GENERALIZED FRATTINI SUBGROUPS OF FINITE GROUPS. II

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1. Introduction. The theory of generalized Frattini subgroups of a finite group is continued in this paper. Several equivalent conditions are given for a proper normal subgroup H of a finite group G to be a generalized Frattini subgroup of G. One such condition on H is that K is nilpotent for each normal subgroup K of G such that K/H is nilpotent. From this result, it follows that the weakly hyper-central normal subgroups of a finite non-nilpotent group G are generalized Frattini subgroups of G.

Let H be a generalized Frattini subgroup of G and let K be a subnormal subgroup of G which properly contains H. Then H is a generalized Frattini subgroup of K.

Let $\phi(G)$ be the Frattini subgroup of G. Suppose that $G/\phi(G)$ is non-nilpotent, but every proper subgroup of $G/\phi(G)$ is nilpotent. Then $\phi(G)$ is the unique maximal generalized Frattini subgroup of G.

A proper normal subgroup K of a group G is said to satisfy property $\mathscr{M}(G)$ if $\phi(G/K) = 1$, G/K contains a unique minimal normal subgroup, and G/K is not of prime order. If a proper normal subgroup K of G satisfies property $\mathscr{M}(G)$, then we denote this fact by $K \in \mathscr{M}(G)$. Let G be a solvable group and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if K is a proper subgroup of the Fitting subgroup $F(G)$ of G.

Let G be a solvable group and let $K \in \mathscr{M}(G)$. For various types of G (i.e., G being an A-group, an E-group, etc.) we consider certain conditions under which K becomes a generalized Frattini subgroup of G. For example, if G is an A-group of nilpotent length two, then K is "generalized Frattini" in G if and only if K is abelian.

2. Properties of generalized Frattini subgroups. The only groups considered in the present paper are finite. It is assumed that the reader is familiar with the notation presented in (**4**). In the present section we give several equivalent conditions for a proper normal subgroup H of a group G to be a generalized Frattini subgroup of G.

Definition 2.1. A proper normal subgroup H of a group G is said to have property (N) if and only if K is a nilpotent normal subgroup for each normal subgroup K of G such that K/H is nilpotent.

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Let H be a proper normal subgroup of G having property (N). Further, let L be a normal subgroup of G and let P be a Sylow p-subgroup of L, p is a prime, such that $G = HN_G(P)$. Then HP/H is a normal Sylow p-subgroup of LH/H, and hence HP/H is a nilpotent normal subgroup of G/H. Therefore, HP is a nilpotent normal subgroup of G. Thus, P is a Sylow p-subgroup of the nilpotent normal subgroup $HP \cap L$, hence $N_G(P) = G$. Therefore, H is a generalized Frattini subgroup of G. From these facts and (**4**, Theorem 3.2) we have the following theorem.

THEOREM 2.1. *Let H be a proper normal subgroup of a group G. Then H is a generalized Frattini subgroup of G if and only if H satisfies condition* (N).

Let G be a non-nilpotent group. Then $L(G)$ is a generalized Frattini subgroup of G (**4**, Theorem 3.5). Since $Z^*(G)$ is contained in $L(G)$, $Z^*(G)$ is also a generalized Frattini subgroup of G (**4**, Theorem 3.1). From this we know that every hypercentral normal subgroup of G is a generalized Frattini subgroup of G.

Baer (**1**) introduced the concept of weakly hypercentral normal subgroup of a finite group G. Let H be a weakly hypercentral normal subgroup of a non-nilpotent group G. By (**1**, p. 636, Proposition 1), H is nilpotent. Hence, H is a proper subgroup of G. By Theorem 2.1 and (**1**, p. 637, Corollary 2), H is a generalized Frattini subgroup of G. We have proved the following corollary.

COROLLARY 2.1.1. *Let H be a weakly hypercentral normal subgroup of a non-nilpotent group G. Then H is a generalized Frattini subgroup of G.*

Definition 2.2. A proper normal subgroup H of a group G is said to satisfy condition (N′) if and only if K is a nilpotent subnormal subgroup of G for each subnormal subgroup K of G such that K/H is nilpotent.

