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On an Exponential Functional Inequality and its Distributional Version

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Abstract. Let G be a group and $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. In this article, as a generalization of the result of Albert and Baker, we investigate the behavior of bounded and unbounded functions $f\colon G \to \mathbb{K}$ satisfying the inequality

$$\Big| f\Big(\sum_{k=1}^{n} x_k\Big) - \prod_{k=1}^{n} f(x_k)\Big| \le \phi(x_2, \dots, x_n), \quad \forall x_1, \dots, x_n \in G,$$

where $\phi\colon G^{n-1} \to [0, \infty)$. Also, as a distributional version of the above inequality we consider the stability of the functional equation

$$u \circ S - \overbrace{u \otimes \cdots \otimes u}^{n\text{-times}} = 0,$$

where u is a Schwartz distribution or Gelfand hyperfunction, $\circ$ and $\otimes$ are the pullback and tensor product of distributions, respectively, and $S(x_1, \dots, x_n) = x_1 + \cdots + x_n$.

1 Introduction

Throughout this paper, we denote by G a group, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, $\phi\colon G^{n-1} \to [0, \infty)$, and $\epsilon \ge 0$. We call $m\colon G \to \mathbb{K}$ an *exponential function* provided that

$$m(x + y) = m(x)m(y)$$

for all $x, y \in G$. Let $f\colon G \to \mathbb{K}$ satisfy the exponential functional inequality

$$|f(x + y) - f(x)f(y)| \le \epsilon \tag{1.1}$$

for all $x, y \in G$. Then f is either an unbounded exponential function or a bounded function satisfying

$$|f(x)| \le \frac{1}{2}(1 + \sqrt{1 + 4\epsilon}) \tag{1.2}$$

for all $x \in G$ (see Baker [3]). In [2], Albert and Baker refined the inequality (1.2) when G is a vector space over the field $\mathbb{Q}$ of rational numbers and proved that if $f\colon G \to \mathbb{R}$ is a bounded function satisfying (1.1) with $0 < \epsilon < \frac{1}{4}$, then f satisfies either

$$-\epsilon \le f(x) \le \frac{1}{2}(1 - \sqrt{1 - 4\epsilon}) \tag{1.3}$$

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for all $x \in G$, or

$$\frac{1}{2}(1+\sqrt{1-4\epsilon}) \le f(x) \le \frac{1}{2}(1+\sqrt{1+4\epsilon}) \tag{1.4}$$

for all $x \in G$. The inequalities (1.3) and (1.4) imply that every bounded function satisfying the inequality (1.1) tends to 0 or 1 (the roots of the algebraic equation $x^2 - x = 0$) as $\epsilon \to 0$.

In this paper, we investigate behaviors of bounded functions and unbounded functions $f\colon G \to \mathbb{K}$ satisfying the exponential functional inequality with n-variables $(n \ge 2)$

$$\Big| f\Big(\sum_{k=1}^{n} x_k\Big) - \prod_{k=1}^{n} f(x_k)\Big| \le \phi(x_2, \dots, x_n) \tag{1.5}$$

for all $x_1, \dots, x_n \in G$. When we consider some exponential functional equations or unbounded solutions of exponential functional inequalities involving n-variables, we can follow the same approach as in the case of 2-variables. However, when we consider bounded solution of exponential functional inequality with n-variables, such as the inequality (1.5), the methods are quite different from that of 2-variables, such as those of Albert and Baker [2].

As a corollary of our main result we obtain that every bounded function $f\colon G \to \mathbb{R}$ satisfying the inequality (1.5) with $\phi(x_2, \dots, x_n) = \epsilon$ for all $x_2, \dots, x_n \in G$ satisfies the following:

> Let $\alpha < \beta < \gamma$ be the positive real roots of the equation $|t^n - t| = \epsilon$. If n is even, then f satisfies either $-\epsilon \le f(x) \le \alpha$ for all $x \in G$, or $\beta \le f(x) \le \gamma$ for all $x \in G$, and if n is odd, then f satisfies $\beta \le f(x) \le \gamma$ for all $x \in G$, $-\alpha \le f(x) \le \alpha$ for all $x \in G$, or $-\gamma \le f(x) \le -\beta$ for all $x \in G$.

