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On the Structure of the Set of Symmetric Sequences in Orlicz Sequence Spaces

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Abstract. We study the structure of the sets of symmetric sequences and spreading models of an Orlicz sequence space equipped with partial order with respect to domination of bases. In the cases that these sets are "small", some descriptions of the structure of these posets are obtained.

1 Introduction

This paper is motivated by the following general problem considered by Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [AOST]. Let $SP_w(X)$ be the partially ordered set of all spreading models $(\tilde{x}_i)$ generated by seminormalized weakly null sequences (x_i) in X. The partial order is defined by domination, that is, $(\tilde{x}_i) \le (\tilde{y}_i)$ if there exists a constant $K \ge 1$ such that $\|\sum_i a_i\tilde{x}_i\| \le K\|\sum_i a_i\tilde{y}_i\|$, for all scalars (a_i). Moreover, identify $(\tilde{x}_i)$ and $(\tilde{y}_i)$ in $SP_w(X)$ if they are equivalent, that is, if $(\tilde{x}_i) \le (\tilde{y}_i)$ and $(\tilde{y}_i) \le (\tilde{x}_i)$. What can be said about the structure of the partially ordered set $(SP_w(X), \le)$?

The following theorem proved in [AOST] asserts that every countable subset of $SP_w(X)$ admits an upper bound in $SP_w(X)$.

Theorem 1.1 *Let $(C_n) \subset (0,\infty)$ such that $\sum_n C_n^{-1} < \infty$ and let X be a Banach space. For all $n \in \mathbb{N}$, let $(x_i^n)_i$ be a normalized weakly null sequence in X having spreading model $(\tilde{x}_i^n)_i$. Then there exists a semi-normalized weakly null basic sequence (y_i) in X such that $(\tilde{y}_i)$ C_n-dominates $(\tilde{x}_i^n)_i$ for all $n \in \mathbb{N}$.*

The purpose of this paper is to study the structure of the set $SP_w(X)$ when X is an Orlicz sequence space. For Orlicz spaces $X = \ell_M$, as we shall see, every spreading model of ℓ_M is actually equivalent to a symmetric sequence in ℓ_M. In particular, for a reflexive ℓ_M, $SP_w(X)$ coincides with the set of symmetric sequences in X. The above quoted theorem takes a simple form for Orlicz spaces and it is particularly well illustrated. One of our main observation is the following. *If a separable Orlicz sequence space ℓ_M contains a symmetric sequence (equivalently, admits a spreading model) (x_i) which dominates (but is not equivalent to) the unit vector basis of ℓ_M, then it contains an uncountable increasing chain of symmetric sequences (equivalently, $SP_w(\ell_M)$ contains an uncountable increasing chain).* As a consequence, we obtain a description of the structure of the set of symmetric sequences of Orlicz sequence spaces ℓ_M which have only countably many of them. We show that in this case the structure of this set (respectively, of $SP_w(\ell_M)$) has a very special form: it contains both the upper

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and the lower bounds and moreover the upper bound is the space ℓ_M itself and the lower bound is some ℓ_p space. Moreover, we also show that if the set of symmetric sequences in ℓ_M is countable, then it cannot contain a strictly increasing infinite chain.

The paper is organized as follows. The main results, mentioned above, are contained in Section 3. Section 2 contains basic definitions and facts about the structure of Orlicz sequence spaces which are followed by some preliminary results.

For more on spreading models and a more general discussion of the structure of $SP_w(X)$ we refer the reader to the paper [AOST]. Here we only recall the definition of a spreading model, which is as much as we shall use.

It is a well-known consequence of Ramsey theory that for every normalized basic sequence (y_i) in a Banach space X and for every $(\varepsilon_n) \searrow 0$ there exist a subsequence (x_i) of (y_i) and a normalized basic sequence $(\tilde{x}_i)$ in some Banach space $\tilde{X}$ such that for all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \in [-1,1]^n$ and $n \leq k_1 < \cdots < k_n$,

$$\Big| \Big\| \sum_{i=1}^n a_i x_{k_i} \Big\| - \Big\| \sum_{i=1}^n a_i \tilde{x}_i \Big\| \Big| < \varepsilon_n.$$

The sequence $(\tilde{x}_i)$ is called the *spreading model* of (x_i) (or a spreading model of X) and it is a suppression 1-unconditional basic sequence if (y_i) is weakly null. The subsequence (x_i) of (y_i) which generates the spreading model $(\tilde{x}_i)$ is called a *good subsequence* and it has the property that every further subsequence of (x_i) generates the same spreading model $(\tilde{x}_i)$.

2 Orlicz Sequence Spaces and Preliminary Results

We recall the basics of Orlicz sequence spaces following the book [LT] with which our notation is consistent.