THEOREM 2.2. *Let H be a proper normal subgroup of G. The following statements are equivalent:*

(a) *H is a generalized Frattini subgroup of G;*

(b) *H satisfies property* (N)*;*

(c) *If K is a normal subgroup of G which contains H, then $F(K/H) = F(K)/H$;*

(d) *$F(G/H) = F(G)/H$;*

(e) *H satisfies property* (N′)*;*

(f) *If K is a subnormal subgroup of G which contains H, then $F(K/H) = F(K)/H$.*

Proof. (a) implies (b). This is a consequence of Theorem 2.1.

(b) implies (c). Assume condition (b). Then H is nilpotent. Let K be a normal subgroup of G which contains H. Then $F(K)$ contains H. There exists a subgroup M of K such that $M/H = F(K/H)$. Since M/H is characteristic in K/H, M is a normal subgroup of G. Hence, M is nilpotent since M/H is nilpotent. This shows that $F(K/H) = F(K)/H$.

(c) implies (d). This is immediate.

(d) implies (e). We assume condition (d). Therefore, H is nilpotent. Let L be a subnormal subgroup of G which contains H and assume that L/H is nilpotent. Let $G = G_0 \supset G_1 \supset \ldots \supset G_{n-1} = L \supset G_n = H$ be a series of subgroups from G to H such that G_i is normal in G_{i-1} for $i = 1, 2, \ldots, n-1$. For $i = 0, 1, 2, \ldots, n-1$, let M_i be a normal subgroup of G_i such that $M_i/H = F(G_i/H)$. We note that $M_{i+1}/H = F(G_{i+1}/H) \subset F(G_i/H) = M_i/H$ for $i = 0, 1, \ldots, n-1$. Therefore, $M_{i+1} \subset M_i$ for $i = 0, 1, 2, \ldots, n-1$. Since $F(G/H) = F(G)/H$, it follows that $F(G)$ contains $M_{n-1} = L$. Hence, L is nilpotent.

(e) implies (f). Assume condition (e). Then H is nilpotent. Let L be a subnormal subgroup of G which contains H. Then $F(L)$ contains H. Moreover, there exists a subgroup M of G which is normal in L and $M/H = F(L/H)$. Since M is subnormal in G, M is a nilpotent normal subgroup of L. Hence, $F(L/H) = F(L)/H$.

(f) implies (a). By Theorem 2.1, it is enough to show that (f) implies (b). Let K be a normal subgroup of G which contains H and assume that K/H is nilpotent. Then $F(K/H) = K/H$ and by condition (f), $K/H = F(K)/H$. Hence, $K = F(K)$ and K is nilpotent.

This completes the proof.

THEOREM 2.3. *Let H be a generalized Frattini subgroup of G and let K be a subnormal subgroup of G which properly contains H. Then H is a generalized Frattini subgroup of K.*

Proof. Let L be a normal subgroup of K which contains H and assume that L/H is nilpotent. Since L is a subnormal subgroup of G and L/H is nilpotent, L is nilpotent by Theorem 2.2 (e). By Theorem 2.1, H is a generalized Frattini subgroup of K. This completes the proof.

Remark. The proof of the above theorem is due to D. C. Dykes of the University of Kentucky.*

The converse of Theorem 2.3 is false in general.

Example 2.1. Let S_3 be the symmetric group on three symbols and let A_3 be the alternating group on three symbols. Let H be a cyclic group of order two and let G be the direct product of S_3 and H. Then $F(G) = A_3 \times H$ and A_3 is a generalized Frattini subgroup of $F(G)$. However, A_3 is not a generalized Frattini subgroup of G.

COROLLARY 2.3.1. *Let H be a generalized Frattini subgroup of G and let K be a normal subgroup of G. If $H \cap K$ is a proper subgroup of K, then $H \cap K$ is a generalized Frattini subgroup of K.*

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Proof. Assume that $H \cap K$ is a proper subgroup of K. Because of (**4**, Theorem 3.1) and Theorem 2.3, the corollary follows.

From (**4**, Theorem 3.1) and Corollary 2.3.1 we obtain the following result.

COROLLARY 2.3.2. *Let H be a generalized Frattini subgroup of G and let K be a non-nilpotent normal subgroup of G. Then $H \cap K$ is a generalized Frattini subgroup of K.*

3. A special type of generalized Frattini subgroup. We recall that a non-trivial normal subgroup H of G is called a minimal normal subgroup of G if it contains no proper non-trivial normal subgroups of G.