As a direct consequence of this result, we also obtain that if n is even, then f satisfies either

$$-\epsilon \le f(x) \le \frac{n}{n-1}\epsilon$$

for all $x \in G$, or

$$-\sqrt[n-1]{n\epsilon} \le f(x) - 1 \le \frac{\epsilon}{n-1}$$

for all $x \in G$, and if n is odd, then f satisfies

$$-\frac{n}{n-1}\epsilon \le f(x) \le \frac{n}{n-1}\epsilon$$

for all $x \in G$,

$$-\sqrt[n-1]{n\epsilon} \le f(x) - 1 \le \frac{\epsilon}{n-1}$$

for all $x \in G$, or

$$-\frac{\epsilon}{n-1} \le f(x) + 1 \le \sqrt[n-1]{n\epsilon}$$

for all $x \in G$. We also consider the unbounded functions $f\colon G \to \mathbb{K}$ satisfying (1.5) and prove that if there exist $q_1, q_2, \dots, q_n \in G$ such that

$$\big| f(q_1)\big(|f(q_2)\cdots f(q_n)| - 1\big) \big| > \phi(q_2, \dots, q_n),$$

then the function f satisfying (1.5) is unbounded and has the form $f(x) = Cm(x)$, where $C \in \mathbb{K}$ with $C^{n-1} = 1$ and m is an exponential function. In the last section of the paper, as a distributional version of the inequality (1.5), we consider the inequality

$$\| u \circ S - \overbrace{u \otimes \cdots \otimes u}^{n\text{-times}} \| \le \epsilon, \tag{1.6}$$

where u is a Schwartz distribution[6] or Gelfand hyperfunction [4,5], $\circ$ and $\otimes$ denote the pullback and the tensor product of distributions, respectively, and $\|\cdot\| \le \epsilon$ means that $|\langle \cdot, \varphi\rangle| \le \epsilon \|\varphi\|_{L^1}$ for all test functions φ (see Section 3). As a result, we prove that if u satisfies (1.6), then either u is a bounded measurable function satisfying

$$\|u\|_{L^\infty} \le \gamma,$$

where $\gamma > 1$ is the root of the algebraic equation $z^n - z = \epsilon$, or

$$u = e^{\frac{i2k\pi}{n-1}} e^{c\cdot x}$$

for some $k \in \{0, 1, 2, \dots, n-2\}$, $c \in \mathbb{C}^n$. We refer the reader to [7–9, 11–14] for related results of Hyers–Ulam stability of functional equations.

2 Classical Solutions of (1.5)

In this section we investigate behaviors of bounded functions and unbounded functions $f\colon G \to \mathbb{K}$ satisfying the exponential functional inequality (1.5). We first investigate behaviors of bounded functions satisfying the inequality (1.5).

Lemma 2.1 *Let $f\colon G \to \mathbb{K}$ be a bounded function satisfying the inequality* (1.5). *Then f satisfies*

$$\big| f(x_1)\big(1 - |f(x_2)\cdots f(x_n)|\big) \big| \le \phi(x_2, \dots, x_n) \tag{2.1}$$

for all $x_1, \dots, x_n \in G$.

Proof Let $M = \sup_{x\in G} |f(x)|$. Using the triangle inequality with (1.5) we have

$$|f(x_1)f(x_2)\cdots f(x_n)| \le |f(x_1+\cdots+x_n)| + \phi(x_2, \dots, x_n) \le M + \phi(x_2, \dots, x_n) \tag{2.2}$$

for all $x_1, \dots, x_n \in G$. From (2.2) we have

$$M|f(x_2)\cdots f(x_n)| = \sup_{x_1\in G} |f(x_1)|\,|f(x_2)\cdots f(x_n)| \le M + \phi(x_2, \dots, x_n) \tag{2.3}$$

for all $x_2, \dots, x_n \in G$. Thus from (2.3), we get

$$M\big(|f(x_2)\cdots f(x_n)| - 1\big) \le \phi(x_2, \dots, x_n) \tag{2.4}$$

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for all $x_2, \ldots, x_n \in G$. Replacing x_1 by $x_1 - x_2 - \cdots - x_n$ in (1.5) and using the triangle inequality with the result we have

$$\begin{aligned} |f(x_1)| &\le |f(x_1 - x_2 - \cdots - x_n)|\,|f(x_2)\cdots f(x_n)| + \phi(x_2, \ldots, x_n) \\ &\le M|f(x_2)\cdots f(x_n)| + \phi(x_2, \ldots, x_n) \end{aligned} \tag{2.5}$$

for all $x_1, \ldots, x_n \in G$. From (2.5) we have

$$M = \sup_{x_1 \in G} |f(x_1)| \le M|f(x_2)\cdots f(x_n)| + \phi(x_2, \ldots, x_n)$$

for all $x_2, \ldots, x_n \in G$, which implies

$$M\big(1 - |f(x_2)\cdots f(x_n)|\big) \le \phi(x_2, \ldots, x_n) \tag{2.6}$$

for all $x_2, \ldots, x_n \in G$. Thus, from (2.4) and (2.6) we have

$$M\big|1 - |f(x_2)\cdots f(x_n)|\big| \le \phi(x_2, \ldots, x_n)$$

for all $x_2, \ldots, x_n \in G$, which implies (2.1). This completes the proof. ■

