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VARIETIES THAT MAKE ONE CROSS

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Abstract

An example is constructed of a locally finite variety of non-associative algebras which satisfies the maximal condition on subvarieties but not the minimal condition. Based on this, counter-examples to various conjectures concerning varieties generated by finite algebras are constructed. The possibility of finding a locally finite variety of algebras which satisfies the minimal condition on subvarieties but not the maximal is also investigated.

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1. Introduction

One method that has often been used in the proof that a finite algebra has a finite basis for its laws is to embed that algebra in a Cross variety (see, for example, Macdonald, 1973). Recall that a *proper section* of an algebra A is H/ρ, where H is a subalgebra of A, ρ a congruence on H and H/ρ is not the whole of A, and that a *critical algebra* is a finite algebra which is not contained in the variety generated by its proper sections. Then a Cross variety has to satisfy three conditions; it must be locally finite, contain, up to isomorphism, only finitely many critical algebras, and have a finite basis for its laws. As is well known (Birkhoff, 1935), the variety generated by a finite algebra is locally finite, but the relationship (if any) between the second and third conditions has not been clear. All the well-known types of finite algebras with finite bases for their laws turn out to generate Cross varieties (even those such as lattices, in which the Cross variety method of proof was not used) so it seemed possible that these two conditions were not independent. However, in this paper we shall construct examples of varieties generated by finite algebras:

(i) with a finite basis for its laws, but containing infinitely many critical algebras;
(ii) with an infinite basis for its laws, but containing only finitely many critical algebras,

thus showing that the two conditions are in fact independent (R. E. Park, 1976, also has an example of type two, a four-element upper bound algebra.)

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The algebras involved all have modular congruence lattices, and this enables us to produce a counterexample to the following conjecture (given in Macdonald, 1973):

CONJECTURE 1. *If $\mathfrak{V}$ is a variety of algebras whose congruence lattices are modular, then every finite algebra in $\mathfrak{V}$ has a finite basis for its laws* (and, *a fortiori*, a counterexample to the even more optimistic conjecture of S. Burris reported in the same paper). Polin (1976) produced the first such example and it was news of this that motivated our construction.

In Section 2 we prove a theorem which gives several equivalents to the existence of only finitely many critical algebras in a locally finite variety, one of these being that the variety satisfies both the maximal and minimal condition on subvarieties, and again one might wonder if both chain conditions are necessary. The major part of the paper is devoted to constructing an example of a variety generated by a finite algebra which satisfies Max but not Min. It is on this example that the other examples mentioned above have been based.

Unfortunately, we have not succeeded in constructing an example of a variety generated by a finite algebra which satisfies Min but not Max. There is some evidence to suggest that the variety generated by the three-element algebra proved by Murskii (1965) to be infinitely based may have this property. This is discussed in Section 4.

2. Equivalents to finitely many critical algebras

Clearly a critical algebra belongs neither to the variety generated by its proper subalgebras, nor to that generated by its proper quotient algebras, in other words it is both **S**-critical and **Q**-critical. **Q**-critical algebras are the easiest to work with as they are subdirectly irreducible; unfortunately **Q**-criticality is not a sufficiently strong condition, indeed (as can be deduced from Example 51.33 of Neumann, 1967) any finite group with non-abelian Sylow subgroups generates a variety containing infinitely many **Q**-critical groups. As condition (a) of the following theorem shows, it is **S**-criticality that is important.

2.1 THEOREM. *If $\mathfrak{V}$ is a locally finite variety then the following conditions are equivalent.*

(a) $\mathfrak{V}$ *has only finitely many* **S**-*critical algebras.*
(b) $\mathfrak{V}$ *has only finitely many critical algebras.*
(c) $\mathfrak{V}$ *has only finitely many subvarieties.*
(d) $\mathfrak{V}$ *satisfies the maximal and minimal conditions on subvarieties.*

Before we prove the theorem we make some comments on condition (d). The minimal condition on subvarieties of $\mathfrak{V}$ is equivalent to the condition that every subvariety of $\mathfrak{V}$ has a finite basis for its laws *as a subvariety*. This means that every subvariety of $\mathfrak{V}$ is determined by the laws of $\mathfrak{V}$ together with a finite additional set of laws. It does not mean that $\mathfrak{V}$ is itself finitely based, and indeed we shall see that there exist locally finite varieties which are not finitely based, but which satisfy the conditions of the theorem. If $\mathfrak{V}$ is a locally finite variety then the maximal condition on subvarieties of $\mathfrak{V}$ is equivalent to the condition that every subvariety of $\mathfrak{V}$ (including $\mathfrak{V}$ itself) is generated by a finite algebra. To see this suppose that $\mathfrak{V}$ is a variety which satisfies the maximal condition on subvarieties and let $\mathfrak{W}$ be a subvariety of $\mathfrak{V}$. For each $n = 1, 2, \ldots$ let $\mathfrak{W}_n$ be the subvariety of $\mathfrak{W}$ generated by its n generator algebras. Then

$$\mathfrak{W}_1 \leqslant \mathfrak{W}_2 \leqslant \ldots \leqslant \mathfrak{W}_n \leqslant \ldots$$

and so for some n, $\mathfrak{W}_m = \mathfrak{W}_n$ for all $m \geqslant n$. This implies that $\mathfrak{W}_n = \mathfrak{W}$, and so $\mathfrak{W}$ is generated by $F_n(\mathfrak{W})$, the free algebra of $\mathfrak{W}$ of rank n. If $\mathfrak{V}$ is locally finite then $F_n(\mathfrak{W})$ is finite, and so $\mathfrak{W}$ is generated by a finite algebra. On the other hand, suppose that every subvariety of $\mathfrak{V}$ is generated by a finite algebra, and let