Definition 3.1. A proper normal subgroup of a group G is said to satisfy property $\mathscr{M}(G)$ (i.e., $K \in \mathscr{M}(G)$) if and only if

(a) $\phi(G/K) = 1$;

(b) G/K contains a unique minimal normal subgroup;

(c) G/K is not of prime order.

Let H be a subgroup of G. We recall that the core of H, denoted core(H), is the intersection of all conjugates of H in G. We note that core(H) is the largest normal subgroup of G contained in H; cf. (**8**, p. 53).

Let H be a self-normalizing maximal subgroup of a solvable group G. In Theorem 4.1 of the next section we shall show that core$(H) \in \mathscr{M}(G)$.

We now give a lemma that will be useful throughout this paper.

LEMMA 3.1. *Let K be a proper normal subgroup of G such that $K \in \mathscr{M}(G)$. Then K contains $\phi(G)$ and G/K is non-nilpotent. In particular, G is non-nilpotent.*

Proof. Since $\phi(G/K) = 1$, it follows that K contains $\phi(G)$. Suppose that G/K is nilpotent and let A/K be the unique minimal normal subgroup of G/K. Since G/K is nilpotent, A/K is an abelian p-group for some prime p. Because of (**8**, Theorem 7.4.15), if follows that $G/K = F(G/K) = A/K$, and therefore G/K is abelian. Hence, G/K is of prime order, which is impossible. Therefore, G/K is non-nilpotent. This completes the proof of the lemma.

THEOREM 3.1. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if K is a proper subgroup of the Fitting subgroup $F(G)$ of G.*

Proof. Let A/K denote the unique minimal normal subgroup of G/K. Since G is solvable, $F(G/K) = A/K$ (**8**, Theorem 7.4.15).

Suppose that K is a generalized Frattini subgroup of G. By (**4**, Corollary 3.2.1), $F(G)/K = A/K$; hence, $F(G) = A$. Thus, K is a proper subgroup of $F(G)$.

Conversely, let K be a proper subgroup of $F(G)$. Then $F(G)/K = A/K = F(G/K)$. Let L be a normal subgroup of G such that L contains K and L/K is nilpotent. Then $L/K \subseteq F(G/K) = F(G)/K$, hence $L \subseteq F(G)$. By Theorem 2.1, K is a generalized Frattini subgroup of G.

From the proof of Theorem 3.1 we obtain the following theorem.

THEOREM 3.2. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let A/K be the unique minimal normal subgroup of G/K. Then K is a generalized Frattini subgroup of G if and only if $A = F(G)$.*

Let G be a solvable group and let $K \in \mathscr{M}(G)$. Assume that K is a generalized Frattini subgroup of G. Then K is a maximal generalized Frattini subgroup of G. For, let H be a generalized Frattini subgroup of G which contains K. By (**4**, Theorem 3.1), $F(G)$ contains H. Now let A/K be the unique minimal normal subgroup of G/K. By Theorem 3.2, $A = F(G)$. Therefore, $H = K$ or $H = F(G)$. Assume that $H = F(G)$. By (**4**, Theorem 3.6), every solvable normal subgroup of G is nilpotent. However, G is solvable, hence nilpotent. This contradicts Lemma 3.1. Hence, we have the following result.

COROLLARY 3.2.1. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. If K is a generalized Frattini subgroup of G, then K is a maximal generalized Frattini subgroup of G.*

THEOREM 3.3. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let H be a normal subgroup of G which properly contains K. If K is a generalized Frattini subgroup of H, then $F(H) = F(G)$ and K is a generalized Frattini subgroup of G.*

Proof. Assume that K is a generalized Frattini subgroup of H. Because of (**4**, Theorems 3.1 and 3.6), K is a proper subgroup of $F(H)$. Since H is normal in G, $F(H)$ is a normal subgroup of G; hence, $F(G)$ contains $F(H)$. Therefore, K is a proper subgroup of $F(G)$, and thus K is a generalized Frattini subgroup of G by Theorem 3.1. Let A/K be the unique minimal normal subgroup of G/K. Then $A/K \subseteq H/K$. Because of (**8**, Theorem 7.4.15) and Theorem 2.2, $F(G/K) = F(G)/K = A/K = F(H)/K = F(H/K)$, hence $F(H) = F(G)$.

