so as to yield strong soundness and completeness for systems with added primitive disjunctive rules. The following definition allows for easier reference to such systems.

DEFINITION 2.10. *If L is a system with* $\{\rho_1, \dots, \rho_n\} \subseteq \{R1 - R7, \mathfrak{c}6\}$ *as rules, then* L^d *is the system obtained by adding* $\rho_1^d - \rho_n^d$ *as further rules, where* ρ_i^d *is the disjunctive version of the rule* ρ_i.

DEFINITION 2.11. $\mathbf{L}^d$ *is the logic obtained from any system* M^d, *where M axiomatizes* **L**.

PROPOSITION 2.1. $\mathbf{L}^d$ *is well defined.*

Proof. It suffices to show that the disjunctive rules of a system M^d are derivable using the system N^d, where M and N both axiomatize **L**. This follows easily from the fact that for any system L, if $A_1, \dots, A_n \vdash^{\mathbb{O}}_{L^d} B$, then $A_1 \vee C, \dots, A_n \vee C \vdash^{\mathbb{O}}_{L^d} B \vee C$. □

PROPOSITION 2.2. $\mathbf{E} = \mathbf{E}^d$.

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Table 1. *Axioms and rules*

(A1)	$A \to A$	
(A2)	$A \to A \vee B$ and $B \to A \vee B$	
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$	
(A4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	
(A5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	
(A6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	
(A7)	$\sim\sim A \to A$ and $A \to \sim\sim A$	
(A8)	$(A \to B) \to (\sim B \to \sim A)$	
(A9)	$(A \to B) \to ((C \to A) \to (C \to B))$	
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	
(A11)	$(A \to (A \to B)) \to (A \to B)$	
(A12)	$(A \to \sim A) \to \sim A$	
(A13)	$((A \to A) \wedge (B \to B) \to C) \to C$	
(A14)	$A \to ((A \to B) \to B)$	
(A15)	$A \vee \sim A$	
(R1)	$\{A, B\} \Vdash A \wedge B$	
(R2)	$\{A, A \to B\} \Vdash B$	
(R3)	$\{A \to B\} \Vdash (C \to A) \to (C \to B)$	
(R4)	$\{A \to B\} \Vdash (B \to C) \to (A \to C)$	
(R5)	$\{A \to B\} \Vdash \sim B \to \sim A$	
(R6)	$\{A\} \Vdash (A \to B) \to B$	
(R7)	$\{A, \sim A \vee B\} \Vdash B$	
($\mathfrak{c}$1)	$\mathfrak{c}$	
($\mathfrak{c}$2)	$\mathfrak{c} \to (A \to A)$	
($\mathfrak{c}$3)	$\mathfrak{c} \to (A \vee \sim A)$	
($\mathfrak{c}$4)	$(\mathfrak{c} \to \mathfrak{c}) \to \mathfrak{c}$	
($\mathfrak{c}$5)	$(\mathfrak{c} \to A) \to A$	
($\mathfrak{c}$6)	$\{A\} \Vdash \mathfrak{c} \to A$	
($\mathfrak{c}$7)	$A \to (\mathfrak{c} \to A)$	
($\mathfrak{c}$8)	$A \wedge \sim A \wedge \mathfrak{c} \to B$	
($\mathfrak{c}$9)	$\mathfrak{c} \wedge \sim \mathfrak{c} \to A$	
($\mathfrak{c}$10)	$A_1 \to (\ldots \to (A_n \to \mathfrak{c}) \ldots)$	$(1 \leq n)$

Proof. This follows from the fact that R^d1 and R^d2—disjunctive adjunction and disjunctive modus ponens—are derivable rules of **E**. That this is the case follows easily from the fact that reasoning by cases holds in **E**, so that if $\{A\} \models^{\mathbb{O}}_{E} C$ and $\{B\} \models^{\mathbb{O}}_{E} C$, then also $\{A \vee B\} \models^{\mathbb{O}}_{E} C$ (cf. [24, p. 461]). □

A quick note on commonly used propositional constants is in order. The *Church constant* $\top$ is often added to relevant logics with the intended reading of being a "trivial" truth. The frame condition corresponding to $\mathfrak{c}$10 ensures that $\mathfrak{c}$ holds throughout the frame and ensures, then, that any $\mathfrak{c}$ so axiomatized is indeed a trivial truth. The more interesting category of propositional constants often goes by the name of the *Ackermann constant*. There are two "standard" readings of such a constant. The first is as the conjunction of the theory at hand. If $\mathfrak{w}$ is to be read as such a constant, one may

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consider $\mathfrak{w}$ as an axiom, along with the rule $\{A\} \Vdash \mathfrak{w} \to A$, or its stronger axiomatic form $A \to (\mathfrak{w} \to A)$. The other standard reading is of a logical truth of some sort. If $\mathfrak{t}$ is to express *logical* truth, then it seems unfitting in the context of Official consequence to have the rule $\{A\} \Vdash \mathfrak{t} \to A$ become derivable. One may therefore consider axiomatizing it using $\mathfrak{t} \to (A \to A)$ (and in some cases also further axioms such as $\mathfrak{t} \to (A \vee {\sim}A)$) to ensure that $\mathfrak{t}$ relevantly implies every logical truth, along with either $\mathfrak{t}$ or, if a modal reading is sought after, $\Box\mathfrak{t}$, to ensure that $\mathfrak{t}$ itself comes out as (necessarily) true.[14]

§3. Models for Official entailment. The semantics set forth in [17] is a frame semantics in which the evaluation points are to be thought of as *theories*. There is a distinguished ground theory in every frame which truth in a model is defined with respect to. Fine uses '*l*' to designate this ground theory. Seeing as this theory ends up being the set of logical theorems of the logic in question, the name can be read as short for *logic*. The ground theory when Official consequence is the study object, however, ends up containing not only the set of logical theorems, but every Official consequence of the set of premises. So as to avoid confusion, I will therefore use '@' to refer to the distinguished theory of any frame. I follow Fine's example in letting T be the set of theories (the evaluation points) of a frame, and S its set of saturated such. The theories in S are called *saturated* because they "contain a disjunct of any contained disjunction" and as such "answer every either-or question they raise" [17, p. 349]. Every frame comes with what has become known as the *Routley star* operator which interprets negation,[15] and a partial order on the set of theories for the relation of subtheory-hood. The last ingredient of a frame is a binary function on theories intended to be interpreted as *theory application*. For theories t and u, $t \cdot u$ is committed to a proposition P just in case there is a proposition Q in u such that t is committed to Q being sufficient for P.[16] I follow Fine in letting $t, u, v, \ldots$ range over elements of T, whereas $a, b, c, \ldots$ range over elements of S. I will therefore suppress restricted quantifiers to T and S whenever this increases readability.[17]

DEFINITION 3.1. *A B*-FRAME *is a sextuple*

$$\mathcal{F} = \langle T, S, @, \cdot, *, \leqslant \rangle$$

such that for every $a, b, c \in S$ and $t, u, v, w \in T$,

14 See [1] for Anderson and Belnap's discussion of such constants, and [27] for a discussion of their view. Propositional constants often come in pairs, one positive and one negative. For want of space, this paper focuses on positive constants. Negative one—those constants $\mathfrak{c}_i$ for which $A \to \mathfrak{c}_i$ is to read as some form of negation of A—can be modeled using the defined *compatibility* relation $tC_iu := \exists v(t \cdot u \leqslant v \;\&\; v \notin \mathbb{C}_i)$ in line with [27] with, then, $t \vDash A \to \mathfrak{c}_i \Leftrightarrow \forall u(tCu \Rightarrow u \nvDash A)$. For more on such a compatibility relations, see Dunn [15], Berto [6], Berto and Restall [7], and Restall [33, sec. 11.4].