From now on, for each integer $n \ge 2$, we denote by $c_n := (n-1)n^{-\frac{n}{n-1}}$ and $D := \{x \in G : \phi(x, \ldots, x) < c_n\}$. Note that c_n is the (local) maximum of the polynomial $p(t) := t - t^n$. One can see that $\frac{1}{4} \le c_n < c_{n+1} < 1$ for all $n = 2, 3, 4, \ldots$. It is easy to see that for each $x \in G$, the equation

$$|t^n - t| = \phi(x, \ldots, x) \tag{2.7}$$

has only one real root $\gamma(x) > 1$, and for each $x \in D$, the equation (2.7) has the three positive real roots $\alpha(x) < \beta(x) < \gamma(x)$. Note that $0 < \alpha(x_1) < n^{-\frac{1}{n-1}} < \beta(x_2) < 1 < \gamma(x_3)$ for all $x_1, x_2, x_3 \in D$. In particular, we denote by $\alpha < \beta < \gamma$ the positive real roots of the equation $|t^n - t| = \epsilon$ when $\epsilon < c_n$.

As a main result of this section we have the following.

Theorem 2.2 *Let $f\colon G \to \mathbb{K}$ be a bounded function satisfying the inequality (1.5). Then f satisfies*

$$|f(x)| \le \gamma(x) \tag{2.8}$$

for all $x \in G$. Furthermore, f satisfies either

$$|f(x)| \le \alpha(x) \tag{2.9}$$

for all $x \in D$, or

$$\beta(x) \le |f(x)| \le \gamma(x) \tag{2.10}$$

for all $x \in D$.

Proof Replacing $x_1, x_2, \ldots, x_n$ by x in (2.1) we have

$$\big||f(x)| - |f(x)|^n\big| \le \phi(x, \ldots, x) \tag{2.11}$$

for all $x \in G$. From (2.11), for each $x \in G$, $|f(x)|$ satisfies

$$|f(x)| \le \gamma(x),$$

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which gives (2.8). For each $x \in D$, $f(x)$ satisfies either

$$|f(x)| \le \alpha(x) \tag{2.12}$$

or

$$\beta(x) \le |f(x)| \le \gamma(x). \tag{2.13}$$

Now, we prove that f satisfies (2.12) for all $x \in D$ or (2.13) for all $x \in D$. Assume that there exist $y_1, y_2 \in D$ such that

$$|f(y_1)| \le \alpha(y_1), \quad \beta(y_2) \le |f(y_2)|. \tag{2.14}$$

Putting $x_1 = y_2$ and $x_2 = x_3 = \cdots = x_n = y_1$ in (2.1) we have

$$|f(y_2)|\big(1 - |f(y_1)|^{n-1}\big) \le \phi(y_1, \dots, y_1). \tag{2.15}$$

On the other hand, from (2.14) we have

$$\begin{aligned} |f(y_2)|\big(1 - |f(y_1)|^{n-1}\big) &\ge \beta(y_2)\big(1 - \alpha(y_1)^{n-1}\big) \\ &> \alpha(y_1)\big(1 - \alpha(y_1)^{n-1}\big) = \phi(y_1, \dots, y_1), \end{aligned}$$

which contradicts (2.15). Thus, we get (2.9) or (2.10). This completes the proof. ■

Let $\phi(x_2, \dots, x_n) = \epsilon < c_n$ for all $x_2, \dots, x_n \in G$ in Theorem 2.2. Then we have the following.

Corollary 2.3 *Let $f\colon G \to \mathbb{K}$ be a bounded function satisfying the inequality (1.5). Then f satisfies either*

$$|f(x)| \le \alpha \tag{2.16}$$

for all $x \in G$, or

$$\beta \le |f(x)| \le \gamma \tag{2.17}$$

for all $x \in G$.

In particular, if G is 2-divisible, $\mathbb{K} = \mathbb{R}$ and $\phi(x_2, \dots, x_n) = \epsilon < c_n$ for all $x_2, \dots, x_n \in G$, then we have the following.