Because of Theorems 2.3 and 3.3 we have the result which follows.

COROLLARY 3.3.1. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let H be a normal subgroup of G which properly contains K. If K is a generalized Frattini subgroup of G, then $F(H) = F(G)$.*

4. Core of a self-normalizing maximal subgroup. Let G be a solvable group and let H be a self-normalizing maximal subgroup of G. It is well known that the index $[G:H]$ of H in G is a power of a prime p. Let K be the core of H in G. Hall (**7**, p. 511) showed that G/K contains a unique minimal normal subgroup A/K whose order is the index $[G:H]$ of H in G. We also note that $\phi(G/K) = 1$ and G/K is non-abelian. Hence, we have the following theorem.

THEOREM 4.1. *Let G be a solvable group and let H be a self-normalizing maximal subgroup of G. Then* $\text{core}(H) \in \mathscr{M}(G)$.

Because of Theorems 3.1 and 4.1 we obtain the following theorem.

THEOREM 4.2. *Let H be a self-normalizing maximal subgroup of a solvable group G and let K denote the core of H in G. Then K is a generalized Frattini subgroup of G if and only if K is a proper subgroup of $F(G)$.*

Let H be a self-normalizing maximal subgroup of a solvable group G. Let K be the core of H in G and assume that K and G' are nilpotent. Since G/K is non-abelian, K does not contain G'; hence, K is a proper subgroup of $F(G)$. Because of Theorem 4.2 we obtain the next two corollaries.

COROLLARY 4.2.1. *Let G be a solvable group and let K be the core of a self-normalizing maximal subgroup of G. If K and G' are nilpotent, then K is a generalized Frattini subgroup of G.*

COROLLARY 4.2.2. *Let G be a solvable group and let H be a nilpotent self-normalizing maximal subgroup of G whose core in G is K. If G' is nilpotent, then K is a generalized Frattini subgroup of G.*

COROLLARY 4.2.3. *Let G be a group and let H be a self-normalizing maximal subgroup which is nilpotent of class less than three. If G' is nilpotent, then the core of H in G is a generalized Frattini subgroup of G.*

Proof. This is a direct consequence of (**5**, Theorem 1) and Corollary 4.2.2.

Let G be a supersolvable group and let $K \in \mathscr{M}(G)$. Because of Lemma 3.1, K does not contain G'. Assume that K is nilpotent. We note that G' is nilpotent (**8**, Theorem 7.2.13); hence, KG' is nilpotent (**8**, Theorem 7.4.1). Therefore, K is a proper subgroup of $F(G)$ and by Theorem 3.1 we obtain the following theorem.

THEOREM 4.3. *Let G be a supersolvable group and let $K \in \mathscr{M}(G)$. If K is nilpotent, then K is a generalized Frattini subgroup of G.*

The assumption in Theorem 4.3 that G is supersolvable cannot be omitted.

Example 4.1. *Let S_4 be the symmetric group on four symbols and let H be the normal subgroup of S_4 which is isomorphic to the Klein four-group. We note that S_4 is not supersolvable, $H \in \mathscr{M}(G)$, and H is abelian. However, H is not a generalized Frattini subgroup of G.*

COROLLARY 4.3.1. *Let G be a supersolvable group and let K be the core of a nilpotent self-normalizing maximal subgroup of G. Then K is a generalized Frattini subgroup of G.*

Proof. This follows from Theorems 4.1 and 4.3.

Let G be a non-nilpotent group all of whose proper subgroups are nilpotent. By (**8**, Theorem 6.5.7), G is a solvable group. Let K be a maximal generalized Frattini subgroup of G. Because of (**4**, Theorem 3.1), K contains $\phi(G)$. Suppose that K properly contains $\phi(G)$. Then there exists a maximal subgroup M such that $G = KM$; hence, G/K is nilpotent. By (**4**, Theorem 3.2), G is a nilpotent group; hence, $K = \phi(G)$. We have established the result which follows.