$$\mathfrak{W}_1 \leqslant \mathfrak{W}_2 \leqslant \ldots \leqslant \mathfrak{W}_n \leqslant \ldots$$

be an ascending chain of subvarieties of $\mathfrak{V}$. Suppose that $\mathfrak{W}_n$ is generated by the finite algebra A_n for $n = 1, 2, \ldots$, and let the variety generated by $\{A_n \colon n = 1, 2, \ldots\}$ be generated by the finite algebra A. Then $A \in \mathbf{QSC}\{A_n \colon n = 1, 2, \ldots\}$ (here $\mathbf{C}$ denotes the cartesian product). Let $A \in \mathbf{QS}B$ where $B \in \mathbf{C}\{A_n \colon n = 1, 2, \ldots\}$. Since A is finite, A is a homomorphic image of a finitely generated subalgebra of B. Since $\mathfrak{V}$ is locally finite this subalgebra is finite, and so is isomorphic to a subalgebra of a finite Cartesian product of copies of algebras in $\{A_n \colon n = 1, 2, \ldots\}$. So A is in the variety generated by a finite subset of $\{A_n \colon n = 1, 2, \ldots\}$, which implies that $A \in \mathfrak{W}_n$ for some n. This implies that $\mathfrak{W}_m = \mathfrak{W}_n$ for $m \geqslant n$.

Proof of Theorem 2.1. Any critical algebra is certainly S-critical and so (a) implies (b). Any subvariety of $\mathfrak{V}$ is generated by its critical algebras, that is, by some subset of the critical algebras of $\mathfrak{V}$. Hence (b) implies (c). Trivially (c) implies (d). The non-trivial part of the proof is that (d) implies (a). Let $\mathfrak{V}$ be a locally finite variety which satisfies the maximal and minimal conditions on subvarieties. Suppose that $\mathfrak{V}$ has infinitely many S-critical algebras, and let $\mathfrak{W}$ be a subvariety of $\mathfrak{V}$ which is minimal with respect to containing infinitely many S-critical algebras. $\mathfrak{V}$ satisfies the maximal condition on subvarieties and so, for some n, $\mathfrak{W}$ is generated by its free algebra of rank n, $F_n(\mathfrak{W})$. If $\mathfrak{U}$ is any proper subvariety of $\mathfrak{W}$ then $F_n(\mathfrak{U})$ must be a proper homomorphic image of $F_n(\mathfrak{W})$, and so $\mathfrak{U}$ satisfies some n variable law which is not satisfied by $\mathfrak{W}$. This implies

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that the maximal proper subvarieties of $\mathfrak{W}$ are determined by n variable laws (together with the laws of $\mathfrak{W}$). Since $\mathfrak{V}$ is locally finite it follows that $\mathfrak{W}$ has only finitely many maximal proper subvarieties. By assumption each of these contains only finitely many S-critical algebras, and so $\mathfrak{W}$ contains infinitely many S-critical algebras which do not lie in any of its proper subvarieties.

Let A be one of these S-critical algebras. Then the proper subalgebras of A must lie in some maximal proper subvariety $\mathfrak{U}$ of $\mathfrak{W}$. As shown above, $\mathfrak{U}$ is determined by the laws of $\mathfrak{W}$ together with some n variable law. Since $A \in \mathfrak{W}$ and $A \notin \mathfrak{U}$, A must fail to satisfy this n variable law. However, every proper subalgebra of A lies in $\mathfrak{U}$, and so A can be generated by n elements. But $\mathfrak{V}$ is locally finite, and so has only finitely many n generator algebras. This leads to a contradiction, and so our assumption that $\mathfrak{V}$ contains infinitely many S-critical algebras is false, and the theorem is proved.

3. Max but not Min

We now give an example of a locally finite variety $\mathfrak{V}$ of non-associative algebras which satisfies the maximal condition on subvarieties, but not the minimal condition. So every subvariety of $\mathfrak{V}$ (including $\mathfrak{V}$ itself) is generated by a finite algebra, but $\mathfrak{V}$ has subvarieties which are not finitely based. By Zorn's Lemma $\mathfrak{V}$ has a "just non-finitely based" subvariety $\mathfrak{W}$. That is, $\mathfrak{V}$ has a subvariety $\mathfrak{W}$ which is not finitely based, but all of whose proper subvarieties are finitely based. Since $\mathfrak{W}$ is a subvariety of $\mathfrak{V}$, $\mathfrak{W}$ satisfies the maximal condition on subvarieties, and $\mathfrak{W}$ satisfies the minimal condition on subvarieties since all the proper subvarieties of $\mathfrak{W}$ are finitely based. So, by the theorem, $\mathfrak{W}$ is generated by a finite algebra, $\mathfrak{W}$ has only finitely many critical algebras and only finitely many subvarieties, and all the proper subvarieties of $\mathfrak{W}$ are Cross.

Let F be a finite field with q elements. A is a non-associative algebra over F if A is a vector space over F with a bilinear product. Since F is finite the variety of non-associative algebras over F is finitely based. If A is any non-associative algebra we use a left normed convention for products of elements of A. Thus abc denotes $(ab)c$. For $i = 0, 1, 2, \dots$ we define ab^i inductively by $ab^0 := a$, $ab^{i+1} := (ab^i)b$. Thus $ab^2 = (ab)b = abb$.

3.1. THEOREM. *Let $\mathfrak{V}$ be the variety of non-associative algebras over F determined by the laws*

$$x_1(x_2x_3) = 0,$$

$$x_1x_2x_3x_4x_5 = x_1x_2x_4x_3x_5,$$

$$x_1x_2x_3^qx_4 = x_1x_2x_3x_4.$$

Then $\mathfrak{V}$ is a locally finite variety satisfying the maximal condition on subvarieties but not the minimal condition.