15 The Routley star in the form stated below dates back to [36]. It is, however, a mere notational variant of the semantic clause for negation given in [8] (cf. [16, sec. 3.4]) Fine calls the star-mate of a theory its *co-theory*.

16 For more on how to interpret the semantics philosophically, the reader is referred to [19] and [21].

17 Normally a class of frames is defined with reference to a logic rather than a system as will be the case in this paper. The reason frames are defined this way here is because it's convenient to differentiate frames fit for L-systems and L^d-systems even though $\mathbf{L}$ and $\mathbf{L}^d$ might be the same logic as is the case for $\mathbf{E}$.

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1. $@ \in T$
2. $S \subseteq T$
3. $\cdot : T^2 \mapsto T$
4. $* : S \mapsto S$
5. $\leqslant$ *is a reflexive, transitive and antisymmetrical relation on* T
6. $u \leqslant t \,\&\, v \leqslant w \Rightarrow u \cdot v \leqslant t \cdot w$
7. $t \cdot u \leqslant a \Rightarrow \exists b \exists c((t \leqslant b \,\&\, u \leqslant c) \,\&\, (b \cdot u \leqslant a \,\&\, t \cdot c \leqslant a))$
8. $@ \cdot t = t$
9. $a^{**} = a$
10. $a \leqslant b \Rightarrow b^* \leqslant a^*$.

It will also be convenient to use the following abbreviations:

- $S(t) := \{a \in S \mid t \leqslant a\}$
- $\mathbb{L} := \{t \mid \forall u(u \leqslant t \cdot u)\}$.

DEFINITION 3.2. *If* $\mathcal{F}$ *is any frame for a system* L, *and* $\mathfrak{c}_i$ *is a propositional constant not occurring in the language of* L, *then a frame for* L *augmented by this propositional constant is obtained by adding a* TRUTH SET*— a frame element* $\mathbb{C}_i$, *which can be any subset of* T *that satisfies the condition that*

$$\forall t(t \in \mathbb{C}_i \Leftrightarrow S(t) \subseteq \mathbb{C}_i).$$

I'll use '$\mathbb{C}$' for the truth set of the generic propositional constant '$\mathfrak{c}$.'

DEFINITION 3.3. *A function* $\phi : T \times At \mapsto \{0, 1\}$ *is an* EVALUATION FUNCTION *for a frame provided it satisfies the condition that*

$$\phi(t, p) = 1 \Leftrightarrow \forall a(t \leqslant a \Rightarrow \phi(a, p) = 1)$$

for every $t \in T$ *and* $p \in At$ $(and\ a \in S)$.

If ϕ *is an evaluation function on a frame* $\mathcal{F}$, *then* $\mathfrak{M} = \langle \mathcal{F}, \phi \rangle$ *is called a* MODEL *over* $\mathcal{F}$.

DEFINITION 3.4. *For every model there is a commitment relation* $\vDash$ *generated as follows*:

(i)	$t \vDash p$	$\Leftrightarrow$	$\phi(t, p) = 1$
(ii)	$t \vDash A \wedge B$	$\Leftrightarrow$	$t \vDash A \,\&\, t \vDash B$
(iii)	$t \vDash A \vee B$	$\Leftrightarrow$	$\forall a(t \leqslant a \Rightarrow (a \vDash A \text{ or } a \vDash B))$
(iv)	$t \vDash {\sim} A$	$\Leftrightarrow$	$\forall a(t \leqslant a \Rightarrow a^* \nvDash A)$
(v)	$t \vDash A \rightarrow B$	$\Leftrightarrow$	$\forall u(u \vDash A \Rightarrow t \cdot u \vDash B)$
(vi)	$t \vDash \mathfrak{c}_i$	$\Leftrightarrow$	$t \in \mathbb{C}_i$.

DEFINITION 3.5.

- *A formula* A *is* TRUE IN A MODEL $\mathfrak{M}$—$\mathfrak{M} \vDash A$—*just in case* $@ \vDash A$.
- *A rule* $\{A_1, \dots, A_n\} \Vdash B$ PRESERVES TRUTH IN $\mathfrak{M}$, *just in case* $(\forall i \leq n : \mathfrak{M} \vDash A_i) \Rightarrow \mathfrak{M} \vDash B$.
- *A formula (rule)* HOLDS IN A FRAME $\mathcal{F}$ *just in case it is true (preserves truth) in every model* $\mathfrak{M}$ *over* $\mathcal{F}$.
- OFFICIAL SEMANTIC CONSEQUENCE IN A MODEL: *for any set of formulas* $\Theta \cup \{A\}$,

$$\Theta \vDash^{\mathbb{O}}_{\mathfrak{M}} A \Leftrightarrow (\mathfrak{M} \vDash B \text{ for every } B \in \Theta \Rightarrow \mathfrak{M} \vDash A).$$

- *An* L-MODEL *is a model which satisfies all the frame conditions corresponding to the axioms and rules of* L.

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Table 2. *Fine's frame conditions*

	Frame condition
$\mathcal{F}(A8)$	$t \cdot a \leqslant b \Rightarrow t \cdot b^* \leqslant a^*$
$\mathcal{F}(A9)$	$t \cdot (u \cdot v) \leqslant (t \cdot u) \cdot v$
$\mathcal{F}(A10)$	$t \cdot (u \cdot v) \leqslant (u \cdot t) \cdot v$
$\mathcal{F}(A11)$	$(t \cdot u) \cdot u \leqslant t \cdot u$
$\mathcal{F}(A12)$	$a \cdot a^* \leqslant a$
$\mathcal{F}(A14)$	$t \cdot u \leqslant u \cdot t$
$\mathcal{F}(A15)$	$\forall a(@ \leqslant a \Rightarrow a^* \leqslant a)$
$\mathcal{F}(R6)$	$t \cdot @ \leqslant t$
$\mathcal{F}(R7)$	$\forall a(@ \leqslant a \Rightarrow \exists b(@ \leqslant b \;\&\; b \leqslant a \;\&\; b \leqslant b^*))$

- Official semantic consequence *for a system* L: *for any set of formulas* $\Theta \cup \{A\}$,

$$\Theta \models^{\mathbb{O}}_{L} A \Leftrightarrow \Theta \models^{\mathbb{O}}_{\mathfrak{M}} A \textit{ for every } L\textit{-model } \mathfrak{M}.$$

Lemma 3.1 (Saturatedness lemma). *For any model* $\mathfrak{M}$, *with* $t \in T$ *and* A *any formula*,

$$t \models A \Longleftrightarrow \forall a(a \in S(t) \Rightarrow a \models A).$$

Proof. By induction on the complexity of A.