An Orlicz function M is a real valued continuous non-decreasing and convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. If $M(t) = 0$ for some $t > 0$, M is said to be a degenerate function.

To any Orlicz function M we associate the space ℓ_M of all sequences of scalars $x = (a_1, a_2, \dots)$ such that $\sum_{n=1}^\infty M(|a_n|/\rho) < \infty$ for some $\rho > 0$. The space ℓ_M is equipped with the norm

$$\|x\| = \inf\Big\{ \rho > 0 : \sum_{n=1}^\infty M(|a_n|/\rho) \leq 1 \Big\},$$

which makes ℓ_M into a Banach space called an Orlicz sequence space.

The subspace h_M of ℓ_M consisting of those sequences $x = (a_1, a_2, \dots) \in \ell_M$ for which $\sum_{n=1}^\infty M(|a_n|/\rho) < \infty$ for every $\rho > 0$ is closed and the unit vectors $\{e_n\}_{n=1}^\infty$ form a symmetric basis of h_M.

An Orlicz function M is said to satisfy the Δ_2-condition at zero if

$$\limsup_{t\to 0} \frac{M(2t)}{M(t)} < \infty.$$

Some other conditions, each of which is equivalent to the Δ_2-condition [LT, Proposition 4.a.4], are :

(i) $\ell_M = h_M$,
(ii) ℓ_M does not contain a subspace isomorphic to ℓ_∞,
(iii) the unit vectors form a boundedly complete symmetric basis of ℓ_M.

Two Orlicz functions M_1 and M_2 are *equivalent at zero* if there exist positive constants K, k, t_0 such that $K^{-1}M_2(k^{-1}t) \le M_1(t) \le KM_2(kt)$ for all $0 < t \le t_0$. When M_1 or M_2 satisfies the Δ_2-condition, they are equivalent (at zero) if there exist constants $K > 0$ and $t_0 > 0$ such that $K^{-1} \le M_1(t)/M_2(t) \le K$ for all $0 < t \le t_0$. This is the case if and only if ℓ_{M_1} and ℓ_{M_2} consist of the same sequences, that is, the unit vector bases in ℓ_{M_1} and ℓ_{M_2} are equivalent.

For an Orlicz function M consider the following subsets of the Banach space $C(0, \frac{1}{2})$ of all real valued continuous functions on $(0, \frac{1}{2})$;

$$E_{M,\Lambda} = \overline{\Big\{ \frac{M(\lambda t)}{M(\lambda)} \,;\, 0 < \lambda < \Lambda \Big\}}, \quad E_M = \bigcap_{0<\Lambda} E_{M,\Lambda}$$

$$C_{M,1} = \overline{\mathrm{conv}} E_{M,1}, \quad C_M = \overline{\mathrm{conv}} E_M,$$

where the closure is taken in the norm topology of $C(0, \frac{1}{2})$. Then $E_{M,1}$, E_M, $C_{M,1}$ and C_M are non-empty norm compact subsets of $C(0, \frac{1}{2})$ consisting entirely of Orlicz functions [LT, Lemma 4.a.6].

The importance of these sets is due to the following result [LT, Proposition 4.a.7, Theorem 4.a.8].

Theorem 2.1 *For every Orlicz function M the following assertions are true.*

(i) *Every infinite-dimensional subspace Y of h_M contains a closed subspace Z which is isomorphic to some Orlicz sequence space h_N.*

(ii) *Let $X \subset h_M$ with a subsymmetric basis $\{x_i\}$. Then X is isomorphic to some Orlicz sequence space h_N and $\{x_i\}$ is equivalent to the unit vector basis of h_N.*

(iii) *An Orlicz sequence space h_N is isomorphic to a subspace of h_M if and only if N is equivalent to some function in $C_{M,1}$.*

By (ii) of the above theorem, every subsymmetric basic sequence in an Orlicz sequence space is symmetric.

Finally we recall that every Orlicz sequence space h_M contains isomorphic copies of some ℓ_p or c_0. Moreover the set of p's for which ℓ_p is contained in h_M is a closed interval [LT, Theorem 4.a.9].

By Theorem 2.1, the set $C_{M,1}$ "coincides" (*i.e.,* there is a one-to-one correspondence) with the collection of all subspaces of h_M which have a subsymmetric (or symmetric) basis. The following proposition asserts that the collection $SP_w(h_M)$ of all spreading models of h_M generated by seminormalized weakly null basic sequences is also "contained" in the set $C_{M,1}$. The proof is a simple generalization of the argument given in [LT, Proposition 4.a.7].