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Theorem 3.1 is an immediate consequence of Theorems 3.2, 3.4 and 3.10. We give the proofs of these in the case when $q = 2$, that is, when $F = Z_2$. The proofs in the general case are essentially the same but the details are more complicated.

3.2. THEOREM. $\mathfrak{B}$ *is locally finite.*

PROOF. Let A be an algebra in $\mathfrak{B}$ and suppose that A is generated by $a_1, a_2, \ldots, a_n$. Then using the law $x_1(x_2 x_3) = 0$ we see that A is spanned by elements of the form

$$a_{i(1)} a_{i(2)} \cdots a_{i(m)}$$

with $m \geqslant 1$ and $i(1), i(2), \ldots, i(m) \in \{1, 2, \ldots, n\}$. Using the law

$$x_1 x_2 x_3 x_4 x_5 = x_1 x_2 x_4 x_3 x_5$$

we see that if $m \geqslant 5$ then we may assume that

$$i(3) \leqslant i(4) \leqslant \ldots \leqslant i(m-1).$$

Using the law $x_1 x_2 x_3^q x_4 = x_1 x_2 x_3 x_4$ in the case $q = 2$ we see that we may assume that

$$i(3) < i(4) < \ldots < i(m-1).$$

Now there are only finitely many sequences $(i(1), i(2), \ldots, i(m))$ such that $i(1), i(2), \ldots, i(m) \in \{1, 2, \ldots, n\}$ and $i(3) < i(4) < \ldots < i(m-1)$, and so A is finite dimensional as a vector space over F. Since F is finite this implies that A is finite.

3.3. THEOREM. $\mathfrak{B}$ *is generated by a finite algebra.*

PROOF. Let A be the non-associative algebra over F defined as follows.

A is generated by elements a, b, c, d.

A has basis $a, b, c, d, ab, abc, abd, abcb, abcd, abdb, abdc, abcdb$ as a vector space over F.

If x and y are members of this basis then $xy = 0$ unless

$$x \in \{a, ab, abc, abd, abcb, abcd, abdb, abdc, abcdb\}$$

and $y \in \{b, c, d\}$. These products are given by the following table.

	b	c	d
a	ab	0	0
ab	ab	abc	abd
abc	$abcb$	abc	$abcd$
abd	$abdb$	$abdc$	abd
$abcb$	$abcb$	abc	$abcd$
$abcd$	$abcdb$	$abdc$	$abcd$
$abdb$	$abdb$	$abdc$	abd
$abdc$	$abcdb$	$abdc$	$abcd$
$abcdb$	$abcdb$	$abdc$	$abcd$

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It is routine to check that A satisfies the laws of $\mathfrak{V}$. Now let $F(\mathfrak{V})$ be the free algebra of $\mathfrak{V}$ generated by $x_1, x_2, \ldots$. As in the proof that $\mathfrak{V}$ is locally finite we see that $F(\mathfrak{V})$ is spanned (as a vector space) by monomials of the form

$$x_{i(1)}\, x_{i(2)} \ldots x_{i(m)}$$

with $m \geqslant 1$, $i(3) < i(4) < \ldots < i(m-1)$. Let S be the set of finite sequences of positive integers of the form $(i(1), i(2), \ldots, i(m))$ with $m \geqslant 1$, $i(3) < i(4) < \ldots < i(m-1)$. If $s = (i(1), i(2), \ldots, i(m)) \in S$ let $w(s) = x_{i(1)}\, x_{i(2)} \ldots x_{i(m)}$. Then $F(\mathfrak{V})$ is spanned by $\{w(s) : s \in S\}$. We show that if $s_1, s_2, \ldots, s_k$ are distinct elements of S and if $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ then

$$\alpha_1 w(s_1) + \alpha_2 w(s_2) + \ldots + \alpha_k w(s_k) = 0$$

is a law of A only if $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. This proves that A generates $\mathfrak{V}$. It also proves that if s and t are distinct elements of S then $w(s) \neq w(t)$, and that $\{w(s) : s \in S\}$ is a basis of $F(\mathfrak{V})$ as a vector space over F. First we need to establish some notation. If $s = (i_1, i_2, \ldots, i_m) \in S$ and i is a positive integer then we say that s has degree r in i if i occurs r times in the sequence $(i_1, i_2, \ldots, i_m)$. Note that r can be at most 4. We say that s involves i if it has degree at least 1 in i. The degree of s is defined to be m. For each positive integer i we let δ_i be the endomorphism of $F(\mathfrak{V})$ which maps x_i to 0 and maps x_j to x_j for $j \neq i$. If $s \in S$ and s involves i then $w(s)\,\delta_i = 0$. If s does not involve i then $w(s)\,\delta_i = w(s)$. Now suppose that A satisfies a law

$$\alpha_1 w(s_1) + \alpha_2 w(s_2) + \ldots + \alpha_k w(s_k) = 0$$

for some non-zero elements $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ and some distinct elements $s_1, s_2, \ldots, s_k \in S$. Since $F = Z_2$, $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 1$. So we can write this law in the form

$$\sum_{s \in P} w(s) = 0,$$

where P is a non-empty finite subset of S. We show that this implies that A satisfies a law of the form above where, for some $m > 0$, P consists of elements which are of degree 1 in each of $1, 2, \ldots, m$, and which involve no other integers. Then we obtain a contradiction by showing that A cannot satisfy a law of this form.