THEOREM 4.4. *Let G be a non-nilpotent group all of whose proper subgroups are nilpotent. Then G is solvable and $\phi(G)$ is the unique maximal generalized Frattini subgroup of G.*

Let G be a non-nilpotent group all of whose proper subgroups are nilpotent. Let K be the core of a self-normalizing maximal subgroup of G. By Theorem 4.4, G is solvable; hence, G' is nilpotent. Because of Corollary 4.2.2, K is a generalized Frattini subgroup of G. Since K contains $L(G)$, by Theorem 4.4, $K = L(G) = \phi(G)$. Therefore, we have the next result.

COROLLARY 4.4.1. *Let G be a non-nilpotent group all of whose proper subgroups are nilpotent. If K is the core of a self-normalizing maximal subgroup, then $K = L(G) = \phi(G)$.*

THEOREM 4.5. *Let G be a non-nilpotent group such that every proper subgroup of $G/\phi(G)$ is nilpotent. Then*

(a) *$G/\phi(G)$ is non-nilpotent;*

(b) *G is solvable;*

(c) *$\phi(G)$ is the unique maximal generalized Frattini subgroup of G;*

(d) *Every proper normal subgroup of G is nilpotent.*

Proof. (a) This follows from (**8**, Theorem 6.4.14).

(b) Because of (**8**, Theorem 7.3.14), and Theorem 4.4, G is solvable.

(c) Let K be a maximal generalized Frattini subgroup of G. By (**4**, Theorem 3.1), K contains $\phi(G)$; hence, $K/\phi(G)$ is a generalized Frattini subgroup of $G/\phi(G)$ (**4**, Theorem 3.4). Because of Theorem 4.4, $K = \phi(G)$; hence, $\phi(G)$ is the unique maximal generalized Frattini subgroup of G.

(d) By (**8**, Theorem 7.3.2), $\phi(G)$ is a small subgroup of G. Since $\phi(G)$ is a generalized Frattini subgroup of G, the result follows from (**4**, Theorem 4.4).

From Corollary 4.4.1 and Theorem 4.5, we obtain the following result.

COROLLARY 4.5.1. *Let G be a non-nilpotent group such that every proper subgroup of $G/\phi(G)$ is nilpotent. If K is the core of a self-normalizing maximal subgroup of G, then $K = \phi(G)$.*

5. Generalized Frattini subgroups of A-groups. Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let H be a subgroup of G which contains K and let $F(H)$ be abelian. Assume that K is a generalized Frattini subgroup of H. By (**4**, Theorems 3.1 and 3.6), K is a proper subgroup of $F(H)$. We note that $F(G)$ contains K. Suppose that $F(G) = K$. Then $F(G)$ is contained in $F(H)$. Since $F(H)$ is abelian, $F(H) = F(G)$ (**8**, Theorem 7.4.7); hence, $K = F(H)$ which is a contradiction. Therefore, K is a proper subgroup of $F(G)$, and therefore K is a generalized Frattini subgroup of G by Theorem 3.1. We have established the result which follows.

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THEOREM 5.1. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let H be a subgroup of G which contains K and $F(H)$ is abelian. If K is a generalized Frattini subgroup of H, then K is a generalized Frattini subgroup of G.*

Because of Theorems 4.1 and 5.1, we obtain the next result.

COROLLARY 5.1.1. *Let G be a solvable group. The core of an abelian self-normalizing maximal subgroup of G is a generalized Frattini subgroup of G.*

COROLLARY 5.1.2. *Let G be a solvable group and let H be an abelian self-normalizing maximal subgroup of G. Then H does not contain $F(G)$.*

Proof. Suppose that H contains $F(G)$. Since $\text{core}(H) = K$ is the largest normal subgroup of G which is contained in H, it follows that K contains $F(G)$. Because of Corollary 5.1.1 and (**4**, Theorem 3.1), $F(G)$ is a generalized Frattini subgroup of G. This fact contradicts (**4**, Theorem 3.6).

A solvable group G is called an A-group if all of the Sylow subgroups of G are abelian; cf. (**9**).