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Proposition 2.2 *Let M be an Orlicz function. Let $(\tilde{x}_i)$ be a spreading model generated by a normalized weakly null sequence (x_i) in h_M. Then there exists $N \in C_{M,1}$ such that $(\tilde{x}_i)$ is equivalent to the unit vector basis of h_N. Moreover, $(\tilde{x}_i)$ is equivalent to a subsequence of (x_i).*

Proof Let (y_i) be the good subsequence of (x_i) which generates $(\tilde{x}_i)$. Since (x_i) is weakly null, by passing to a further subsequence if necessary we can assume that (y_i) is a block basic sequence of the unit vector basis of h_M.

For each $i = 1, 2, \dots$, let $y_i = \sum_{l=n_{i-1}+1}^{n_i} c_l e_l$. To every vector y_i we associate the function $M_i(t) = \sum_{l=n_{i-1}+1}^{n_i} M(|c_l|t)$. Since y_i is normalized, $\sum_{l=n_{i-1}+1}^{n_i} M(|c_l|) = 1$ and hence the functions $\{M_i\}_{i=1}^{\infty}$, as elements of $C(0, \frac{1}{2})$, belong to the set $C_{M,1}$.

Now by the norm compactness of $C_{M,1}$ (in $C(0, \frac{1}{2})$), there exists a subsequence $\{M_{i_n}\}_{n=1}^{\infty}$ of $\{M_i\}$ and an Orlicz function $N \in C_{M,1}$, which might be degenerate, so that $|M_{i_n}(t) - N(t)| \le 2^{-n}$ for $0 \le t \le 1/2$ and $n = 1, 2, \dots$. Assume for simplicity of notation that the subsequence $\{M_{i_n}\}_{n=1}^{\infty}$ coincides with the whole sequence $\{M_i\}$.

Thus for any $a = (a_i)_{i=1}^m \in c_{00}$, we have

$$\begin{aligned}\Big\| \sum_{i=1}^m a_i \tilde{x}_i \Big\| &= \lim_{k_1 \to \infty} \cdots \lim_{k_m \to \infty} \Big\| \sum_{i=1}^m a_i y_{k_i} \Big\| \\ &= \lim_{k_1 \to \infty} \cdots \lim_{k_m \to \infty} \inf \Big\{ \rho : \sum_{i=1}^m M_{k_i}(|a_i|/\rho) \le 1 \Big\} \\ &= \inf \Big\{ \rho : \sum_{i=1}^m N(|a_i|/\rho) \le 1 \Big\} = \Big\| \sum_{i=1}^m a_i e_i \Big\|_{h_N}.\end{aligned}$$

Moreover, the above argument yields that $(\tilde{x}_i)$ is actually equivalent to a subsequence of (x_i). Indeed, since $|M_{i_n}(t) - N(t)| \le 2^{-n}$ for $0 \le t \le 1/2$ and $n = 1, 2, \dots$, it follows that $\sum_{n=1}^{\infty} M_{i_n}(|a_n|) < \infty$ if and only if $\sum_{n=1}^{\infty} N(|a_n|) < \infty$, provided that N is non-degenerate. Hence the corresponding subsequence (y_{i_n}) is equivalent to unit vector basis of h_N [LT, Proposition 4.a.7]. If $N(t) = 0$ for some $t > 0$, then (y_{i_n}) is equivalent to unit vector basis of c_0 which, in this case, is isomorphic to h_N. ■

Obviously, by Theorem 2.1, for every $N \in C_{M,1}$, h_N is a spreading model of h_M. Hence, with some abuse of notation, we can write $SP_w(h_M) \subset C_{M,1} \subset SP(h_M)$, where $SP(h_M)$ denotes the set of all spreading models of h_M.

We recall the following well-known fact [LT, Proposition 4.a.5].

Proposition 2.3 *Let M_1 and M_2 be two Orlicz functions. Then the unit vector basis of h_{M_1} dominates the unit vector basis of h_{M_2} if and only if there exist constants $K > 0$, $k > 0$ and $t_0 > 0$ such that $M_2(t) \le K M_1(kt)$ for all $0 < t \le t_0$.*

Definition 2.4 Let N_1 and N_2 be two Orlicz functions. We say that N_1 *dominates* N_2 and denote by $N_2 \le N_1$ if there exist constants $K > 0, k > 0$ and $t_0 > 0$ such that $N_2(t) \le K N_1(kt)$ for all $0 < t \le t_0$. We write $N_2 < N_1$ if $N_2 \le N_1$ but $N_1 \not\le N_2$.

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Obviously, $N_2 \leq N_1$ and $N_1 \leq N_2$ mean that N_1 is equivalent to N_2. Thus by Proposition 2.3, we have $N_2 \leq N_1$ if and only if $h_{N_2} \leq h_{N_1}$, where by the latter relation we mean that the unit vector basis of h_{N_1} dominates the unit vector basis of h_{N_2}.