So suppose that A satisfies the law $\sum_{s \in P} w(s) = 0$, where P is some non-empty finite subset of S. Then for each positive integer i

$$\sum_{s \in P} w(s) - \sum_{s \in P} w(s)\,\delta_i = 0$$

is a law of A. Now $w(s)\,\delta_i = 0$ if s involves i, and otherwise $w(s)\,\delta_i = w(s)$. So A satisfies the law $\sum_{s \in Q} w(s) = 0$, where Q is the subset of P consisting of elements which involve i. By repeated use of this argument we see that if I is any finite subset of the positive integers then $\sum_{s \in R} w(s) = 0$ is a law of A, where R is the subset of P consisting of elements which involve the integers in I and no other integers. So we may assume that A satisfies a law of the form $\sum_{s\in P} w(s) = 0$ where P is a non-empty finite subset of S whose elements all involve precisely the same set of integers. There is no loss in generality in taking this set of integers to be $1, 2, \ldots, n$. Let m be the maximum of the degrees of the elements of P, and let $t \in P$ have degree m. If $m > n$ then t cannot be of degree 1 in all of $1, 2, \ldots, n$. We suppose that t has degree r in i with $r > 1$. Let θ be the endomorphism of $F(\mathfrak{B})$ which maps x_i to $x_i + x_{n+1}$ and which maps x_j to x_j for $j \neq i$. Then

$$w(t)\,\theta = \sum_{s\in Q} w(s)$$

where Q is a 2^r element subset of S consisting of elements of degree m, and where $2^r - 2$ of the elements in Q involve $1, 2, \ldots, n+1$. If $u \in P$ and $u \neq t$ then

$$w(u)\,\theta = \sum_{s\in R} w(s),$$

where R is a finite subset of S whose elements have the same degree as u and where, importantly, $Q \cap R = \varnothing$. So $(\sum_{s\in P} w(s))\,\theta = \sum_{s\in T} w(s)$, where T is a finite subset of S *which contains* Q and which consists of elements of degree at most m. This implies that $\sum_{s\in T} w(s) = 0$ is a law of A, and hence that $\sum_{s\in U} w(s) = 0$ is a law of A, where U is the subset of T consisting of elements involving $1, 2, \ldots, n+1$. By repeated use of this argument we see that A satisfies a law of the form $\sum_{s\in P} w(s) = 0$ where P is a non-empty finite subset of S consisting of elements of degree m which involve $1, 2, \ldots, m$. Such a law is of the form

$$\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} \ldots x_{m\sigma} = 0,$$

where π is a non-empty set of permutations of $\{1, 2, \ldots, m\}$ with the property that if $\sigma \in \pi$ then $3\sigma < 4\sigma < \ldots < (m-1)\,\sigma$. We obtain a contradiction by showing that A cannot satisfy a law of this form.

If $m \geqslant 4$ and $\tau \in \pi$ let φ be the homomorphism from $F(\mathfrak{B})$ to A which maps $x_{1\tau}$ to a, $x_{2\tau}$ to b, $x_{m\tau}$ to c, and x_j to d for $j \neq 1\tau, 2\tau, m\tau$. Then

$$(x_{1\tau} x_{2\tau} \ldots x_{m\tau})\,\varphi = abdc,$$

and if $\sigma \in \pi$, $\sigma \neq \tau$ then

$$(x_{1\sigma} x_{2\sigma} \ldots x_{m\sigma})\,\varphi = 0 \quad \text{or} \quad abcd.$$

So

$$\Big(\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} \ldots x_{m\sigma}\Big)\,\varphi = abdc \quad \text{or} \quad abdc + abcd.$$

Since $abdc$ and $abdc + abcd$ are both non-zero this shows that $\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} \ldots x_{m\sigma} = 0$ is not a law in A.

If $m = 3$ and $\tau \in \pi$ we let φ be a homomorphism from $F(\mathfrak{B})$ to A which maps $x_{1\tau}$ to a, $x_{2\tau}$ to b, $x_{3\tau}$ to c. Then

$$(x_{1\tau} x_{2\tau} x_{3\tau})\,\varphi = abc,$$

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and if $\sigma \in \pi$, $\sigma \neq \tau$, then

$$(x_{1\sigma} x_{2\sigma} x_{3\sigma})\varphi = 0.$$

So

$$\left(\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} x_{3\sigma}\right)\varphi = abc \neq 0,$$

which implies that $\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} x_{3\sigma} = 0$ is not a law in A.

If $m = 2$ and $\tau \in \pi$ we let φ be a homomorphism from $F(\mathfrak{V})$ to A which maps $x_{1\tau}$ to a and $x_{2\tau}$ to b. Then

$$x_{1\tau} x_{2\tau} \varphi = ab$$

and if $\sigma \neq \tau$ then $x_{1\sigma} x_{2\sigma} \varphi = 0$. So $(\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma})\varphi = ab \neq 0$, which implies that $\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} = 0$ is not a law of A.

Finally, if $m = 1$ then $\sum_{\sigma\in\pi} x_{1\sigma} x_{2\sigma} \dots x_{m\sigma}$ must be x_1, and clearly $x_1 = 0$ is not a law of A.

This completes the proof that A generates $\mathfrak{V}$.

3.4. THEOREM. *If $\mathfrak{W}$ is any subvariety of $\mathfrak{V}$ then $\mathfrak{W}$ is generated by a finite algebra, and hence $\mathfrak{V}$ satisfies the maximal condition on subvarieties.*

PROOF. First we show how the result will follow from the existence of a well order $\leqslant$ and a quasi order $\preccurlyeq$ on S with appropriate properties. The definitions of these are given later.