Corollary 2.4 *Assume that G is 2-divisible and $f\colon G \to \mathbb{R}$ is a bounded function satisfying the inequality*

$$\Big| f\Big(\sum_{k=1}^{n} x_k\Big) - \prod_{k=1}^{n} f(x_k) \Big| \le \epsilon \tag{2.18}$$

for all $x_1, \dots, x_n \in G$. If n is even, then f satisfies either

$$-\epsilon \le f(x) \le \alpha \tag{2.19}$$

for all $x \in G$, or

$$\beta \le f(x) \le \gamma \tag{2.20}$$

for all $x \in G$. If n is odd, then f satisfies (2.20) for all $x \in G$,

$$-\alpha \le f(x) \le \alpha \tag{2.21}$$

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for all $x \in G$, *or*

$$-\gamma \le f(x) \le -\beta \tag{2.22}$$

for all $x \in G$.

Proof Replacing x_1, x_2 by $\frac{x}{2}$ and putting $x_3 = x_4 = \ldots = x_n = 0$ in (2.18) we have

$$f\Big(\frac{x}{2}\Big)^2 f(0)^{n-2} - \epsilon \le f(x) \le f\Big(\frac{x}{2}\Big)^2 f(0)^{n-2} + \epsilon \tag{2.23}$$

for all $x \in G$. We first consider the case when n is even or $f(0) \ge 0$. From (2.23) we have

$$-\epsilon \le f\Big(\frac{x}{2}\Big)^2 f(0)^{n-2} - \epsilon \le f(x) \tag{2.24}$$

for all $x \in G$. Note that

$$\epsilon = \alpha - \alpha^n < \alpha. \tag{2.25}$$

From (2.16), (2.24), and (2.25) we have

$$-\epsilon \le f(x) \le \alpha \tag{2.26}$$

for all $x \in G$, or from (2.17), (2.24), and (2.25) we have

$$\beta \le f(x) \le \gamma \tag{2.27}$$

for all $x \in G$. Thus, if n is even, from (2.26) and (2.27) we get (2.19) or (2.20). Now, we consider the case when n is odd and $f(0) < 0$. From (2.24) we have

$$f(x) \le f\Big(\frac{x}{2}\Big)^2 f(0)^{n-2} + \epsilon \le \epsilon \tag{2.28}$$

for all $x \in G$. Thus, from (2.16), (2.25), and (2.28), we have

$$-\alpha \le f(x) \le \epsilon \tag{2.29}$$

for all $x \in G$, or from (2.17), (2.25), and (2.28) we have

$$-\gamma \le f(x) \le -\beta \tag{2.30}$$

for all $x \in G$. Thus, if n is odd, from (2.26), (2.27), (2.29), and (2.30), we get (2.20), (2.21), or (2.22). This completes the proof. ∎

Note that α, β, γ satisfy

$$0 < \alpha < \frac{n}{n-1}\epsilon, \quad 1 - \sqrt[n-1]{n\epsilon} < \beta < 1, \quad 1 < \gamma < 1 + \frac{\epsilon}{n-1}. \tag{2.31}$$

As a consequence of the Corollary 2.4 together with the inequality (2.31), we have the following.

Corollary 2.5 *Assume that G is 2-divisible and* $f\colon G \to \mathbb{R}$ *is a bounded function satisfying the inequality* (2.18) *for* $\epsilon < c_n$. *If n is even, then f satisfies either*

$$-\epsilon \le f(x) \le \frac{n}{n-1}\epsilon$$

for all $x \in G$, *or*

$$-\sqrt[n-1]{n\epsilon} \le f(x) - 1 \le \frac{\epsilon}{n-1}$$

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for all $x \in G$. If n is odd, then f satisfies

$$-\frac{n}{n-1}\epsilon \le f(x) \le \frac{n}{n-1}\epsilon$$

for all $x \in G$,

$$-\sqrt[n-1]{n\epsilon} \le f(x) - 1 \le \frac{\epsilon}{n-1}$$

for all $x \in G$, or

$$-\frac{\epsilon}{n-1} \le f(x) + 1 \le \sqrt[n-1]{n\epsilon}$$

for all $x \in G$.

Remark 2.6 From Corollary 2.5, if n is even, every bounded solution of (2.18) tends to 0 or 1 as $\epsilon \to 0$, and if n is odd, every bounded solution of (2.18) tends to 0, 1, or -1 as $\epsilon \to 0$.

If $n = 2$ and $0 < \epsilon < \frac{1}{4}$, then it is easy to see that

$$\alpha = \frac{1}{2}(1 - \sqrt{1-4\epsilon}), \quad \beta = \frac{1}{2}(1 + \sqrt{1-4\epsilon}), \quad \gamma = \frac{1}{2}(1 + \sqrt{1+4\epsilon}).$$

Thus, by Corollary 2.5 we obtain a improved version of the result of Albert and Baker for vector space [2].