COROLLARY 5.1.3. *Let G be a solvable group and let $K \in \mathscr{M}(G)$. Let H be an* A-*subgroup of G containing K. If K is a generalized Frattini subgroup of H, then K is a generalized Frattini subgroup of G.*

Proof. This follows from (**9**, Theorem 3.3) and Theorem 5.1.

Let G be a solvable group. The lower nilpotent series for G is the series

$$G = L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n = 1,$$

where L_{i+1} is the smallest normal subgroup of L_i such that L_i/L_{i+1} is nilpotent, $i = 0, 1, 2, \ldots, n - 1$. It is well known that each L_i exists and is unique ($i = 1, 2, \ldots, n - 1$). The positive integer n is called the nilpotent length of G. For an A-group, the lower nilpotent series and the derived series of G are the same. This fact is a consequence of (**9**, Theorem 3.4). Hence, the nilpotent length of an A-group is just the derived length of G.

THEOREM 5.2. *Let G be an* A-*group of nilpotent length two and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if K is abelian.*

Proof. Assume that K is a generalized Frattini subgroup of G. By (**4**, Theorem 3.1), K is nilpotent; hence, K is abelian (**9**, Theorem 3.3).

Conversely, assume that K is abelian. Further, suppose that K is not a generalized Frattini subgroup of G. Because of Theorem 3.1 and (**9**, Theorem 5.4), $K = Z(G) \times G'$. Hence, G/K is abelian which contradicts Lemma 3.1. Therefore, K is a generalized Frattini subgroup of G.

A group G is called a complemented group if to every subgroup H of G there exists at least one subgroup K such that $G = HK$ and $H \cap K = 1$; cf. (**6**). Hall (**6**) showed that a complemented group is an A-group all of whose Sylow subgroups are elementary abelian. He also showed that the nilpotent length of a complemented group is two. Because of Theorem 5.2, we obtain the following result.

COROLLARY 5.2.1. *Let G be a complemented group and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if K is abelian.*

We now turn our attention to more general A-groups.

THEOREM 5.3. *Let G be an* A-*group such that $Z(G)$ is a non-trivial subgroup and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G.*

Proof. Assume that K is a generalized Frattini subgroup of G. From (**4**, Theorem 3.1), it follows that every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G.

Now assume that every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G. Further, suppose that K is not a generalized Frattini subgroup of G. Since K is solvable, K' is a proper subgroup of K which is normal in G, hence K is nilpotent (**4**, Theorem 3.2). Therefore, K is abelian (**9**, Theorem 3.3) and because of Theorem 3.1, $K = F(G)$.

Let $s + 1$ be the derived length of G. Because of (**9**, Theorem 5.4),

$$K = Z(G) \times Z(G') \times \ldots \times Z(G^{(s)}).$$

Since $Z(G) \neq 1$, $Z(G^{(s)}) = G^{(s)}$ is a proper subgroup of K which is normal in G; hence, $G^{(s)}$ is a generalized Frattini subgroup of G. Because of (**4**, Theorem 3.2; **9**, Theorem 3.3), $G^{(s-1)}$ is abelian; hence, $G^{(s)} = 1$ and $Z(G^{(s-1)}) = G^{(s-1)}$. Proceeding in this way we show that $K = Z(G) \times G'$. Hence, G/K is abelian which contradicts Lemma 3.1. This completes the proof of the theorem.

THEOREM 5.4. *Let H be a generalized Frattini subgroup of G such that G/H is an* A-*group and $Z(G/H)$ is non-trivial. Let $K \in \mathscr{M}(G)$ and let K properly contain H. Then K is a generalized Frattini subgroup of G if and only if every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G.*

Proof. We first note that G is solvable (**4**, Theorem 3.1). Since G/K is isomorphic to $(G/H)/(K/H)$, $K/H \in \mathscr{M}(G/H)$.

Suppose that K is a generalized Frattini subgroup of G. Then every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G (**4**, Theorem 3.1).

Conversely, let M/H be a normal subgroup of G/H such that M/H is properly contained in K/H. Then M is a generalized Frattini subgroup of G; hence, M/H is a generalized Frattini subgroup of G/H (**4**, Theorem 3.4). By Theorem 5.3, K/H is a generalized Frattini subgroup of G/H; hence, K is a generalized Frattini subgroup of G (**4**, Theorem 3.4).