Every non-zero element of $F(\mathfrak{V})$ can be written uniquely in the form

$$w(s_1) + w(s_2) + \dots + w(s_k)$$

with $s_1, s_2, \dots, s_k \in S$ and $s_1 > s_2 > \dots > s_k$. We define the weight of this element to be s_1. The quasi order $\preccurlyeq$ has the property that if $v \in F(\mathfrak{V})$ has weight b and if $a \preccurlyeq b$ then there is an element of weight a in the fully invariant ideal generated by v. We let $F(\mathfrak{W})$ be the free algebra of $\mathfrak{W}$ generated by $y_1, y_2, \dots$, and we let π be the homomorphism from $F(\mathfrak{V})$ to $F(\mathfrak{W})$ which maps x_i to y_i for $i = 1, 2, \dots$. Then if $w \in F(\mathfrak{W})$ and $w \neq 0$ we define the weight of w to be the minimum with respect to $\leqslant$ of the weights of elements $v \in F(\mathfrak{V})$ such that $v\pi = w$. We let T be the set of weights of non-zero elements of $F(\mathfrak{W})$. We show that the quasi order is a well quasi order, and this implies that there is a finite subset T_0 of T with the property that if $b \in T$ then there is an element $a \in T_0$ with $a \preccurlyeq b$. The well order $\leqslant$ has the property that there are only finitely many elements of $F(\mathfrak{V})$ of any given weight, and so there are only finitely many elements in $F(\mathfrak{V})$ whose weight lies in T_0. This means that there is an integer N with the property that if the weight of v lies in T_0 then $v \in F_N(\mathfrak{V})$. ($F_N(\mathfrak{V})$ is the subalgebra of $F(\mathfrak{V})$ generated by $x_1, x_2, \dots, x_N$.)

Now suppose that $v = 0$ is *not* a law in $\mathfrak{W}$. Then $v\pi \neq 0$. Let $v\pi$ have weight b, and let u be an element of weight b in $F(\mathfrak{V})$ such that $u\pi = v\pi$. Let a be an element

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of T_0 such that $a \preccurlyeq b$. Then there is an element $w \in F(\mathfrak{W})$ of weight a, and an element $p \in F(\mathfrak{V})$ of weight a such that $p\pi = w$. Also there is an element u^* of weight a in the fully invariant ideal generated by u. So the elements p and u^* of $F(\mathfrak{V})$ both have weight a and it follows that $p - u^*$ has weight less than a. Hence $w - u^*\pi = (p - u^*)\pi$ has weight less than a which is the weight of w. This implies that $u^*\pi \neq 0$. Now the weight of u^* lies in T_0 and so $u^* \in F_N(\mathfrak{V})$, which implies that $u^*\pi$ is a non-zero element of $F_N(\mathfrak{W})$. But u^* is in the fully invariant ideal of $F(\mathfrak{V})$ generated by u and so it follows that $u = 0$ is not a law in $F_N(\mathfrak{W})$. Finally, the fact that $u\pi = v\pi$ implies that $u = v$ is a law of $\mathfrak{W}$, and so the fact that $u = 0$ is not a law of $F_N(\mathfrak{W})$ implies that $v = 0$ is not a law of $F_N(\mathfrak{W})$. To summarize: if $v = 0$ is not a law of $\mathfrak{W}$ then it is not a law of $F_N(\mathfrak{W})$. This implies that $\mathfrak{W}$ is generated by the finite algebra $F_N(\mathfrak{W})$.

3.5. DEFINITION. The well order $\leqslant$ is defined as follows. Let $s, t \in S$ and let $s = (i_1, i_2, \ldots, i_m)$, $t = (j_1, j_2, \ldots, j_n)$. Let I be the set of integers involved in s, that is the set $\{i_1, i_2, \ldots, i_m\}$, and let J be the set of integers involved in t. Let $s < t$ if I is a proper subset of J, or if $I \nsubseteq J$ and $J \nsubseteq I$ and $\max(I \backslash J) < \max(J \backslash I)$, or if $I = J$ and

$m \leqslant 2$ and $m < n$, or
$m = n \leqslant 2$ and $i_1 < j_1$, or
$m = n = 2$ and $i_1 = j_1, i_2 < j_2$, or
$m, n > 2$ and $i_1 < j_1$, or
$m, n > 2$ and $i_1 = j_1$, $i_2 < j_2$, or
$m, n > 2$ and $i_1 = j_1$, $i_2 = j_2$, $i_m < j_n$, or
$m, n > 2$ and $i_1 = j_1$, $i_2 = j_2$, $i_m = j_n$ and s has smaller degree in i_1 than t, or
$m, n > 2$ and $i_1 = j_1$, $i_2 = j_2$, $i_m = j_n$ and s has the same degree as t in i_1, but smaller degree than t in i_2, or
$m, n > 2$ and $i_1 = j_1$, $i_2 = j_2$, $i_m = j_n$ and s has the same degree as t in i_1 and i_2, but smaller degree than t in i_m.

3.6. LEMMA. *$\leqslant$ is a well order.*

PROOF. If $s = (i_1, i_2, \ldots, i_m) \in S$ then s is determined by the set $\{i_1, i_2, \ldots, i_m\}$, the integers i_1, i_2, i_m, and the degrees of s in i_1, i_2, i_m, and so $\leqslant$ defines an order on S. Also the first two conditions in the definition of $\leqslant$ imply that if $s \in S$ then there are only finitely many elements $t \in S$ with $t < s$. This shows that $\leqslant$ is a well order, and also shows that there are only finitely many elements in $F(\mathfrak{V})$ of any given weight.

3.7. DEFINITION. The quasi order $\preccurlyeq$ is defined as follows. If $(i_1, i_2, \ldots, i_m)$, $(j_1, j_2, \ldots, j_n) \in S$ let $(i_1, i_2, \ldots, i_m) \preccurlyeq (j_1, j_2, \ldots, j_n)$ if one of the following conditions is satisfied.

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1. Both m and n are greater than 2, and there is a map φ from $\{j_1, j_2, \ldots, j_n\}$ onto $\{i_1, i_2, \ldots, i_m\}$ such that

$$r\varphi = i_1 \text{ if and only if } r = j_1,$$
$$r\varphi = i_2 \text{ if and only if } r = j_2,$$
$$r\varphi = i_m \text{ if and only if } r = j_n,$$
$$\{j_3, j_4, \ldots, j_{n-1}\}\varphi = \{i_3, i_4, \ldots, i_{m-1}\},$$
$$r\varphi \leqslant s\varphi \text{ if } r \leqslant s.$$

(Note that we do not insist that φ is one–one.)