1. The atomic case is by definition of an evaluation function at the commitment relation for such.
2. $\mathfrak{c}$: Trivial given the requirement on $\mathbb{C}$ that $\forall t(t \in \mathbb{C} \Leftrightarrow S(t) \subseteq \mathbb{C})$.
3. $\vee$: From Definition 3.4(iii) using the fact that $\leqslant$ is reflexive.
4. $\sim$, $\wedge$ and $\rightarrow$: See [17]. □

Lemma 3.2 (Persistence lemma). *For any model* $\mathfrak{M}$, *with* $t, u \in T$ *and* A *any formula*,

$$t \leqslant u \;\&\; t \models A \Longrightarrow u \models A.$$

Proof. Since $t \leqslant u \Rightarrow S(u) \subseteq S(t)$, the claim follows from Lemma. 3.1. □

To obtain, then, a frame for any logic extending **B** axiomatized using any of the axioms and rules displayed above, one adds the associated frame conditions. These are listed in Tables 2 and 3. To obtain a frame for any such system augmented with the disjunctive version of every one of its primitive rules—if the system in question is one of the L^d-systems of Definition 2.10—one must in addition add the frame condition $\mathcal{F}(^d)$.

Fine makes use of a defined disjunction. The following lemma simply shows that the commitment clause used here is equivalent to the defined one used by Fine.

Lemma 3.3. *For any* t *in any model*, $t \models A \vee B \Leftrightarrow t \models {\sim}({\sim}A \wedge {\sim}B)$.

Proof. Left for the reader. □

Lemma 3.4. *For any model* $\mathfrak{M}$,

1. $@ \in \mathbb{L}$
2. $t \in \mathbb{L} \;\&\; t \leqslant u \Rightarrow u \in \mathbb{L}$
3. $t \in \mathbb{L} \Rightarrow t \models A \rightarrow A$
4. $a \models A \vee B \Leftrightarrow a \models A$ *or* $a \models B$.

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Table 3. *Frame conditions for A13, $\mathfrak{c}$-principles, and disjunctedness*

	Frame condition
$\mathcal{F}(A13)$	$\forall t \exists u(u \in \mathbb{L}\ \&\ t \cdot u \leqslant t)$
$\mathcal{F}(\mathfrak{c}1)$	$@ \in \mathbb{C}$
$\mathcal{F}(\mathfrak{c}2)$	$\mathbb{C} \subseteq \mathbb{L}$
$\mathcal{F}(\mathfrak{c}3)$	$\mathbb{C} \subseteq \{t \mid \forall a(t \leqslant a \Rightarrow a^* \leqslant a)\}$
$\mathcal{F}(\mathfrak{c}4)$	$\forall t(\forall u(u \in \mathbb{C} \Rightarrow t \cdot u \in \mathbb{C}) \Rightarrow t \in \mathbb{C}))$
$\mathcal{F}(\mathfrak{c}5)$	$\forall t \exists u(u \in \mathbb{C}\ \&\ t \cdot u \leqslant t)$
$\mathcal{F}(\mathfrak{c}6)$	$\mathbb{C} \subseteq \{t \mid @ \leqslant t\}$
$\mathcal{F}(\mathfrak{c}7)$	$\mathbb{C} \subseteq \{t \mid \forall u(u \leqslant u \cdot t)\}$
$\mathcal{F}(\mathfrak{c}8)$	$\forall t(t \in \mathbb{C} \Rightarrow \exists a(t \leqslant a\ \&\ t \leqslant a^*))$
$\mathcal{F}(\mathfrak{c}9)$	$\forall t(t \in \mathbb{C} \Rightarrow \exists a(t \leqslant a\ \&\ a^* \in \mathbb{C}))$
$\mathcal{F}(\mathfrak{c}10)$	$\forall t(t \in \mathbb{C})$
$\mathcal{F}(^d)$	$@ \in S$

Proof.

1. From Definition 3.1(5 & 8).
2. Assume that $t \in \mathbb{L}\ \&\ t \leqslant u$. For $u \in \mathbb{L}$, it must be the case that $v \leqslant u \cdot v$, for every $v \in T$. Since $t \in \mathbb{L}$, $v \leqslant t \cdot v$, and since $t \leqslant u$, it follows from Definition 3.1(5–6) that $t \cdot v \leqslant u \cdot v$. Since $\leqslant$ is transitive, it therefore follows that $v \leqslant u \cdot v$.
3. Assume that $t \in \mathbb{L}$. To show that $t \vDash A \rightarrow A$, let $u \in T$ be such that $u \vDash A$. Since $t \in \mathbb{L}$, it follows that $u \leqslant t \cdot u$, and so $t \cdot u \vDash A$ by Lemma 3.2. But then $t \vDash A \rightarrow A$ by Definition 3.4(v).
4. See [17, p. 350]. □

Before we turn to soundness, note the following.

Proposition 3.1. *$\mathbb{L} \subseteq \mathbb{C}$ in any $\mathcal{F}(\mathfrak{c}5)$-frame.*

Proof. Assume that $t \in \mathbb{L}$. To show that $t \in \mathbb{C}$ it suffices by Definition 3.2 to show that $a \in \mathbb{C}$ for every $a \in S(t)$. By $\mathcal{F}(\mathfrak{c}5)$ there is some $u \in \mathbb{C}$ such that $a \cdot u \leqslant a$. Since $t \in \mathbb{L}$ and $t \leqslant a$, it easily follows that $a \in \mathbb{L}$. But then $u \leqslant a \cdot u$, and so $u \leqslant a$. By Definition 3.2 it now follows that $a \in \mathbb{C}$. □

§4. Soundness. The goal of this section is simply to prove the strong soundness theorem. The main work that goes into this is simply to check that the frame conditions are sufficient for ensuring that the axioms and rules hold true.

Lemma 4.1.

- *The axioms and rules of B are true / preserve truth in any model.*
- *For $8 \leq n \leq 15$ and $n \neq 13$: An holds true in any model which satisfies $\mathcal{F}(An)$.*
- *For $6 \leq n \leq 8$: Rn preserves truth in any model which satisfies $\mathcal{F}(Rn)$.*

Proof. See [17].[18] □

[18] There is a slight lacuna in Fine's soundness proof for $R7$ (cf. [17, p. 358]) in that it does not cover the case in which $S(@) = \varnothing$. Note, however, that if so then $S = \varnothing$—if $a \in S$,

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The following lemma allows for slightly shorter proofs in that in order to establish that $@ \vDash A \to B$ it suffices to show that $t \vDash B$ for any $t \in T$ such that $t \vDash A$. I will in the following do so without reference to the lemma.

LEMMA 4.2. $@ \vDash A \to B \Longleftrightarrow \forall t(t \vDash A \Rightarrow t \vDash B)$.

Proof. Trivial given Definitions 3.1(8) and 3.4(v). □

LEMMA 4.3 ($\mathcal{F}(A13) \rightsquigarrow$ A13). *$A13$ holds in any model which satisfies $\mathcal{F}(A13)$.*

Proof. In order to show that $@ \vDash (((A \to A) \wedge (B \to B)) \to C) \to C$, assume that $t \vDash ((A \to A) \wedge (B \to B)) \to C$. It suffices to show that $t \vDash C$. By $\mathcal{F}(A13)$ there is some u such that $u \in \mathbb{L}$ and $t \cdot u \leqslant t$. That $u \vDash (A \to A) \wedge (B \to B)$ follows from Lemma 3.4(3), and so by the commitment clause Definition 3.4(v) it follows that $t \cdot u \vDash C$. That $t \vDash C$ follows then from Lemma 3.2. □

LEMMA 4.4 ($\mathcal{F}(\mathfrak{c}1) \rightsquigarrow \mathfrak{c}1$). *$\mathfrak{c}1$ holds in any $\mathcal{F}(\mathfrak{c}1)$-frame.*