COROLLARY 5.4.1. *Let G be a non-nilpotent group such that $L(G)$ properly contains $\phi(G)$. Let $G/\phi(G)$ be an* A-*group and let $K \in \mathscr{M}(G)$. Then K is a generalized Frattini subgroup of G if and only if every proper subgroup of K which is normal in G is a generalized Frattini subgroup of G.*

Proof. By (**3**, Theorem 2.2), $Z(G/\phi(G)) = L(G)/\phi(G) \neq 1$. Hence, the corollary follows from (**4**, Corollary 3.1.1) and Theorem 5.4.

6. Generalized Frattini subgroups of E-groups. A group G is called an elementary group if $\phi(H) = 1$ for each subgroup H of G. A group G is elementary if and only if all the Sylow p-subgroups of G, p a prime, are elementary p-abelian (**2**, Corollary 2.3). Hence, a solvable elementary group is an A-group. For further properties of elementary groups, the reader is referred to Bechtell's results (**2**).

A group G is called an E-group if $\phi(G)$ contains $\phi(H)$ for each subgroup H of G. A group G is an E-group if and only if $G/\phi(G)$ is elementary. This is the content of (**2**, Theorem 3.1). For several other interesting properties of E-groups, see (**2**).

Let H and K be subgroups of a group G. Then $[H, K]$ will denote the subgroup of G generated by the set of all commutators $[h, k] = h^{-1}k^{-1}hk$ with $h \in H$ and $k \in K$ (**8**, p. 58).

Let G be a group. Denote by $K(G)$ the normal subgroup of least order generated by $[\phi(P), G]$ for each Sylow p-subgroup P of G and all primes p dividing the order of G. Further, denote by $E(G)$ the normal subgroup of least order of G which contains $\phi(P)$ for all Sylow p-subgroups P of G and each prime p dividing the order of G; see (**2**). By (**2**, Theorem 4.1), $E(G)$ is the smallest normal subgroup of G such that $G/E(G)$ is an elementary group.

We note that $E(G)$ contains $K(G)$ and also $E(G)/K(G)$ is contained in the centre of $G/K(G)$. Our aim in this section is to consider conditions under which $K(G)$ and/or $E(G)$ becomes generalized Frattini subgroups of G. We begin with the following theorem.

THEOREM 6.1. *Let G be a group such that $E(G)$ is a proper subgroup of G. Then*

(a) *If G is an* E-*group, then $E(G)$ is a generalized Frattini subgroup of G;*

(b) *$E(G)$ is a generalized Frattini subgroup of G if and only if $K(G)$ is a generalized Frattini subgroup of G.*

Proof. (a) Assume that G is an E-group. Because of (**2**, Theorem 4.1), $E(G) = \phi(G)$, and therefore $E(G)$ is a generalized Frattini subgroup of G (**4**, Corollary 3.1.1).

(b) Assume that $E(G)$ is a generalized Frattini subgroup of G. Then $K(G)$ is a generalized Frattini subgroup of G (**4**, Theorem 3.1).

Conversely, let $K(G)$ be a generalized Frattini subgroup of G. We note that $Z(G/K(G))$ contains $E(G)/K(G)$, and $E(G)/K(G)$ is a proper subgroup of $G/K(G)$. If $G/K(G)$ is abelian, then $E(G)/K(G)$ is a generalized Frattini

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subgroup of $G/K(G)$; hence, $E(G)$ is a generalized Frattini subgroup of G (**4**, Theorem 3.4). Hence, assume that $G/K(G)$ is non-abelian. By (**4**, Corollary 3.1.1), $Z(G/K(G))$ is a generalized Frattini subgroup of $G/K(G)$; hence, $E(G)$ is a generalized Frattini subgroup of G (**4**, Theorems 3.1 and 3.4). This completes the proof.

We now give two examples that will help illustrate the theory of this section.

Example 6.1. *Let* S_4 *and* H *be as in Example* 4.1. *Then* $E(S_4) = K(S_4) = H \in \mathscr{M}(S_4)$ and $E(S_4)$ is not a generalized Frattini subgroup of S_4. We note that $L(S_4) = \phi(S_4) = 1$.