As mentioned earlier, it is shown in [AOST] that for an arbitrary Banach space X every countable subset of $SP_w(X)$ admits an upper bound in $SP_w(X)$. When X is an Orlicz sequence space, the corresponding result becomes an easy observation. Before stating this result we need the following lemma, which will be used in the sequel.

Lemma 2.5 *Let M be an Orlicz function. The unit vector basis (e_i) of h_M is weakly null if and only if h_M is not isomorphic to ℓ_1 if and only if $\lim_{t\to 0} M(t)/t = 0$. In particular, $h_N \in SP_w(h_M)$ if and only if $N \in C_{M,1}$ and $\lim_{t\to 0} N(t)/t = 0$.*

Proof The first equivalence follows from standard known results: if h_M is isomorphic to ℓ_1, since ℓ_1 has a unique symmetric basis, then the unit vector basis (e_i) of h_M is equivalent to the unit vector basis of ℓ_1 and hence it is not weakly null. Moreover, if (e_i) is not weakly null, since it is symmetric, it is equivalent to the unit vector basis of ℓ_1 [LT, Proposition 3.b.5].

For the second equivalence, first we note that for every Orlicz function M, $\lim_{t\to 0} M(t)/t$ exists. This follows from the fact that the function $M(t)/t$ is monotone. Indeed, by convexity of M, for all $0 < t < s$, we have $M(t) \leq (t/s)M(s) + (1 - t/s)M(0) = (t/s)M(s)$, *i.e.*, $M(t)/t \leq M(s)/s$.

Moreover, for all n, by definition of the norm of h_M, we have

$$\frac{\| \sum_{i=1}^{n} e_i \|_{h_M}}{n} = \frac{1}{nM^{-1}(1/n)} = \frac{M(t_n)}{t_n},$$

where M^{-1} is the inverse function of M and for all n, $M^{-1}(1/n) = t_n$. (Note also that t_n tends to zero.) It follows that $\lim_{n\to\infty} \| \sum_{i=1}^{n} e_i \|_{h_M}/n$ exists as well. Now recall a well-known fact [LT] that a symmetric (even subsymmetric) basis (y_i) is equivalent to the unit vector basis of ℓ_1 if and only if $\lim_{n\to\infty} \| \sum_{i=1}^{n} y_i \|/n > 0$. Since the unit vector basis (e_i) of h_M is symmetric, consequently it follows that the unit vector basis (e_i) of h_M is not equivalent to the unit vector basis of ℓ_1 if and only if $\lim_{n\to\infty} \| \sum_{i=1}^{n} e_i \|_{h_M}/n = 0$ if and only if $\lim_{t\to 0} M(t)/t = 0$.

Finally, if $h_N \in SP_w(h_M)$, then by the remark following Proposition 2.2 the unit vector basis of h_N is equivalent to a subsequence of the generating weakly null basic sequence in h_M, therefore it is weakly null, and by the above, $\lim_{t\to 0} N(t)/t = 0$. ■

Remark It follows from the above Lemma and the remark following Proposition 2.2 that if an Orlicz sequence space h_M does not contain an isomorphic copy of ℓ_1, then the sets $SP_w(h_M)$ and $C_{M,1}$ coincide, that is, $SP_w(h_M) = C_{M,1}$.

Proposition 2.6 *Let M be an Orlicz function. Suppose that $h_{N_1}, h_{N_2}, \ldots \in SP_w(h_M)$. Then there exists $h_{N_0} \in SP_w(h_M)$ such that h_{N_0} dominates h_{N_i} for every $i \in \mathbb{N}$.*

Proof By Lemma 2.5, $N_1, N_2, \dots \in C_{M,1}$ and $\lim_{t\to 0} N_i(t)/t = 0$ for all i. Define $N_0(t) = \sum_{i=1}^{\infty} 2^{-i} N_i(t)$; then clearly $N_0 \in C_{M,1}$. For every $i \in \mathbb{N}$, $N_0(t) \geq 2^{-i} N_i(t)$ for all $t > 0$. Hence h_{N_0} dominates h_{N_i} for every $i \in \mathbb{N}$. It remains to show that $\lim_{t\to 0} N_0(t)/t = 0$.

Observe that, since $N_i(t)/t$ is non-decreasing, $N_i(t)/t \leq 2N_i(1/2) \leq 2$ for all $i \in \mathbb{N}$ and $0 < t \leq 1/2$.