Proof. Trivial. □

LEMMA 4.5 ($\mathcal{F}(\mathfrak{c}2) \rightsquigarrow \mathfrak{c}2$). *$\mathfrak{c}2$ holds in any $\mathcal{F}(\mathfrak{c}2)$-frame.*

Proof. From Lemma 3.4(3). □

LEMMA 4.6 ($\mathcal{F}(\mathfrak{c}3) \rightsquigarrow \mathfrak{c}3$). *$\mathfrak{c}3$ holds in any $\mathcal{F}(\mathfrak{c}3)$-frame.*

Proof. To show that $@ \vDash \mathfrak{c} \to (A \vee {\sim}A)$, let $t \vDash \mathfrak{c}$. Then $t \in \mathbb{C}$. We must show that $t \vDash A \vee {\sim}A$, so let $a \in S(t)$. We must show that $a \vDash A$ or $a \vDash {\sim}A$, so assume that $a \nvDash {\sim}A$. By the commitment clause for negation, there is some $b \geqslant a$ such that $b^* \vDash A$. Since $a \leqslant b$, it follows by Definition 3.1(10) that $b^* \leqslant a^*$. By $\mathcal{F}(\mathfrak{c}3)$ it then follows that $b^* \leqslant a$, and so Lemma 3.2 yields that $a \vDash A$. □

LEMMA 4.7 ($\mathcal{F}(\mathfrak{c}4) \rightsquigarrow \mathfrak{c}4$). *$\mathfrak{c}4$ holds in any $\mathcal{F}(\mathfrak{c}4)$-frame.*

Proof. To show that $@ \vDash (\mathfrak{c} \to \mathfrak{c}) \to \mathfrak{c}$, let t be any theory such that $t \vDash \mathfrak{c} \to \mathfrak{c}$. Then for every $u \in \mathbb{C}$, $t \cdot u \in \mathbb{C}$. By $\mathcal{F}(\mathfrak{c}4)$, then, $t \in \mathbb{C}$, and $t \vDash \mathfrak{c}$ by the commitment clause for $\mathfrak{c}$. □

LEMMA 4.8 ($\mathcal{F}(\mathfrak{c}5) \rightsquigarrow \mathfrak{c}5$). *$\mathfrak{c}5$ holds in any $\mathcal{F}(\mathfrak{c}5)$-frame.*

Proof. Similar to the case for $A13$. □

LEMMA 4.9 ($\mathcal{F}(\mathfrak{c}6) \rightsquigarrow \mathfrak{c}6$). *$\mathfrak{c}6$ holds in any $\mathcal{F}(\mathfrak{c}6)$-frame.*

Proof. Assume that $@ \vDash A$. To establish that $@ \vDash \mathfrak{c} \to A$, let t be any theory such that $t \vDash \mathfrak{c}$. Then $t \in \mathbb{C}$, and so $@ \leqslant t$ by $\mathcal{F}(\mathfrak{c}6)$. Lemma 3.2 yields then that $t \vDash A$ from the assumption that $@ \vDash A$. □

LEMMA 4.10 ($\mathcal{F}(\mathfrak{c}7) \rightsquigarrow \mathfrak{c}7$). *$\mathfrak{c}7$ holds in any $\mathcal{F}(\mathfrak{c}7)$-frame.*

Proof. To show that $@ \vDash A \to (\mathfrak{c} \to A)$, let u be any theory such that $u \vDash A$. To show that $u \vDash \mathfrak{c} \to A$, we must show that $u \cdot t \vDash A$ for any theory t such that $t \vDash \mathfrak{c}$. If $t \vDash \mathfrak{c}$, then $t \in \mathbb{C}$, and so $u \leqslant u \cdot t$ by $\mathcal{F}(\mathfrak{c}7)$. By Lemma 3.2 it therefore follows that $u \cdot t \vDash A$. □

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then $@ \cdot a \leqslant a$ and so by Definition 3.1(7) there is some $b \in S$ such that $@ \leqslant b$. It is easily realized that the only possible assignment function for such a frame is the one for which $\phi(t, p) = 1$ for every t and p. An easy induction then yields that $t \vDash A$ for every t and A. It follows that $R7$ preserves truth in such a frame.

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LEMMA 4.11 ($\mathcal{F}(\mathfrak{c}8) \rightsquigarrow \mathfrak{c}8$). *$\mathfrak{c}8$ holds in any $\mathcal{F}(\mathfrak{c}8)$-frame.*

Proof. Similar to the case for $\mathfrak{c}9$. □

LEMMA 4.12 ($\mathcal{F}(\mathfrak{c}9) \rightsquigarrow \mathfrak{c}9$). *$\mathfrak{c}9$ holds in any $\mathcal{F}(\mathfrak{c}9)$-frame.*

Proof. To show that $@ \vDash \mathfrak{c} \wedge {\sim}\mathfrak{c} \rightarrow A$, assume for contradiction that there is a theory t such that $t \vDash \mathfrak{c} \wedge {\sim}\mathfrak{c}$. Then $t \vDash \mathfrak{c}$ and so $t \in \mathbb{C}$. By $\mathcal{F}(\mathfrak{c}9)$ there is some $a \in S(t)$ such that $a^* \in \mathbb{C}$, and so $a^* \vDash \mathfrak{c}$. However, since $t \vDash {\sim}\mathfrak{c}$ it follows that $a^* \nvDash \mathfrak{c}$. Contradiction. Trivially, therefore, $t \vDash \mathfrak{c} \wedge {\sim}\mathfrak{c} \Rightarrow t \vDash A$. □

LEMMA 4.13 ($\mathcal{F}(\mathfrak{c}10) \rightsquigarrow \mathfrak{c}10$). *$\mathfrak{c}10$ holds in any $\mathcal{F}(\mathfrak{c}10)$-frame.*

Proof. Since $(\dots(@ \cdot t_1) \cdot \dots) \cdot t_n \in T = \mathbb{C}$, it easily follows that $@ \vDash A_1 \rightarrow (\dots \rightarrow (A_n \rightarrow \mathfrak{c})\dots)$. □

LEMMA 4.14. *For any rule ρ and model $\mathfrak{M}$ based on a frame $\mathcal{F}$: if ρ preserves truth in $\mathfrak{M}$ and $\mathcal{F}(^d)$ holds true in $\mathcal{F}$, then ρ^d also preserves truth in $\mathfrak{M}$.*

Proof. Let ρ be some rule $\{A_1, \dots, A_n\} \Vdash B$. To show, then, that ρ^d, that is, $\{A_1 \vee C, \dots, A_n \vee C\} \Vdash B \vee C$, preserves truth, assume that $@ \vDash A_i \vee C$ for every $i \leq n$. Since $@ \in S$, it follows using Lemma. 3.4(4) that either $@ \vDash A_i$ for every $i \leq n$, or $@ \vDash C$. If the latter is the case, then it follows that $@ \vDash B \vee C$. If $@ \vDash A_{i \leq n}$, then since ρ preserves truth in the model in question, it follows that $@ \vDash B$, and therefore that $@ \vDash B \vee C$. □

We have now seen that the axioms and rules all hold true provided the corresponding frame conditions are enforced. As an easy corollary, then, we have the following result.

THEOREM 4.1 (Strong soundness).

$$\Theta \vdash_{L}^{\mathbb{O}} A \Longrightarrow \Theta \vDash_{L}^{\mathbb{O}} A,$$

where L is any system obtainable from one listed in Definition 2.5 by adding any number of the axioms and rules listed in Table 1 as well as the disjunctive extension of any such.

§5. Completeness.