Corollary 2.7 *Let $0 < \epsilon < \frac{1}{4}$. Assume that G is a 2-divisible group and $f \colon G \to \mathbb{R}$ is a bounded function satisfying the inequality*

$$|f(x+y) - f(x)f(y)| \le \epsilon$$

for all $x, y \in G$. Then f satisfies either

$$-\epsilon \le f(x) \le \frac{1}{2}(1 - \sqrt{1-4\epsilon})$$

for all $x \in G$ or

$$\frac{1}{2}(1 + \sqrt{1-4\epsilon}) \le f(x) \le \frac{1}{2}(1 + \sqrt{1+4\epsilon})$$

for all $x \in G$.

Finally, we investigate the unbounded solutions of the inequality (1.5).

Theorem 2.8 *Let $f \colon G \to \mathbb{K}$ satisfy the inequality (1.5). Assume that there exist $q_1, q_2, \ldots, q_n \in G$ such that*

$$\Big| f(q_1)\big(|f(q_2)\cdots f(q_n)| - 1\big)\Big| > \phi(q_2, \ldots, q_n). \tag{2.32}$$

Then f is unbounded and there exists an exponential function $m \colon G \to \mathbb{K}$ and $C \in \mathbb{K}$ with $C^{n-1} = 1$ such that

$$f(x) = Cm(x) \tag{2.33}$$

for all $x \in G$.

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Proof By Lemma 2.1, we can see that if f satisfies (2.32), then f is unbounded and $f(0) \neq 0$. Let $z_k \in G$, $k = 1, 2, 3, \dots$ be a sequence such that $|f(z_k)| \to \infty$ as $k \to \infty$. Replacing x_1 by z_k, $k = 1, 2, 3, \dots$, x_2 by x, putting $x_3 = \cdots = x_n = 0$ in (1.5) and dividing the result by $|f(z_k)|$ we have

$$\Big| f(x) f(0)^{n-2} - \frac{f(z_k + x)}{f(z_k)} \Big| \leq \frac{\phi(x, 0, \dots, 0)}{|f(z_k)|}. \tag{2.34}$$

Letting $k \to \infty$ in (2.34) we have

$$f(x) f(0)^{n-2} = \lim_{k\to\infty} \frac{f(z_k + x)}{f(z_k)} \tag{2.35}$$

for all $x \in G$. Thus, using (1.5) and (2.35) we have

$$\begin{aligned} f(x+y) &= \frac{1}{f(0)^{n-2}} \lim_{k\to\infty} \frac{f(z_k + x + y)}{f(z_k)} \\ &= \frac{1}{f(0)^{n-2}} \lim_{k\to\infty} \frac{f(z_k + x) f(y) f(0)^{n-2}}{f(z_k)} \\ &= f(y) \lim_{k\to\infty} \frac{f(z_k + x)}{f(z_k)} \\ &= f(y) f(x) f(0)^{n-2} \end{aligned} \tag{2.36}$$

for all $x, y \in G$. Putting $y = 0$ in (2.36) we have

$$f(0)^{n-1} = 1. \tag{2.37}$$

Dividing (2.36) by $f(0)^n$ and using (2.37) we have

$$\frac{f(x+y)}{f(0)} = \frac{f(x)}{f(0)} \cdot \frac{f(y)}{f(0)} \tag{2.38}$$

for all $x, y \in G$. From (2.38) we get (2.33). This completes the proof. ■

3 Distributions and Hyperfunctions

We briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions and the space $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$ of Gelfand hyperfunctions. Here we use the notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$. We also denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Definition 3.1 A distribution u is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

Definition 3.2 We denote by $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n)$ the space of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ satisfying the following; there exist positive constants A and B such that

$$\|\varphi\|_{A,B} := \sup_{x\in\mathbb{R}^n,\alpha,\beta\in\mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|}B^{|\beta|}\alpha!^{1/2}\beta!^{1/2}} < \infty. \tag{3.1}$$

The topology on the space $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n)$ is defined by the seminorms $\|\cdot\|_{A,B}$ in the left-hand side of (3.1) and we denote by $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$ the dual space of $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n)$ and the elements of $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$ are called *Gelfand hyperfunctions.*

It is known that the space $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ that can be continued to an entire function satisfying

$$|\varphi(x+iy)| \le C\exp(-a|x|^2 + b|y|^2) \tag{3.2}$$

for some $a, b > 0$.