Let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon/4$. Since $\lim_{t\to 0} N_i(t)/t = 0$ for all i, there exists $t_\varepsilon > 0$ such that for all $0 < t < t_\varepsilon$, $\sum_{i=1}^{m} 2^{-i} \frac{N_i(t)}{t} < \varepsilon/2$. Then for all $0 < t < t_\varepsilon$,

$$\frac{N_0(t)}{t} = \sum_{i=1}^{m} 2^{-i} \frac{N_i(t)}{t} + \sum_{i=m+1}^{\infty} 2^{-i} \frac{N_i(t)}{t} < \frac{\varepsilon}{2} + 2 \sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon.$$

Consequently, $\lim_{t\to 0} N_0(t)/t = 0$, as desired. ■

3 The Structure of $SP_w(\ell_M)$

We have seen by Proposition 2.2 that every spreading model $(\tilde{x}_i)$ of an Orlicz sequence space h_M generated by a weakly null sequence in h_M corresponds to a function N in $C_{M,1}$. This reduces the study of the partially ordered set $SP_w(h_M)$ to the study of the partially ordered set $C_{M,1}$. Hence our next results are on the structure of the set $C_{M,1}$.

We start with an easy observation.

Lemma 3.1 *Let M be an Orlicz function satisfying the Δ_2-condition. Then for all $N \in C_{M,1}$, there exists a sequence (G_n) of Orlicz functions which belong to the equivalence class of M in $C_{M,1}$ such that (G_n) converges uniformly in the norm topology of $C(0, \frac{1}{2})$ to N.*

Proof The fact that for every $N \in C_{M,1}$ there exists a sequence (G_n) of the form

$$G_n = \sum_{i\in\sigma_n} \alpha_i^{(n)} \frac{M(\lambda_i^{(n)} t)}{M(\lambda_i^{(n)})}$$

for some finite subset $\sigma_n \in \mathbb{N}$ and scalars $\alpha_i^{(n)}$ with $\sum_{i\in\sigma_n} \alpha_i^{(n)} = 1$ and $0 < \lambda_i \leq 1/2$ so that the (G_n) converges uniformly to N (in the norm topology of $C(0, \frac{1}{2})$), follows from the definition of $C_{M,1}$.

To show that G_n is equivalent to M for every $n \in \mathbb{N}$, it is sufficient to show that the functions $\frac{M(\lambda t)}{M(\lambda)}$ $(0 < \lambda \leq 1/2)$ are equivalent to M. Since M satisfies the Δ_2-condition and it is non-decreasing, it follows that for every $\lambda > 2^{-m}$ we have

$$\frac{M(t)}{K^m M(\lambda)} \leq \frac{M(\lambda t)}{M(\lambda)} \leq \frac{M(t)}{M(\lambda)},$$

where K is the Δ_2-condition constant. Also due to the Δ_2-condition, M is not degenerate, hence $M(\lambda) \neq 0$. This concludes that the functions $\frac{M(\lambda t)}{M(\lambda)}$ and hence G_n's are equivalent to M, for every $n \in \mathbb{N}$. ■

For our main result on the structure of the set $C_{M,1}$, we also need the following lemma, which is a reformulation in our context of [AOST, Proposition 3.7].

Lemma 3.2 *Let $C \subset C_{M,1}$ be a non-empty subset satisfying the following two conditions:*

(i) *C does not have a maximal element with respect to domination.*

(ii) *For every $(N_i) \subset C$ there exists $N \in C$ such that $N_i \leq N$ for every $i \in \mathbb{N}$.*

Then for all ordinals $\alpha < \omega_1$, there exists $N^\alpha \in C$ such that if $\alpha < \beta < \omega_1$ then $N^\alpha < N^\beta$.

Sketch of the proof We use transfinite induction. Suppose that N^α has been constructed for $\alpha < \beta < \omega_1$. Then N^β is chosen using (i) if β is a successor ordinal. If β is a limit ordinal, then use (ii) to choose N^β and use (i) to show that $N^\alpha < N^\beta$ for $\alpha < \beta < \omega_1$. ■

The following theorem gives an important criterion on the structure of the set $C_{M,1}$.

Theorem 3.3 *Let M be an Orlicz function satisfying the Δ_2-condition. Suppose that there exists $N_0 \in C_{M,1}$ such that N_0 is not dominated by M. Then the set $C_{M,1}$ contains an uncountable increasing chain of mutually non-equivalent Orlicz functions.*

Proof We will show that there exists a subset C of $C_{M,1}$ which satisfies the conditions (i) and (ii) of Lemma 3.2.

First, we observe that the assumption implies that there exists $N_0' \in C_{M,1}$ satisfying $N_0' \not\leq M$ which is, additionally, of the form

$$\sum_{i=1}^{\infty} c_i \frac{M(\lambda_i t)}{M(\lambda_i)},$$

for some $c_i > 0$ with $\sum_i c_i = 1$, and for $0 < \lambda_i < 1/2$.