The goal of this section is to prove that for any set of formulas $\Theta \cup \{A\}$,

$$\Theta \vDash_{L}^{\mathbb{O}} A \Longrightarrow \Theta \vdash_{L}^{\mathbb{O}} A,$$

where L is as in the above soundness theorem. It will be assumed, therefore, that $\Theta \cup \{A\}$ is any set of formulas such that $\Theta \nvdash_{L}^{\mathbb{O}} A$ and use this to construct a model in which Θ holds true, but A does not. The construction is that of [17], taking cues for how to generalize this to the Official setting from [31].

DEFINITION 5.1. *For any set of formulas Π, Δ and Θ:*

- *$\vec{\Pi}$ is the set of all members of Π on the form $A \rightarrow B$.*
- *Δ is a Π-theory :=*
  1. $A, B \in \Delta \Rightarrow A \wedge B \in \Delta$
  2. $\vec{\Pi} \vdash_{L}^{\mathbb{O}} A \rightarrow B \Rightarrow (A \in \Delta \Rightarrow B \in \Delta)$
- *Δ is prime* $:= A \vee B \in \Delta \Rightarrow (A \in \Delta$ *or* $B \in \Delta)$
- $\Delta \cdot \Theta := \{B \mid \exists A(A \in \Theta \;\&\; A \rightarrow B \in \Delta)\}$

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- Δ *is* Π*-deductively closed* $:= \Delta \cup \vec{\Pi} \vdash_L^{\mathbb{O}} A \Rightarrow A \in \Delta$
- Δ *is nontrivial* $:= A \notin \Delta$ *for some formula* A, *and* $\Delta \neq \varnothing$.
- Δ *is* Π*-canonical* $:= \Delta$ *is a nontrivial* Π*-theory.*

DEFINITION 5.2 (The canonical frame & model). *The* CANONICAL FRAME *for a set of formulas* $\Theta \cup \{\alpha\}$ *such that* $\Theta \nvdash_L^{\mathbb{O}} \alpha$, *where* $L \neq L^d$, *is defined as follows*:
CANONICAL FRAME $\mathcal{C} = \langle T, S, @, \cdot, *, \leqslant, \mathbb{C}\rangle$, *where*

1. $@ := \{B \mid \Theta \vdash_L^{\mathbb{O}} B\}$
2. $T := \{t \mid t$ *is* $@$*-canonical*$\}$
3. $S := \{t \in T \mid t$ *is prime*$\}$
4. $t \cdot u := \{C \mid \exists B(B \in u \;\&\; B \to C \in t)\}$
5. *for* $a \in S : a^* := \{B \mid {\sim}B \notin a\}$
6. $t \leqslant u \Leftrightarrow t \subseteq u$
7. $\mathbb{C} := \{t \mid \mathfrak{c} \in t\}$.

The canonical frame for L^d *is defined in the same way as in the nondisjunctive case, except that* $@$ *is defined to be some—any will do—disjunctive extension of* $\{B \mid \Theta \vdash_L^{\mathbb{O}} B\}$. *The other elements in the frame are, then, to be* $@$*-canonical relative to this disjunctive extension.*

The CANONICAL MODEL *for the frame defined above is given by the evaluation function* $\phi(a, p) = 1 \Leftrightarrow p \in a$.

LEMMA 5.1. $@$ *in canonical frames for logics axiomatized using disjunctive rules exists.*

Proof. That one may extend a set $t = \{B \mid \Theta \vdash_L^{\mathbb{O}} B\}$, where $\Theta \nvdash_L^{\mathbb{O}} \alpha$, to one prime and deductively closed set $a \supseteq t$ in which α is not a member is the content of the corollary in [31]. □

LEMMA 5.2. *If* t *is any* $@$*-theory,* Γ *is closed under disjunction, and* $\Gamma \cap t = \varnothing$, *then there is a prime* $@$*-theory* $a \supseteq t$ *such that* $\Gamma \cap a = \varnothing$.

Proof. See [31] □

LEMMA 5.3. *If* t *is any* $@$*-theory with some* $A \notin t$, *then there is a* $a \in S(t)$ *such that* $A \notin a$.

Proof. This is an easy corollary to Lemma 2 in [31]. □

THEOREM 5.1. *The L-canonical frame defined above is a frame.*

Proof. We need to check that any canonical frame satisfies Definition 3.1(1–10), and that $\forall t(t \in \mathbb{C} \Leftrightarrow S(t) \subseteq \mathbb{C})$.

(1) $@ \in T$: Trivial.
(2) $S \subseteq T$: Trivial.
(3) $\cdot : T^2 \mapsto T$: See [17].
(4) $* : S \mapsto S$: That any a^* is prime and closed under conjunction is shown in [17]. To show that it is closed under $\vec{@}$-derivable conditionals, assume that $\vec{@} \vdash_L^{\mathbb{O}} A \to B$ and that $A \in a^*$. Then by definition of the Routley star we have that ${\sim}A \notin a$, and furthermore that $\vec{@} \vdash_L^{\mathbb{O}} {\sim}B \to {\sim}A$. Since $a \in T$ it follows that ${\sim}B \notin a$, and therefore that $B \in a^*$.
(5) $\leqslant$ is a reflexive, transitive and antisymmetrical relation on T: Trivial.

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(6) $u \leqslant t \;\&\; v \leqslant w \Rightarrow u \cdot v \leqslant t \cdot w$: Straightforward.
(7) $t \cdot u \leqslant a \Rightarrow \exists b, c((t \leqslant b \;\&\; u \leqslant c) \;\&\; (b \cdot u \leqslant a \;\&\; t \cdot c \leqslant a))$: Let $t \cdot u \subseteq a$.

$$\bar{u} := \{A \mid \exists B[B \notin a \;\&\; A \to B \in t]\}$$
$$\bar{t} := \{A \mid \exists B \exists C[\overrightarrow{@} \vdash^{\mathbb{O}}_{L} A \to (B \to C) \;\&\; B \in u \;\&\; C \notin a]\}.$$

Rather straightforwardly, then, $u \cap \bar{u} = t \cap \bar{t} = \varnothing$. That $\bar{u}$ is closed under disjunction (see [17]). That $\bar{t}$ is closed under disjunction: Assume that $A_1, A_2 \in \bar{t}$. Then there are $B_1, B_2 \in u$ and $C_1, C_2 \notin a$ such that both $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} A_1 \to (B_1 \to C_1)$ and $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} A_2 \to (B_2 \to C_2)$. Since a is prime it follows that $C_1 \vee C_2 \notin a$, but since $u \in T$ that $B_1 \wedge B_2 \in u$. Using the rules of the system B it is easy to establish that $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} A_1 \vee A_2 \to (B_1 \wedge B_2 \to C_1 \vee C_2)$, and so it follows that $A_1 \vee A_2 \in \bar{t}$. By Lemma 5.2 it now follows that there are prime b, c such that $t \subseteq b$ and $u \subseteq c$ which fail to intersect, respectively, $\bar{t}$ and $\bar{u}$. b and c are therefore nontrivial and as such @-canonical theories and therefore members of S. It remains to show that $b \cdot u \subseteq a$ and $t \cdot c \subseteq a$. Suppose first that $B \in b \cdot u$. By canonical $\cdot$ there is some $A \in u$ such that $A \to B \in b$. b does not intersect $\bar{t}$, so $A \to B \notin \bar{t}$, and so by definition of the latter together with the fact that $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} (A \to B) \to (A \to B)$ it follows that $B \in a$. Now suppose that $B \in t \cdot c$. There is, then, some $A \in c$ such that $A \to B \in t$. Since c does not intersect $\bar{u}$, it follows that $A \notin \bar{u}$, and so by definition of the latter it follows that $B \in a$.
(8) $@ \cdot t = t$: Suppose that $B \in @ \cdot t$. Then by definition there is some $A \in t$ with $A \to B \in @$. Since t is @-canonical it follows that $B \in t$. Assume now that $A \in t$. Since $A \to A \in @$ it follows that $A \in @ \cdot t$.
(9) $a^{**} = a$: See [17].
(10) $a \leqslant b \Rightarrow b^* \leqslant a^*$: See [17].