Example 6.2. Let $G = \langle a, b | \, a^9 = b^2 = 1, ba = a^{-1}b\rangle$. Then $L(G) = \phi(G) = K(G) = E(G) = \langle a^3 \rangle$ and $E(G)$ is a maximal generalized Frattini subgroup of G. We note that $E(G) \in \mathscr{M}(G)$ and G is an E-group which is not elementary.

Let G be a non-nilpotent group and let $E(G)$ be a maximal generalized Frattini subgroup of G. Because of (**4**, Theorem 3.10), $E(G)$ contains $L(G)$. By (**2**, Theorem 4.1), $\phi(G) \subseteq E(G) \subseteq R(G)$; hence, $\phi(G) = L(G)$. Now assume that G possesses an $E(G)$-series; cf. (**4**). Then $E(G) = \phi(G) = L(G) = Z^*(G)$ (**4**, Theorem 5.2); hence, G is an E-group (**2**, Theorem 4.1). We have established the theorem which follows.

THEOREM 6.2. *Let* G *be a non-nilpotent group and let* $E(G)$ *be a maximal generalized Frattini subgroup of* G. *Then*

(a) $\phi(G) = L(G)$;

(b) *If* G *has an* $E(G)$*-series, then* G *is an* E*-group and* $E(G) = \phi(G) = L(G) = Z^*(G)$.

Let G be the non-nilpotent group in Example 6.2. Then $E(G)$ is a maximal generalized Frattini subgroup of G, and G is an E-group; however, $Z^*(G) = 1$ and G does not possess an $E(G)$-series.

Let $K(G)$ be a maximal generalized Frattini subgroup of the non-nilpotent group G. By (**4**, Theorem 3.2), $Z(G/K(G))$ is a proper subgroup of $G/K(G)$; hence, $E(G)$ is a proper subgroup of G since $E(G)/K(G)$ is a subgroup of $Z(G/K(G))$. Because of Theorems 6.1 and 6.2, we obtain the next theorem.

THEOREM 6.3. *Let* G *be a non-nilpotent group and let* $K(G)$ *be a maximal generalized Frattini subgroup of* G. *Then*

(a) $K(G) = E(G)$ *and* $L(G) = \phi(G)$;

(b) *If* G *has a* $K(G)$*-series, then* G *is an* E*-group and* $E(G) = K(G) = \phi(G) = L(G) = Z^*(G)$.

THEOREM 6.4. *Let* G *be a solvable group and let* $K(G) \in \mathscr{M}(G)$. *Then*

(a) $K(G) = E(G)$;

(b) *If* $K(G)$ *is a generalized Frattini subgroup of* G, *then* $K(G) = E(G)$ *and* $L(G) = \phi(G)$;

(c) *If* G *is an* E*-group, then* $K(G) = E(G) = \phi(G) = L(G)$.

Proof. (a) Since G is solvable, G contains a normal maximal subgroup; hence, $E(G)$ is a proper subgroup of G (**2**, Theorem 4.1). Suppose that $K(G)$ is a proper subgroup of $E(G)$. Then $Z(G/K(G))$ is non-trivial; hence, $Z(G/K(G))$ contains the unique minimal normal subgroup $A/K(G)$ of $G/K(G)$. Because of (**8**, Theorem 7.4.15), $F(G/K(G)) = A/K(G) = Z(G/K(G))$; hence, $G/K(G)$ is abelian (**8**, Theorem 7.4.7). This contradicts Lemma 3.1, and therefore $K(G) = E(G)$.

(b) Assume that $K(G)$ is a generalized Frattini subgroup of G. By Corollary 3.2.1, $K(G)$ is a maximal generalized Frattini subgroup of G. By Lemma 3.1, G is non-nilpotent; hence, by Theorem 6.3, $K(G) = E(G)$ and $L(G) = \phi(G)$.

(c) Assume that G is an E-group. By (**2**, Theorem 4.1), $E(G) = \phi(G)$, and therefore $E(G)$ is a generalized Frattini subgroup (**4**, Corollary 3.1.1). By the first two parts of this theorem, $K(G) = E(G) = \phi(G) = L(G)$.

This completes the proof.

We note that it is possible for $K(G)$ to be $E(G)$ without $K(G)$ being a generalized Frattini subgroup of G; see Example 6.1.

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