Indeed, let (G_n) be a sequence in the equivalence class of M which converges uniformly to N_0 (Lemma 3.1). Since $N_0 \not\leq M$, there exists a sequence $(t_k) \searrow 0$ such that for all $k \in \mathbb{N}$,

$$\frac{M(t_k)}{N_0(t_k)} < \frac{1}{k2^k}.$$

For every k, let n_k be such that $G_{n_k}(t_k) \geq (1/2)N_0(t_k)$, and put

$$N_0'(t) = \sum_{k=1}^{\infty} 2^{-k} G_{n_k}(t) \in C_{M,1}.$$

Then $N_0'(t_k) \geq 2^{-k} G_{n_k}(t_k) \geq 2^{-(k+1)} N_0(t_k) \geq (k/2) M(t_k)$, *i.e.*, $\limsup_{t\to 0} \frac{N_0'(t)}{M(t)} = \infty$ and hence $N_0' \not\leq M$. And clearly,

$$N_0'(t) = \sum_{k=1}^{\infty} 2^{-k} G_{n_k}(t) = \sum_k 2^{-k} \sum_i \alpha_i^{(n_k)} \frac{M(\lambda_i^{(n_k)} t)}{M(\lambda_i^{(n_k)})} = \sum_i c_i \frac{M(\lambda_i t)}{M(\lambda_i)},$$

for some c_i such that $\sum_i c_i = 1$ and $0 < \lambda_i < 1$.

For convenience of notation we denote N_0' by N_0 again. So suppose that $N_0(t) = \sum_i c_i \frac{M(\lambda_i t)}{M(\lambda_i)}$. Observe that $c_i \neq 0$ for infinitely many i's, due to the assumption that $N_0 \not\precsim M$.

For all n, let s_n be the normalized partial sum,

$$s_n(t) = \frac{1}{\sum_{i=1}^n c_i} \sum_{i=1}^n c_i \frac{M(\lambda_i t)}{M(\lambda_i)}.$$

Then $s_n \in C_{M,1}$. Let $k_0 \in \mathbb{N}$ such that $\sum_{i=1}^{k_0} c_i \geq 1/2$. Then for all $n \geq k_0$, we have $s_n(t) \leq 2N_0(t)$ for all $0 \leq t \leq 1$. Let us relabel the sequence $\{s_n\}_{n=k_0}^\infty$ and denote it again by $\{s_n\}_{n=1}^\infty$.

Let

$$C = \Big\{ \mathcal{N} \in C_{M,1} : \mathcal{N}(t) = \sum_{n=1}^\infty b_n s_n(t), \text{ for some } b_n \geq 0 \text{ and } \sum_n b_n = 1 \Big\}.$$

First, we remark that for all $\mathcal{N} \in C$, we have $N_0 \not\precsim \mathcal{N}$. Indeed, let $\mathcal{N} = \sum_{n=1}^\infty b_n s_n(t) \in C$ for some $b_n \geq 0$ with $\sum_n b_n = 1$ and let $\varepsilon > 0$ be arbitrary. Let $m \in \mathbb{N}$ be such that $\sum_{n=m+1}^\infty b_n < \varepsilon/4$. Using the fact that $\sum_{n=1}^m b_n s_n(t)$ is equivalent to M and $N_0 \not\precsim M$, we pick $t_\varepsilon > 0$ such that $\sum_{n=1}^m b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} < \varepsilon/2$. Then, since $s_n(t) \leq 2N_0(t)$ for all n and t, we have

$$\begin{aligned}\frac{\mathcal{N}(t_\varepsilon)}{N_0(t_\varepsilon)} &= \sum_{n=1}^m b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} + \sum_{n=m+1}^\infty b_n \frac{s_n(t_\varepsilon)}{N_0(t_\varepsilon)} \\ &< \frac{\varepsilon}{2} + 2 \sum_{n=m+1}^\infty b_n < \varepsilon.\end{aligned}$$

That is, $\liminf_{t\to 0} \frac{\mathcal{N}(t)}{N_0(t)} = 0$, and $N_0 \not\precsim \mathcal{N}$.

Now we check the conditions (ii) and (i) of Lemma 3.2 for the set C.

(ii) If $\mathcal{N}_i(t) = \sum_n b_n^{(i)} s_n(t) \in C$ for some $b_n^{(i)} \geq 0$ with $\sum_n b_n^{(i)} = 1$ and $i = 1, 2, \dots$, then we put $\mathcal{N}(t) = \sum_{i=1}^\infty 2^{-i} \mathcal{N}_i(t)$. Then

$$\mathcal{N}(t) = \sum_i 2^{-i} \sum_n b_n^{(i)} s_n(t) = \sum_n c_n s_n(t),$$

where $c_n \geq 0$ with $\sum_n c_n = 1$. That is, $\mathcal{N} \in C$. Moreover, for all i, we have $\mathcal{N}_i \leq \mathcal{N}$.