($\mathbb{C}$) Lastly, we must show that $\forall t(t \in \mathbb{C} \Leftrightarrow S(t) \subseteq \mathbb{C})$. First, assume that $t \in \mathbb{C}$. Then by definition of $\mathbb{C}$ in the canonical model, $\mathfrak{c} \in t$, and so $\mathfrak{c} \in a$ for every $a \in S(t)$. Lastly, assume that $t \notin \mathbb{C}$. Then $\mathfrak{c} \notin t$ by the definition of $\mathbb{C}$ in the canonical model. By Lemma 5.3 there is a prime extension a of t not containing $\mathfrak{c}$, and so $S(t) \nsubseteq \mathbb{C}$. □

LEMMA 5.4. *$A \in t \Leftrightarrow t \vDash A$ for any formula A and any $t \in T$ in any L-canonical theory.*

Proof. The base case for propositional variables and for $\mathfrak{c}$ are immediate from definition of the canonical model.

The inductive case for $B \to C$ is the only one in which the proof given in [17] makes reference to L-derivability (cf. Definition 2.4). It suffices, then, to make sure that this case also holds in the context of Official derivability, so assume for inductive hypothesis that it is true for B and C. Assume first that $B \to C \in t$ and that u is any theory such that $u \vDash B$. By IH we then have that $B \in u$, and so by canonical definition of $\cdot$ it follows that $C \in t \cdot u$. IH then yields that $t \cdot u \vDash C$. Since u was arbitrary it then follows that $t \vDash B \to C$. Lastly assume that $t \vDash B \to C$. Let $u := \{D \mid \overrightarrow{@} \vdash^{\mathbb{O}}_{L} B \to D\}$. u is clearly @-canonical with $B \in u$. By IH it follows that $u \vDash B$, and so $t \cdot u \vDash C$ which by IH yields that $C \in t \cdot u$. By canonical definition of $\cdot$ it follows that there is some $A \in u$ such that $A \to C \in t$. But then $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} B \to A$ and therefore $\overrightarrow{@} \vdash^{\mathbb{O}}_{L} (A \to C) \to (B \to C)$

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since every system L has the suffixing rule R3 as at least derivable. But then $B \to C \in t$ since t is @-canonical. □

LEMMA 5.5. *For any system L which yields $A13$ as a theorem, with $\Box A := (A \to A) \to A$,*

1. $\varnothing \vdash^{\mathbb{O}}_{L} \Box(A \to A)$
2. $\{\Box A, \Box B\} \vdash^{\mathbb{O}}_{L} \Box(A \wedge B)$
3. $\{A \to B, \Box A\} \vdash^{\mathbb{O}}_{L} \Box B$
4. $\{\Box A\} \vdash^{\mathbb{O}}_{L} (A \to B) \to B$.

Proof. See [28]. □

LEMMA 5.6 $(\text{A13} \rightsquigarrow \mathcal{F}(A13))$. *If $A13$ is a theorem, then any canonical frame satisfies $\mathcal{F}(A13)$.*

Proof. Let $\imath := \{B \mid \varnothing \vdash^{\mathbb{O}}_{L} \Box B\}$. To complete the proof it suffices to prove the following three statements:

1. $\imath \in T$ in every canonical frame.
2. $\imath \in \mathbb{L}$ in every canonical frame.
3. $t \cdot \imath \leqslant t$ for every $t \in T$ of any canonical frame.

1. $\imath$ is nontrivial since $A \to A \in \imath$ (Lemma 5.5(1)) and the logics in question are all sublogics of classical logic. That it is closed under adjunction follows from Lemma 5.5(2) and closed under $\vec{@}$-derivable conditionals follows from Lemma 5.5(3). Thus $\imath \in T$.
2. To show that $\imath \in \mathbb{L}$, we must show that for every $u \in T$, $u \subseteq \imath \cdot u$. Let, therefore, $B \in u$. $B \to B \in \imath$ by Lemma 5.5(1), and so $B \in \imath \cdot u$ by the definition of $\cdot$ in the canonical frame.
3. To show that $t \cdot \imath \subseteq t$ for every $t \in T$, let $C \in t \cdot \imath$. By definition of $\cdot$ in the canonical frame there is some $B \in \imath$ such that $B \to C \in t$. But then $\varnothing \vdash^{\mathbb{O}}_{L} \Box B$, and so it follows from the assumption that $t \in T$ together with Lemma 5.5(4) that $C \in t$ and so $t \cdot \imath \subseteq t$. □

LEMMA 5.7 $(\mathfrak{c}1 \rightsquigarrow \mathcal{F}(\mathfrak{c}1))$. *If $\mathfrak{c}1$ is a theorem, then any canonical frame satisfies $\mathcal{F}(\mathfrak{c}1)$.*

Proof. If $\mathfrak{c}$ is a theorem, then $\mathfrak{c} \in @$, and so $@ \vDash \mathfrak{c}$ by Lemma 5.4. Since the canonical model is a model, it follows that $@ \in \mathbb{C}$ by the commitment clause for $\mathfrak{c}$. □

LEMMA 5.8 $(\mathfrak{c}2 \rightsquigarrow \mathcal{F}(\mathfrak{c}2))$. *If $\mathfrak{c}2$ is a theorem, then any canonical frame satisfies $\mathcal{F}(\mathfrak{c}2)$.*

Proof. Let $t \in \mathbb{C}$. To show that $t \in \mathbb{L}$, we must show that $u \subseteq t \cdot u$ for any $u \in T$, so assume that $A \in u$. Since $t \in \mathbb{C}$, $\mathfrak{c} \in t$, and so $A \to A \in t$. But then $A \in t \cdot u$ by the definition of $\cdot$ in the frame-canonical. □

LEMMA 5.9 $(\mathfrak{c}3 \rightsquigarrow \mathcal{F}(\mathfrak{c}3))$. *If $\mathfrak{c}3$ is a theorem, then any canonical frame satisfies $\mathcal{F}(\mathfrak{c}3)$.*

Proof. Let $t \in \mathbb{C}$, and a be any prime @-theory such that $t \subseteq a$. We must show that $a^* \subseteq a$, so assume that $A \in a^*$. By definition of the Routley star in the canonical model, it follows that ${\sim}A \notin a$. Since $t \in \mathbb{C}$, $a \in \mathbb{C}$, and so $A \vee {\sim}A \in a$. But then $A \in a$ since a is prime. □

LEMMA 5.10 $(\mathfrak{c}4 \rightsquigarrow \mathcal{F}(\mathfrak{c}4))$. *If $\mathfrak{c}4$ is a theorem, then any canonical frame satisfies $\mathcal{F}(\mathfrak{c}4)$.*