Definition 3.3 Let $u_j \in \mathcal{D}'(\mathbb{R}^{n_j})$ [resp. $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$] for $j = 1, 2$. Then the tensor product $u_1 \otimes u_2$ of u_1 and u_2, defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2)\rangle = \big\langle u_{x_1}, \langle u_{x_2}, \varphi(x_1, x_2)\rangle\big\rangle$$

for $\varphi(x_1, x_2) \in C_c^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, belongs to $\mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ [resp. $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$].

4 Distributional Solution of (1.6)

In this section, as a distributional version of the functional inequality (1.5) we consider the inequality

$$\|u \circ S - \overbrace{u \otimes \cdots \otimes u}^{n\text{-times}}\| \le \epsilon, \tag{4.1}$$

where $\otimes$ is tensor product of distributions, $S(x_1, \dots, x_n) = x_1 + \cdots + x_n$, the pullback $u \circ S$ is defined by

$$\langle u \circ S, \varphi(x_1, \dots, x_n)\rangle$$
$$= \Big\langle u, \int \varphi(x_1, \dots, x_{n-1}, x - x_1 - \cdots - x_{n-1})\, dx_1 \cdots dx_{n-1}\Big\rangle, \ \ \varphi \in C_c^\infty(\mathbb{R}^{n^2}),$$

and $\|\cdot\| \le \epsilon$ means that $|\langle\,\cdot\,, \varphi\rangle| \le \epsilon\|\varphi\|_{L^1}$ for all test functions $\varphi \in C_c^\infty(\mathbb{R}^n)$ [resp. $(\mathcal{S}^{1/2}_{1/2})(\mathbb{R}^n)$].

We denote by $\delta(x)$ the function on $\mathbb{R}^n$,

$$\delta(x) = \begin{cases} qe^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \ge 1, \end{cases}$$

where

$$q = \Big(\int_{|x|<1} e^{-\frac{1}{1-|x|^2}}\, dx\Big)^{-1}.$$

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It is easy to see that $\delta(x)$ an infinitely differentiable function with support $\{x : |x| \le 1\}$. Now we employ the function $\delta_t(x) := t^{-n}\delta(x/t)$, $t > 0$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then for each $t > 0$, $(u * \delta_t)(x) = \langle u_y, \delta_t(x - y)\rangle$ is a smooth function in $\mathbb{R}^n$ and $(u * \delta_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int (u * \delta_t)(x)\varphi(x)\, dx.$$

We also employ the heat kernel

$$E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0.$$

In view of (3.2) it is easy to see that the heat kernel $E_t(x)$ belongs to $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$ for each $t > 0$. It is well known that the heat kernel satisfies the semigroup property

$$E_t * E_s = E_{t+s}$$

for all $t, s > 0$, which will be useful. We first consider the inequality (4.1) in the space of Schwartz distributions.

Theorem 4.1 *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the inequality (4.1). Then either u is a bounded measurable function satisfying*

$$\|u\|_{L^\infty} \le \gamma, \tag{4.2}$$

where $\gamma > 1$ is the root of the algebraic equation $z^n - z = \epsilon$, or

$$u = e^{\frac{i2k\pi}{n-1}} e^{c\cdot x} \tag{4.3}$$

for some $k \in \{0, 1, 2, \dots, n-2\}$, $c \in \mathbb{C}$.

Proof Convolving $(\delta_{t_1} \otimes \cdots \otimes \delta_{t_n})](x_1, \dots, x_n) := \delta_{t_1}(x_1)\cdots\delta_{t_n}(x_n)$ in each side of (4.1) we have

$$\begin{aligned}
&[(u \circ S) * (\delta_{t_1} \otimes \cdots \otimes \delta_{t_n})](x_1, \dots, x_n)\\
&\quad = \Big\langle u_{\xi_1}, \int \delta_{t_1}(x_1 + \xi_2 + \cdots + \xi_n - \xi_1)\delta_{t_2}(x_2 - \xi_2)\cdots\delta_{t_n}(x_n - \xi_n)\, d\xi_2\cdots d\xi_n\Big\rangle\\
&\quad = \Big\langle u_{\xi_1}, \int (\delta_{t_1} * \delta_{t_2})(x_1 + x_2 + \xi_3 + \cdots + \xi_n - \xi_1)\\
&\qquad\qquad \times \delta_{t_3}(x_3 - \xi_3)\cdots\delta_{t_n}(x_n - \xi_n)\, d\xi_3\cdots d\xi_n\Big\rangle\\
&\quad = \Big\langle u_{\xi_1}, \int (\delta_{t_1} * \cdots * \delta_{t_{n-1}})(x_1 + \cdots + x_{n-1} + \xi_n - \xi_1)\delta_{t_n}(x_n - \xi_n)\, d\xi_n\Big\rangle\\
&\quad = \langle u_{\xi_1}, (\delta_{t_1} * \cdots * \delta_{t_n})(x_1 + \cdots + x_n - \xi_1)\rangle\\
&\quad = (u * \delta_{t_1} * \cdots * \delta_{t_n})(x_1 + \cdots + x_n).
\end{aligned}$$