(i) Suppose that there is a maximal element $\mathcal{M} \in C$. Then $\mathcal{M}(t) = \sum_n b_n s_n(t)$ for some $b_n \geq 0$ such that $\sum_n b_n = 1$. By the above remark, $N_0 \not\precsim \mathcal{M}$, and hence there exists a sequence $(t_k) \searrow 0$ such that for all k,

$$\frac{\mathcal{M}(t_k)}{N_0(t_k)} < \frac{1}{k2^k}.$$

Since the partial sums s_n converge to N_0, for all k we may choose (n_k) such that $s_{n_k}(t_k) \geq (1/2)N_0(t_k)$. Let $\mathcal{M}_0(t) = \sum_k 2^{-k} s_{n_k}(t) \in C$. Then for all k,

$$\mathcal{M}_0(t_k) \geq 2^{-k} s_{n_k}(t_k) \geq 2^{-(k+1)} N_0(t_k) \geq (k/2)\mathcal{M}(t_k).$$

That is, $\limsup_{t\to 0} \frac{\mathcal{M}_0(t)}{\mathcal{M}(t)} = \infty$ and $\mathcal{M}_0 \not\precsim \mathcal{M}$, a contradiction. Therefore, C does not contain a maximal element.

The proof is now complete by Lemma 3.2. ■

Remark As it was observed in [FPR], the set of all block bases (or spreading models generated by block bases) of a Banach space is either countable (up to equivalence) or has cardinality continuum. Thus the following consequence of Theorem 3.3 is immediate.

Corollary 3.4 *Let M be an Orlicz function which satisfies the Δ_2-condition. Suppose that there exists a spreading model generated by a normalized weakly null sequence (or a symmetric sequence) in ℓ_M which is not dominated by the unit vector basis of ℓ_M. Then the set $SP(\ell_M)$ (respectively, the set of all symmetric sequences in ℓ_M) has, up to equivalence, cardinality continuum.*

The next consequence of Theorem 3.3 gives a description of the structure of the set of symmetric sequences (respectively, of $SP_w(\ell_M)$) in ℓ_M for which these sets are "small".

Corollary 3.5 *Let ℓ_M be an Orlicz sequence space which is not isomorphic to ℓ_1. Suppose that the set of symmetric sequences, up to equivalence, (respectively, $SP_w(\ell_M)$) is countable. Then*

(i) *the unit vector basis of ℓ_M is the upper bound of the set of symmetric sequences in ℓ_M (respectively, it is the upper bound of $SP_w(\ell_M)$);*

(ii) *the unit vector basis of ℓ_p for some $1 < p < \infty$ is the lower bound of the set of symmetric sequences in ℓ_M (respectively, it is the lower bound of $SP_w(\ell_M)$).*

Proof Observe that the assumptions immediately imply that M satisfies the Δ_2-condition. Indeed, otherwise ℓ_∞ embeds into ℓ_M, which implies that the set of symmetric sequences (respectively, $SP_w(\ell_M)$) is uncountable, *e.g.*, for all $1 < p < \infty$, $\ell_p \subset \ell_\infty \subset \ell_M$. Moreover, the assumptions also imply that ℓ_M does not contain an isomorphic copy of ℓ_1 (hence it is reflexive). Indeed, if $\ell_1 \subset \ell_M$, since the unit vector basis of ℓ_1 trivially dominates the unit vector basis of $\ell_M(= h_M)$, it follows from Corollary 3.4 that either the set of symmetric sequences in ℓ_M (respectively, $SP_w(\ell_M)$) is uncountable or ℓ_M is isomorphic to ℓ_1.

Therefore by the remark following Lemma 2.5, we have that $SP_w(h_M) = C_{M,1}$. Moreover, by reflexivity, these sets coincide with the set of all symmetric sequences in ℓ_M. That is, the structure of these sets is isomorphic with respect to corresponding partial orders.

(i) By Proposition 2.6, $C_{M,1}$ contains an upper bound. Suppose that there exists $N \in C_{M,1}$ such that N is not equivalent to M and the unit vector basis of h_N is the

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upper bound for the set of symmetric sequences in ℓ_M (respectively, of $SP_w(\ell_M)$). It follows that $N \not\precsim M$ and by Theorem 3.3, $C_{M,1}$ contains uncountable mutually non-equivalent Orlicz functions, and thus the set of symmetric sequences in ℓ_M (respectively, $SP_w(h_M)$) is uncountable, a contradiction. Therefore ℓ_M must be the upper bound.