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Proof. We must show that $\forall t(\forall u(u \in \mathbb{C} \Rightarrow t \cdot u \in \mathbb{C}) \Rightarrow t \in \mathbb{C})$ holds in the canonical model. To that end, let t be such that $t \notin \mathbb{C}$, and $\kappa := \{B \mid \varnothing \models^{\mathbb{O}}_{L} \mathfrak{c} \rightarrow B\}$. It is easy to verify that κ thus defined is @-canonical. That $\kappa \in \mathbb{C}$ is obvious. The proof ends if we can show that $t \cdot \kappa \notin \mathbb{C}$. Assume for contradiction that $t \cdot \kappa \in \mathbb{C}$. There must then be some $B \in \kappa$ such that $B \rightarrow \mathfrak{c} \in t$. But then $\varnothing \models^{\mathbb{O}}_{L} \mathfrak{c} \rightarrow B$, and so $\varnothing \models^{\mathbb{O}}_{L} (B \rightarrow \mathfrak{c}) \rightarrow (\mathfrak{c} \rightarrow \mathfrak{c})$ by the suffixing rule $R4$. Since t is @-canonical it then follows that $\mathfrak{c} \rightarrow \mathfrak{c} \in t$ and by $\mathfrak{c}4$ that $\mathfrak{c} \in t$. Contradiction. □

LEMMA 5.11 ($\mathfrak{c}5 \rightsquigarrow \mathcal{F}(\mathfrak{c}5)$). *If* $\mathfrak{c}5$ *is a theorem, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}5)$.

Proof. Let κ be as in the above lemma. From the proof above, it follows that $\kappa \in \mathbb{C}$, since $\mathfrak{c}4$ is a theorem if $\mathfrak{c}5$ is. The only thing left to check is that $t \cdot \kappa \subseteq t$ for every $t \in T$. Assume to that end that $A \in t \cdot \kappa$. By definition of the application operator, it follows that there is some $B \in \kappa$ such that $B \rightarrow A \in t$. Since $B \in \kappa$, $\varnothing \models^{\mathbb{O}}_{L} \mathfrak{c} \rightarrow B$, from which it easily follows that $\varnothing \models^{\mathbb{O}}_{L} (B \rightarrow A) \rightarrow (\mathfrak{c} \rightarrow A)$. Since $\mathfrak{c}4$ is a logical theorem, it follows that $\varnothing \models^{\mathbb{O}}_{L} (B \rightarrow A) \rightarrow A$. Since t is an @-theory, it now follows that $A \in t$. □

LEMMA 5.12 ($\mathfrak{c}6 \rightsquigarrow \mathcal{F}(\mathfrak{c}6)$). *If* $\mathfrak{c}6$ *is a derivable rule of the logic, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}6)$.

Proof. Let $t \in \mathbb{C}$. Then $t \models \mathfrak{c}$, and so $\mathfrak{c} \in t$ by Lemma 5.4. To show that $@ \subseteq t$, let $A \in @$. Since @ is @-deductively closed, it follows that $\mathfrak{c} \rightarrow A \in @$. Since t is an @-theory, it follows that $A \in t$. □

LEMMA 5.13 ($\mathfrak{c}7 \rightsquigarrow \mathcal{F}(\mathfrak{c}7)$). *If* $\mathfrak{c}7$ *is a theorem, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}7)$.

Proof. Let $t \in \mathbb{C}$ and u be any theory. We must show that for every $u \in T$, $u \subseteq u \cdot t$, so assume that $B \in u$. Since u is a theory and $\mathfrak{c}7$ holds in the logic, it follows that $\mathfrak{c} \rightarrow B \in u$ from which Lemma 5.4 yields that $u \models \mathfrak{c} \rightarrow B$. Since $t \in \mathbb{C}$ it follows that $t \models \mathfrak{c}$ and so $u \cdot t \models B$. Lemma 5.4 then yields that $B \in u \cdot t$ which ends the proof. □

LEMMA 5.14 ($\mathfrak{c}8 \rightsquigarrow \mathcal{F}(\mathfrak{c}8)$). *If* $\mathfrak{c}8$ *is a theorem, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}8)$.

Proof. Let $t \in \mathbb{C}$. Then $\mathfrak{c} \in t$. By Lemma 5.2 there is some nontrivial $a \in S$ such that $t \subseteq a$. We must show that $t \subseteq a^*$, so assume for contradiction that there is some A such that $A \in t$ and $A \notin a^*$. By definition of the Routley star in the canonical frame it follows that ${\sim}A \in a$. But then $A \wedge {\sim}A \wedge \mathfrak{c} \in a$. Since $\mathfrak{c}8$ is a theorem it now follows that a is the trivial theory which it cannot be since $a \in T$. Contradiction. □

LEMMA 5.15 ($\mathfrak{c}9 \rightsquigarrow \mathcal{F}(\mathfrak{c}9)$). *If* $\mathfrak{c}9$ *is a theorem, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}9)$.

Proof. Similar to the proof of Lemma 5.14. □

LEMMA 5.16 ($\mathfrak{c}10 \rightsquigarrow \mathcal{F}(\mathfrak{c}10)$). *If* $\mathfrak{c}10$ *is a theorem, then any canonical frame satisfies* $\mathcal{F}(\mathfrak{c}10)$.

Proof. Let t be any @-canonical theory. t is, then, nonempty, so assume that $A \in t$ for some formula A. Since t is @-canonical and $A \rightarrow \mathfrak{c}$ is an instance of $\mathfrak{c}10$, it follows that $\mathfrak{c} \in t$. By lem. 5.4 it then follows that $t \models \mathfrak{c}$, which by definition of $\mathbb{C}$ yields that $t \in \mathbb{C}$. □

LEMMA 5.17. *The frame condition* $\mathcal{F}(^{d})$*—that* $@ \in S$*—holds true in any canonical frame for any logic* L^{d}.

Proof. By definition of the canonical frame for such logics. □

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We have now seen that the frame conditions hold in the canonical model provided the logic in question validates the corresponding logical axiom/rule. As an easy corollary, then, we have the following result.

THEOREM 5.2 (Strong completeness).

$$\Theta \models^{\mathbb{O}}_{L} A \Longrightarrow \Theta \vdash^{\mathbb{O}}_{L} A,$$

where L is any system obtainable from one listed in Definition 2.5 by adding any number of the axiom and rules listed in Table 1 as well as the disjunctive extension of any such.

§6. Summary. This paper has shown how Fine's "theory-application" type semantics for relevant logics can be set up such as to yield a strong soundness and completeness result relative to what Anderson and Belnap, tongue-in-cheek, called "Official" consequence. Fine's own frame condition, although fitting for its intended purpose of yielding a weak completeness proof for **E**, is too strong to support a strong completeness theorem. The frame condition corresponding to the axiom $(((A \to A) \wedge (B \to B)) \to C) \to C$ was therefore presented. It was furthermore shown how to model propositional constants and how ten different axioms and rules for such constants are to be modeled in Fine's semantics. The completeness result shows then, that it is possible to set up Fine's semantics so as to yield a strong completeness result even for logics—like the contractionless logics **B**, **DW**, **TW**, and **EW**—which fail to have the disjunctive version of every derivable rule as yet another derivable rule. This contrasts with the more familiar Routley–Meyer semantics which, under the standard ways of setting up the semantics, is only weakly complete with regards to these logics. It was also shown that a single frame condition can be added to Fine's semantics to ensure that such disjunctive rules are truth-preserving, allowing, then, for a strong soundness and completeness result for, among others, the above mentioned contractionless logics augmented by the disjunctive versions of their primitive rules. This shows that Fine's semantics is both more flexible and more comprehensive than has previously been noted.