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We also have

$$[(u \otimes \cdots \otimes u) * (\delta_{t_1} \otimes \cdots \otimes \delta_{t_n})](x_1, \ldots, x_n) = (u * \delta_{t_1})(x_1) \cdots (u * \delta_{t_n})(x_n).$$

Thus, the inequality (4.1) is converted to the following inequality

$$|(u * \delta_{t_1} * \cdots * \delta_{t_n})(x_1 + \cdots + x_n) - (u * \delta_{t_1})(x_1) \cdots (u * \delta_{t_n})(x_n)| \le \epsilon \tag{4.4}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^n$, $t_1, \ldots, t_n > 0$. It follows from (4.4) that the limit

$$f(x) := \limsup_{t \to 0^+} (u * \delta_t)(x)$$

exists for all $x \in \mathbb{R}^n$. In (4.4), fixing $x_2, \ldots, x_n$ and letting $t_2, t_3, \ldots, t_n \to 0^+$ so that $(u * \delta_{t_j})(x_j) \to f(x_j)$ as $t_j \to 0^+$ for all $j = 2, 3, \ldots, n$, we have

$$|(u * \delta_{t_1})(x_1 + \cdots + x_n) - (u * \delta_{t_1})(x_1) f(x_2) \cdots f(x_n)| \le \epsilon. \tag{4.5}$$

Replacing x_n by x, letting $x_1 = x_2 = \cdots = x_{n-1} = 0$ and $t_1 \to 0^+$, so that $(u * \delta_{t_1})(0) \to f(0)$ as $n \to \infty$ in (4.5), we have

$$\|u - f(0)^{n-1} f(x)\| \le \epsilon. \tag{4.6}$$

If f is bounded, then from (4.6) u is defined by a bounded measurable function, *i.e.*,

$$\langle u, \varphi \rangle = \int h(x) \varphi(x)\, dx, \quad \varphi \in C_c^\infty(\mathbb{R}^n)$$

for some bounded measurable function h. Now, using the heat kernel E_t instead of δ_t and convolving $(E_{t_1} \otimes \cdots \otimes E_{t_n})(x_1, \ldots, x_n)$ in each side of (4.1), we have

$$|U(x_1 + \cdots + x_n, t_1 + \cdots + t_n) - U(x_1, t_1) \cdots U(x_n, t_n)| \le \epsilon \tag{4.7}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$, $t_1, t_2, \ldots, t_n > 0$, where $U(x,t) = (u * E_t)(x)$. Using the same method as in the proof of Lemma 2.1 with (4.7), we can prove that

$$|U(x_1, t_1)| \big(|U(x_2, t_2) \cdots U(x_n, t_n)| - 1 \big) \le \epsilon \tag{4.8}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$, $t_1, t_2, \ldots, t_n > 0$. Letting $x_1 = x_2 = \cdots = x_n = x$, $t_1 = t_2 = \cdots = t_n = t$ in (4.8) we have

$$|U(x,t)| \le \gamma \tag{4.9}$$

for all $x \in \mathbb{R}^n$, $t > 0$. Letting $t \to 0^+$ in (4.9), we get (4.2). Now, we consider the case when f is unbounded. Let c_k, $k = 1, 2, 3, \ldots$, be a sequence such that $|f(c_k)| \to \infty$ as $k \to \infty$. Replacing $x_2 = \cdots = x_n = c_k$ in (4.5) and dividing the result by $|f(c_k)|^{n-1}$ and letting $k \to \infty$ we have

$$(u * \delta_{t_1})(x_1) = \lim_{n \to \infty} \frac{(u * \delta_{t_1})\big(x_1 + (n-1)c_k\big)}{f(c_k)^{n-1}}. \tag{4.10}$$