2. Both m and n equal 2 and there is a one–one order preserving map φ from $\{j_1, j_2\}$ to $\{i_1, i_2\}$.

3. Both m and n equal 1. In this case to keep the notation consistent we let φ be the map from $\{j_1\}$ to $\{i_1\}$.

3.8. Lemma. *If* $a = (i_1, i_2, \ldots, i_m) \preccurlyeq (j_1, j_2, \ldots, j_n) = b$ *and* $v \in F(\mathfrak{B})$ *has weight* b *then the fully invariant ideal generated by* v *contains an element of weight* a.

Proof. Let φ be the map from $\{j_1, j_2, \ldots, j_n\}$ to $\{i_1, i_2, \ldots, i_m\}$ satisfying the conditions of Definition 3.7 and let θ be a endomorphism of $F(\mathfrak{B})$ which maps x_i to $x_{i\varphi}$ for $i = j_1, j_2, \ldots, j_n$. Then, using the law $x_1 x_2 x_3 x_3 x_4 = x_1 x_2 x_3 x_4$, we see that $w(b)\theta = w(a)$.

If $v = \sum_{s \in P} w(s)$ for some finite subset of P containing b, then, as in the proof of Theorem 3.3, we see that the fully invariant ideal of $F(\mathfrak{B})$ generated by v contains the element $\sum_{s \in Q} w(s)$, where Q is the subset of P consisting of elements involving the integers $j_1, j_2, \ldots, j_n$ and no others. This implies that $\sum_{s \in Q} w(s)\theta$ is in the fully invariant ideal generated by v, and we show that $\sum_{s \in Q} w(s)$ has weight a. Note that $b \in Q$ and that $w(b)\theta = w(a)$. We show that if $s \in Q$ and $s \neq b$ then $w(s)\theta = w(t)$ for some $t \in S$, $t < a$. This is trivial in the case when $m = n = 1$, or in the case when $m = n = 2$, for then φ is a one–one order preserving map. So suppose that $m, n > 2$, and let $s = (k_1, k_2, \ldots, k_p)$. Note that $\{k_1, k_2, \ldots, k_p\} = \{j_1, j_2, \ldots, j_n\}$ since $s \in Q$. Then $w(s)\theta = w(t)$ where $t = (k_1\varphi, k_2\varphi, r_1, r_2, \ldots, r_q, k_p\varphi)$ with $r_1 < r_2 < \ldots < r_q$ and $\{r_1, r_2, \ldots, r_q\} = \{k_3\varphi, k_4\varphi, \ldots, k_{(p-1)}\varphi\}$. First, since s involves the integers $j_1, j_2, \ldots, j_n$ and no other integers, and since φ maps $\{j_1, j_2, \ldots, j_n\}$ onto $\{i_1, i_2, \ldots, i_m\}$ it follows that t involves the integers $i_1, i_2, \ldots, i_m$. This means that t involves the same integers as a. Next, since $r = i_1$ if and only if $r = j_1$, and since $r\varphi \leqslant s\varphi$ if $r \leqslant s$, it follows that if $k_1 < j_1$ then $k_1\varphi < j_1\varphi$. Similarly if $k_2 < j_2$ then $k_2\varphi < j_2\varphi$, and if $k_p < j_n$ then $k_p\varphi < j_n\varphi$. This implies that $t < a$ except in the case when $k_1 = j_1$, $k_2 = j_2$ and $k_p = j_n$.

So suppose that $k_1 = j_1$, $k_2 = j_2$, $k_p = j_n$. Since s and b involve the same integers the condition that $s < b$ must imply that b has higher degree than s in one of the

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integers j_1, j_2, j_n. However, the condition that $r\varphi = i_1$ if and only if $r = j_1$ implies that the degree of b in j_1 is the same as the degree of a in i_1. Similarly the degree of s in j_1 is the same as the degree of t in i_1. So if b has higher degree than s in j_1 then a has higher degree than t in i_1. Similarly if b has higher degree than s in j_2 or j_n then a has higher degree than t in i_2 or i_m (respectively). So $t < a$ in this case also. This completes the proof that $\sum_{s \in Q} w(s)$ has weight a.

3.9. Lemma. $\preccurlyeq$ *is a well quasi order.*

Proof. From Higman (1952), it is sufficient to show that every sequence of elements of S has a subsequence which is ascending with respect to $\preccurlyeq$. Clearly this is so if infinitely many terms of the sequence are of the form (i_1) or of the form (i_1, i_2). So, replacing the original sequence by a subsequence if necessary, we may suppose that all the terms in the sequence have degree at least 3. Let the sequence be $s_1, s_2, \ldots$ and let $s_n = (n_1, n_2, \ldots, n_{m(n)})$ for $n = 1, 2, \ldots$. Now one or more of the following conditions is satisfied by the sequence.

$n_1 > n_2$ for infinitely many terms in the sequence.

$n_1 = n_2$ for infinitely many terms in the sequence.

$n_1 < n_2$ for infinitely many terms of the sequence.

So, replacing the sequence by a subsequence if necessary, we may suppose that $n_1 > n_2$ for all n, or $n_1 = n_2$ for all n, or $n_1 < n_2$ for all n. Similarly we may assume that $n_1 > n_{m(n)}$ for all n, or $n_1 = n_{m(n)}$ for all n, or $n_1 < n_{m(n)}$ for all n and that $n_2 > n_{m(n)}$ for all n or $n_2 = n_{m(n)}$ for all n, or $n_2 < n_{m(n)}$ for all n. Let us suppose for example that $n_1 > n_2 > n_{m(n)}$ for all n.