§A. On the commitment criterion for maximum necessity. Fine [17] considers a "maximum necessity" constant T and sets the commitment clause for it as follows:

$$t \vDash T \Leftrightarrow l \leqslant t.$$

l is the ground theory which in the current context is named '@.' In Fine's set-up, l is the theory which in the canonical model ends up being the set of logically true formulas of the logic in question, and so in this model any theory t is committed to T just in case it contains the set of logical truths. Fine's axiomatization of T consists of T as a sole axiom, as well as the rule $\{A\} \Vdash T \to A$, corresponding, then, to $\mathfrak{c}1$ and $\mathfrak{c}6$ as used in this paper. First of all, note that Fine's commitment clause does hold true in the canonical models for logics in which $\mathfrak{c}$ satisfies Fine's criteria for a "maximum necessity":

PROPOSITION A.1. *$t \vDash \mathfrak{c} \Leftrightarrow @ \leqslant t$ holds in the canonical model of any logic dealt with in this paper provided $\mathfrak{c}$ is a theorem and $\{A\} \Vdash \mathfrak{c} \to A$ is a derivable rule.*

Proof. Left for the reader. □

$\mathfrak{A} = \langle T, S, @, \cdot, *, \leqslant, \phi \rangle$
$T = \{0, 1, 2\} = \mathbb{L}$
$@ = 0 = 0^*$
$S = \{0\}$
$\forall t \in T \forall p \in At : \phi(t, p) = 1$

$\leqslant$	0	1	2
0	+	−	−
1	+	+	−
2	+	−	+

$\cdot$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

Figure 1. An **E**-model.

There is a slight problem with Fine's commitment criterion for T, however: it fails to yield the saturatedness lemma (Lemma 3.1), contra to what is claimed in Fine [17, p. 359]:[19]

PROPOSITION A.2. *The saturatedness lemma fails to hold if the language in question includes a propositional constant* $\mathfrak{c}$ *for which the commitment clause is* $t \vDash \mathfrak{c} \Leftrightarrow @ \leqslant t$.

Proof. Consider the **E**-model displayed in Figure 1.[20] $@ \vDash \mathfrak{c}$ but $1 \nvDash \mathfrak{c}$ by the assumed commitment clause. However, since $S(1) = \{@\}$, it follows that the saturatedness lemma fails. □

Inspecting the model used in the above theorem it is evident that the saturatedness lemma fails even if one were to require that every frame be such that $S \neq \varnothing$ or even that $S(t) \neq \varnothing$ for every $t \in T$. Note, however, that $S(1) \subseteq S(@)$, but $@ \nleqslant 1$. One of the referees suggested that one can hold on to Fine's commitment clause, provided every frame is demanded to satisfy

$$(S\leqslant) \quad \forall t \forall u (S(t) \subseteq S(u) \Rightarrow u \leqslant t)$$

Their proof that $t \vDash \mathfrak{c} \Leftrightarrow S(t) \subseteq S(@)$—which is the needed part in order for the saturatedness lemma to go through—was as follows:

> Suppose that for all $a \in S(t)$, $a \vDash \mathfrak{c}$. Then for all $a \in S(t)$, $@ \leqslant a$, so $S(t) \subseteq S(@)$, hence $@ \leqslant t$. The other direction is trivial.

As the referee also pointed out, the condition holds true in the canonical model. To show this, I'll first prove the following lemma:

LEMMA A.1. *In any canonical frame*, $A \in t \Leftrightarrow A \in \bigcap S(t)$.

---

19 Fine's commitment clause is also used in the "updated" version of his paper [3, sec. 51], and in Mares' recent book (cf. [21, pp. 232f]). The fix mentioned below works in all cases.
I would also like to mention that Fine's commitment clause for his propositional constant is also used in the accounts given of Fine's semantics in Bimbó and Dunn [10, 11]. Note, however, that the only requirement on an evaluation function therein used is the *persistence requirement* familiar from Routley–Meyer semantics—for any propositional variable p and theories t and s, if $\phi(t, p) = 1$ and $t \leqslant s$, then $\phi(s, p) = 1$—rather than the saturatedness requirement of Definition 3.3. As a result, the saturatedness lemma is allowed to fail even for propositional variables. A model in the sense of Bimbó and Dunn [10, 11], therefore, need not be a Fine model. As an example, consider the model displayed in Figure 1, but with the evaluation function ϕ replaced ϕ' specified as $\forall t \in T \forall p \in At(\phi'(t, p) = 1 \Leftrightarrow t \in S)$. This yields a model in the sense of Bimbó and Dunn as it does satisfy the persistence requirement. It is evident, however, that ϕ' fails the saturatedness requirement.

20 The model was found using Mace4 (cf. [22]). In fact it is a model of the logic Æ (cf. [26]) seeing as it validates $\mathfrak{c}2$, $\mathfrak{c}5$, and $\mathfrak{c}9$, provided the truth set $\mathbb{C}$ for $\mathfrak{c}$ is set as $= \mathbb{L}$.

Proof. If A is a member of t, then it's also a member of every prime extension of t. If $A \notin t$, then since $A \vee A \to A$ is a theorem of every logic under question, $\Gamma \cap t = \varnothing$, where Γ is the disjunctive closure of $\{A\}$. It follows from Lemma 5.2 that there is a $a \in S(t)$ such that $\Gamma \cap a = \varnothing$. Thus $A \notin \bigcap S(t)$. □

PROPOSITION A.3. *$(S\leqslant)$ holds in any canonical frame.*

Proof. Assume that $S(t) \subseteq S(u)$. To show that $u \subseteq t$, let $A \in u$. From Lemma A.1 it follows that $A \in \bigcap S(u)$. $\bigcap S(u) \subseteq \bigcap S(t)$ by elementary properties of sets, and so $A \in \bigcap S(t)$ which by Lemma A.1 suffices for concluding that $A \in t$. □

By adding the frame condition, therefore, one can retain the much used commitment clause for an Ackermann constant.

§B. Correspondence. The proof of Lemma 5.6 establishes something stronger than needed, namely that the following condition holds:

$$\mathcal{F}(A13)^\sharp \quad \exists u \forall t (u \in \mathbb{L} \;\&\; t \cdot u \leqslant t).$$

With regards to strong soundness and completeness, then, it would be fine to prune away $\mathcal{F}(A13)$-frames which fail to validate $\mathcal{F}(A13)^\sharp$. There is a sense in which the sharp frame condition is too strong for $A13$, however. This can be brought out using the notion of *correspondence*:

DEFINITION B.1. *A frame condition X* CORRESPONDS *to a logical principle P just in case for every frame $\mathcal{F}$: $\mathcal{F}$ satisfies X if and only if P holds in $\mathcal{F}$.*

PROPOSITION B.1. *$\mathcal{F}(Ax13)^\sharp$ does not correspond to $Ax13$.*

Proof. The frame displayed in Figure 1 satisfies $\mathcal{F}(A13)$, but not $\mathcal{F}(A13)^\sharp$. By Lemma 4.3, however, $A13$ holds in the frame. □

It would take us too far afield to go further into the issue of correspondence.[21] I would like to note, however, the following open problem:

OPEN PROBLEM B.1. *Does $\mathcal{F}(A13)$ correspond to $A13$?*

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[21] For a range of such results, see Restall [33, sec. 11.5].

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