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Multiplying both sides of (4.10) by $f(x_2)\cdots f(x_n)$, and using (4.5) and (4.10), we have

$$
\begin{aligned}
(4.11)\quad (u*\delta_{t_1})(x_1)f(x_2)\cdots f(x_n) &= \lim_{k\to\infty}\frac{(u*\delta_{t_1})\big(x_1+(n-1)c_k\big)f(x_2)\cdots f(x_n)}{f(c_k)^{n-1}}\\
&= \lim_{k\to\infty}\frac{(u*\delta_{t_1})\big(x_1+\cdots+x_n+(n-1)c_k\big)}{f(c_k)^{n-1}}\\
&= \lim_{k\to\infty}\frac{(u*\delta_{t_1})\big(x_1+\cdots+x_n+(n-1)c_k\big)}{f(c_k)^{n-1}}\\
&= (u*\delta_{t_1})(x_1+\cdots+x_n)
\end{aligned}
$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}^n$, $t_1, t_2, \dots, t_n > 0$. Putting $x_2 = x_3 = \cdots = x_{n-1} = 0$ in (4.11) we have

$$(4.12)\qquad (u*\delta_{t_1})(0)f(0)^{n-2}f(x) = (u*\delta_{t_1})(x)$$

for all $x \in \mathbb{R}^n$. Choosing $t_1 > 0$ such that $(u*\delta_{t_1})(0) \neq 0$ and putting (4.12) to (4.11) we have

$$(4.13)\qquad f(x_1)f(x_2)\cdots f(x_n) = f(x_1+\cdots+x_n)$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}^n$. Choosing a sequence s_k, $k = 1, 2, 3, \dots$ so that $(u*\delta_{s_k})(0) \to f(0)$ as $k \to \infty$, replacing t_1 by s_k in (4.12) and letting $k \to \infty$ we have

$$
\begin{aligned}
(4.14)\qquad \langle u,\varphi\rangle &= \lim_{k\to\infty}\int (u*\delta_{s_k})(x)\varphi(x)\,dx\\
&= \lim_{k\to\infty}\int (u*\delta_{s_k})(0)f(0)^{n-2}f(x)\varphi(x)\,dx\\
&= f(0)^{n-1}\int f(x)\varphi(x)\,dx = \int f(x)\varphi(x)\,dx
\end{aligned}
$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Now, it is easy to see that the solution f of (4.13), being a measurable function, is given by

$$(4.15)\qquad f(x) = f(0)e^{c\cdot x} = e^{\frac{i2k\pi}{n-1}}e^{c\cdot x}$$

for some $k \in \{0, 1, 2, \dots, n-2\}$, $c \in \mathbb{C}^n$. Thus, from (4.14) and (4.15), we get (4.3). This completes the proof. ■

Note that every locally integrable function f defines a distribution via the correspondence

$$\varphi \longrightarrow \int f(x)\varphi(x)\,dx.$$

As a direct consequence of the above result we obtain the following.

Corollary 4.2 *Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function satisfying*

$$\|f(x_1+\cdots+x_n) - f(x_1)\cdots f(x_n)\|_{L^\infty(\mathbb{R}^n)} \le \epsilon.$$

Then either f is a bounded measurable function satisfying

$$\|f(x)\|_{L^\infty} \le \gamma,$$

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where $\gamma > 1$ *is the root of the algebraic equation* $z^n - z = \epsilon$*, or*

$$f(x) = e^{\frac{i2k\pi}{n-1}} e^{c\cdot x}$$

for almost every $x \in \mathbb{R}^n$*, where* $k \in \{0, 1, 2, \dots, n-2\}$*,* $c \in \mathbb{C}^n$*.*

As a consequence of the method of proof of Theorem 4.1 we obtain the stability of the inequality (4.1) in the space $(\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$ of Gelfand hyperfunctions.

Theorem 4.3 *Let* $u \in (\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$ *satisfy the inequality* (4.1)*. Then either* u *is a bounded measurable function satisfying*

$$\|u\|_{L^\infty} \le \gamma,$$

where $\gamma > 1$ *is the root of the algebraic equation* $z^n - z = \epsilon$*, or*

$$u = e^{\frac{i2k\pi}{n-1}} e^{c\cdot x}$$

for some $k \in \{0, 1, 2, \dots, n-2\}$*,* $c \in \mathbb{C}^n$*.*

Proof Let $u \in (\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^n)$. Then using the heat kernel E_t instead of δ_t and convolving $(E_{t_1} \otimes \cdots \otimes E_{t_n})(x_1, \dots, x_n)$ in each side of (4.1) we have

$$|U(x_1 + \cdots + x_n, t_1 + \cdots + t_n) - U(x_1, t_1) \cdots U(x_n, t_n)| \le \epsilon$$

for all $x_1, x_2, \dots, x_n \in \mathbb{R}^n$, $t_1, t_2, \dots, t_n > 0$, where $U(x, t) = (u * E_t)(x)$. Using the same method as in the proof of Theorem 4.1, we get the result. ∎

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