Then let A_n be the set of integers involved in s_n which are greater than n_1. Let B_n be the set of integers which are involved in s_n and which lie between n_1 and n_2. Let C_n be the set of integers which are involved in s_n and lie between n_2 and $n_{m(n)}$. Let D_n be the set of integers which are involved in s_n and are less than $n_{m(n)}$. Then replacing the sequence by a subsequence if necessary, we may suppose that A_n is empty for all n, or that A_n is non-empty for all n and that $|A_n| \leqslant |A_{n+1}|$ for all n. Similarly we may suppose that $|B_n| = 0$ for all n, or that $0 < |B_n| \leqslant |B_{n+1}|$ for all n, and that $|C_n| = 0$ for all n, or that $0 < |C_n| \leqslant |C_{n+1}|$ for all n, and that $|D_n| = 0$ for all n, or that $0 < |D_n| \leqslant |D_{n+1}|$ for all n. But then we can find order preserving maps from A_{n+1} onto A_n, and from B_{n+1} onto B_n, and from C_{n+1} onto C_n, and from D_{n+1} onto D_n. Combining these maps we can find an order preserving map φ_n from the set of integers involved in s_{n+1} to the set of integers involved in s_n which maps $(n+1)_1$ to n_1, $(n+1)_2$ to n_2, $(n+1)_{m(n+1)}$ to $n_{m(n)}$ and maps A_{n+1} onto A_n, B_{n+1} onto B_n, C_{n+1} onto C_n and D_{n+1} onto D_n. The map φ_n satisfies the conditions in the definition of $\preccurlyeq$ and so $s_1 \preccurlyeq s_2 \preccurlyeq \ldots$. This shows that the original sequence has an ascending subsequence. We obtain ascending subsequences by the same method whatever the relative magnitudes of $n_1, n_2, n_{m(n)}$.

This completes the proof that $\preccurlyeq$ is a well quasi order, and also completes the proof that $\mathfrak{V}$ satisfies the maximal condition on subvariates.

3.10. THEOREM. *$\mathfrak{V}$ does not satisfy the minimal condition on subvarieties.*

PROOF. We prove this by exhibiting an example of a subvariety of $\mathfrak{V}$ which does not have a finite basis for its laws. Let $\mathfrak{W}$ be the subvariety of $\mathfrak{V}$ determined by the laws $w_n = 0$ for $n = 6, 7, \ldots$, where

$$w_n := (x_1 x_2 x_3 - x_1 x_3 x_2) x_6 x_7 \ldots x_n x_4 x_5 - (x_1 x_2 x_3 - x_1 x_3 x_2) x_6 x_7 \ldots x_n x_5 x_4.$$

Consider the fully invariant ideal generated by w_n. If u is any element in $F(\mathfrak{V})$ then $uw_n = 0$ by the law $x_1(x_2 x_3) = 0$. Also $w_n u = 0$ by the law

$$x_1 x_2 x_3 x_4 x_5 = x_1 x_2 x_4 x_3 x_5.$$

So the fully invariant ideal generated by w_n is spanned by elements of the form

$$v = (v_1 v_2 v_3 - v_1 v_3 v_2) v_6 v_7 \ldots v_n v_4 v_5 - (v_1 v_2 v_3 - v_1 v_3 v_2) v_6 v_7 \ldots v_n v_5 v_4,$$

where $v_1, v_2, \ldots, v_n \in F(\mathfrak{V})$. Since w_n is linear in $x_1, x_2, \ldots, x_n$ we may assume that $v_1, v_2, \ldots, v_n$ are monomials. But if any of $v_2, v_3, \ldots, v_n$ is a monomial which is a product of two or more of the generators of $F(\mathfrak{V})$ then $v = 0$ by the law $x_1(x_2 x_3) = 0$. If v_1 is a monomial which is a product of two or more of the generators of $F(\mathfrak{V})$ then $v = 0$ by the law $x_1 x_2 x_3 x_4 x_5 = x_1 x_2 x_4 x_3 x_5$. Hence we may assume that $v_1, v_2, \ldots, v_n$ are generators of $F(\mathfrak{V})$, which implies that v is a linear combination of monomials each of which is a product of at most n generators of $F(\mathfrak{V})$. So any element of the fully invariant ideal generated by w_n is a linear combination of monomials each of which is a product of at most n generators of $F(\mathfrak{V})$. Now suppose that $\mathfrak{W}$ is finitely based. Then, for some n, w_n is in the fully invariant closure of $w_1, w_2, \ldots, w_{n-1}$. Hence w_n can be expressed in the form

$$\alpha_1 m_1 + \alpha_2 m_2 + \ldots \alpha_k m_k$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ and for some monomials $m_1, m_2, \ldots, m_k$, each of which is a product of at most $n-1$ generators of $F(\mathfrak{V})$. For $i = 1, 2, \ldots, n$ let δ_i be the endomorphism of $F(\mathfrak{V})$ which maps x_i to 0 and maps x_j to x_j for $j \neq i$. Then $w_n \delta_i = 0$ for $i = 1, 2, \ldots, n$. Also if $1 \leqslant j \leqslant k$ then m_j is a product of at most $n-1$ generators of $F(\mathfrak{V})$ and so $m_j \delta_i = m_j$ for some $i\,(1 \leqslant i \leqslant n)$. Let μ be the map $(1-\delta_1)(1-\delta_2)\ldots(1-\delta_n)$ from $F(\mathfrak{V})$ to $F(\mathfrak{V})$. (Note that although μ is not an endomorphism of $F(\mathfrak{V})$as an algebra, it is a linear transformation of $F(\mathfrak{V})$ as a vector space.) Then $w_n \mu = w_n$, and $m_j \mu = 0$ for $1 \leqslant j \leqslant k$. Hence

$$w_n = w_n \mu = \sum_{1 \leqslant j \leqslant k} \alpha_j (m_j \mu) = 0.$$

However, if A is the algebra defined above which generates $\mathfrak{V}$ and if θ is the homomorphism from $F(\mathfrak{V})$ to A which maps x_1 to a, x_2 to b, x_5 to d and x_j to c for $j \neq 1, 2, 5$ then

$$w_n\,\theta = abcd - abdc \neq 0.$$

This implies that $w_n \neq 0$, and so $\mathfrak{W}$ cannot be finitely based.