(ii) Since the set of p's for which ℓ_p embeds into ℓ_M is a closed interval [LT, Theorem 4.a.9], it follows from the assumption that this set is a singleton. (Hence there exists a unique $1 < p < \infty$ such that $\ell_p \in SP_w(h_M)$.) Moreover, it follows from Theorem 2.1 that ℓ_M is ℓ_p-saturated. That is, every subspace of ℓ_M has a further subspace which contains an isomorphic copy of ℓ_p. For Orlicz sequence spaces, by Theorem 2.1, ℓ_p embeds into h_M if and only if $t^p \in C_{M,1}$. In particular, for all $N \in C_{M,1}$, the function t^p belongs to $C_{N,1}$. Moreover, the assumption that M satisfies the Δ_2-condition implies that N also satisfies the Δ_2-condition for all $N \in C_{M,1}$.

If (the unit vector basis of) ℓ_p is not the lower bound of the set of symmetric sequences in ℓ_M (respectively, of $SP_w(\ell_M)$), then there exists $N \in C_{M,1}$ such that $t^p \not\precsim N$. But, by the above, $t^p \in C_{N,1}$, hence it follows from Theorem 3.3 that $C_{N,1} \subset C_{M,1}$ is uncountable. This implies that the set of symmetric sequences in $h_N \subset \ell_M$ (respectively, $SP_w(h_N) \subset SP_w(\ell_M)$) is uncountable, a contradiction. Therefore ℓ_p must be the lower bound. ■

Remark It is also worth noting the following. *If ℓ_M is non-reflexive then either ℓ_M is isomorphic to ℓ_1 or $SP_w(\ell_M)$ is uncountable.* This is obvious if M does not satisfy the Δ_2-condition for then ℓ_M contains ℓ_∞. On the other hand, if M satisfies the Δ_2-condition and ℓ_M is non-reflexive then it is known [LT, Proposition 4.a.4] that ℓ_M contains ℓ_1. By the first part of the proof of Corollary 3.5, either ℓ_M is isomorphic to ℓ_1 or $SP_w(\ell_M)$ is uncountable.

We give only a sketch of the argument for our next result as it follows along similar lines to the proof of Theorem 3.3.

Theorem 3.6 *Let M be an Orlicz function satisfying the Δ_2-condition. Suppose that $C_{M,1}$ contains a strictly increasing infinite sequence $M_1 < M_2 < \cdots$. Then the set $C_{M,1}$ contains an uncountable increasing well-ordered chain of mutually non-equivalent Orlicz functions.*

Sketch of the proof By passing to a subsequence, if necessary, assume that (M_n) converges (uniformly) to some $N_0 \in C_{M,1}$. First, assume that there exists a constant $K \geq 1$ such that $M_n(t) \leq K N_0(t)$, for all $t > 0$.

We proceed as in the proof of Theorem 3.3 by defining

$$C = \Big\{ \mathcal{N} \in C_{M,1} : \mathcal{N}(t) = \sum_{n=1}^{\infty} b_n M_n(t) \text{ for some } b_n \geq 0 \text{ and } \sum_n b_n = 1 \Big\}.$$

Next, using the above assumption and the fact that for all $m \in \mathbb{N}$ and $b_n > 0$, $\sum_{n=1}^{m} b_n M_n(t)$ is equivalent to $M_m(t)$ and $M_m < N_0$ (due to the fact that (M_n) is strictly increasing), we show that for all $\mathcal{N} \in C$, $N_0 \not\precsim \mathcal{N}$ (in fact, $\mathcal{N} < N_0$).

Finally, using the fact that (M_n) converges to N_0, one can show, similarly as in the proof of Theorem 3.3, that C satisfies (i) and (ii) of Lemma 3.2.

If the assumption that there exists $K \geq 1$ such that $M_n(t) \leq KN_0(t)$ for all $t > 0$ fails, then put $N_0'(t) = \sum_{n=1}^{\infty} 2^{-n} M_n(t)$. Now take

$$M_n'(t) = \frac{1}{\sum_{k=1}^{n} 2^{-k}} \sum_{k=1}^{n} 2^{-k} M_k(t).$$

Note that $M_n'(t) \leq 2N_0'(t)$ for all $t > 0$ and, of course, M_n' converges (uniformly) to N_0'. So now replace (M_n) in the first part of the proof by M_n' and N_0 by N_0'. This finishes the proof. ∎

Corollary 3.7 *Let ℓ_M be an Orlicz sequence space. Suppose that the set of symmetric sequences, up to equivalence, (respectively, $SP_w(\ell_M)$) is countable. Then the set of symmetric sequences in ℓ_M (respectively, $SP_w(\ell_M)$) cannot contain a strictly increasing infinite sequence.*

Question Does there exist an Orlicz sequence space ℓ_M so that the set of symmetric sequences in ℓ_M, up to equivalence, (respectively, the set $SP_w(\ell_M)$) is precisely countably infinite?

We have recently [S] extended Corollary 3.7 to arbitrary Banach spaces X for $SP_w(X)$. For a recent discussion of more general form of the above question see [DOS].

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