As we have proved above, $\mathfrak{V}$ contains finite algebras which do not have finite bases for their laws, but which generate varieties with only finitely many subvarieties. Here is an example of such an algebra. Let B be the non-associative algebra over F defined as follows.

B is generated by elements a, b, c.

As a vector space over F, B has basis $a, b, c, ab, ac, abc, acb, abcb$.

If x, y are members of this basis then $xy = 0$ unless $x \in \{a, ab, ac, abc, acb, abcb\}$ and $y = b$ or c. These products are given by the following table.

	b	c
a	ab	ac
ab	ab	abc
ac	acb	ac
abc	$abcb$	abc
acb	acb	$abc+acb-abcb$
$abcb$	$abcb$	abc

The following laws are a basis for the laws of the variety generated by B (as a subvariety of $\mathfrak{V}$).

$$(x_1x_2x_3 - x_1x_3x_2)x_6x_7\ldots x_nx_4x_5 - (x_1x_2x_3 - x_1x_3x_2)x_6x_7\ldots x_nx_5x_4 = 0 \quad \text{for } n = 6, 7, \ldots,$$

$$(x_1x_2x_3 - x_1x_3x_2)x_4x_5 - (x_1x_2x_3 - x_1x_3x_2)x_5x_4 = 0,$$

$$x_1x_2x_3x_4 + x_1x_3x_4x_2 + x_1x_4x_2x_3 - x_1x_2x_4x_3 - x_1x_4x_3x_2 - x_1x_3x_2x_4 = 0,$$

$$x_1x_2^qx_3^q - x_1x_2^qx_3 - x_1x_2x_3^q + x_1x_2x_3 = 0,$$

$$x_1x_2^qx_3x_4 - x_1x_2^qx_4x_3 - x_1x_2x_3x_4 + x_1x_2x_4x_3 = 0,$$

$$x_1x_2x_3x_4^q - x_1x_3x_2x_4^q - x_1x_2x_3x_4 + x_1x_3x_2x_4 = 0.$$

Every subvariety of the variety generated by B is determined by four variable laws as a subvariety, and this implies that the variety generated by B has only finitely many subvarieties.

To summarize, our collection of examples is as follows:

$\mathfrak{V}$ is a locally finite variety that satisfies Max but not Min.

$\mathfrak{V}$ is a variety generated by a finite algebra which has a finite basis for its laws, but contains infinitely many critical algebras.

Var(B), the variety generated by the finite algebra B, has an infinite basis for its laws, but contains only finitely many critical algebras.

Since all the algebras in $\mathfrak{B}$, being linear algebras, have modular (indeed, permutable) congruence lattices, B also provides a counter-example to conjecture 1. (As mentioned in the introduction this was inspired by an example of Polin (1976).)

4. Min but not Max?

Murskii (1965) proved that the three element groupoid M defined by the multiplication table shown below has an infinite basis for its laws. (Murskii treated it

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

as a groupoid with a single binary operation, but his results go through when it is regarded as a groupoid with zero; we shall regard it thus as this simplifies many of the calculations.)

4.1 THEOREM. Var(M) *contains an infinite ascending chain of critical algebras*

$$G_1 < G_2 < \ldots < G_n < \ldots.$$

PROOF. The G_n are of the form $G_n = \{0, g_1, \ldots, g_n\}$: $0g_i = g_i 0 = 0$, $g_i g_i = 0$, $g_i g_j = g_i$ $(i \neq j)$. These are constructed as follows:

For $n = 1$ the subgroupoid $\{0, 1\}$ of M satisfies these conditions.

For $n > 1$, let H_n be the direct product of n copies of M. Consider the equivalence relation ρ on H_n defined by $(a_1, \ldots, a_n)\,\rho(b_1, \ldots, b_n)$ if

$$\text{either} \quad \text{there exists } i, j \text{ such that } a_i = b_j = 0$$
$$\text{or} \quad a_i = b_i \ (i = 1, \ldots, n).$$

ρ is clearly a congruence relation. Let G_n be the subgroupoid of H_n/ρ whose elements are $0, g_1, \ldots, g_n$, where

$$0 = [(0, \ldots, 0)]_\rho, \quad g_i = [(2, \ldots, \underset{i}{1}, \ldots, 2)]_\rho.$$

Then

$$0g_i = g_i 0 = 0, \quad g_i g_i = [(2, \ldots, \underset{i}{0}, \ldots, 2)]_\rho = 0, \quad g_i g_j = [(2, \ldots, \underset{i}{1}, \ldots, 2)]_\rho = g_i,$$

so G_n has the required properties.

It remains to show that G_n is critical. First note that there are no proper non-trivial congruences on G_n; for, if $(0, g_i) \in \sigma$, then $(0, g_j) \in \sigma$, and if $(g_i, g_j) \in \sigma$ $(i \neq j)$

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then $(0, g_i) \in \sigma$. Thus it is sufficient to show that G_n does not belong to the variety generated by its proper subalgebras. Since these are precisely 0 and G_m for $m<n$, it is sufficient to construct a law $w(x_1, \ldots, x_n) = 0$ which holds in G_m for $m<n$, but not in G_n. We define w inductively as follows:

$$w(x_1, x_2) = x_2 x_1,$$
$$w(x_1, \ldots, x_n) = x_n(x_{n-1}(x_n(x_{n-2} \ldots (x_n(x_1(w(x_1, \ldots, x_{n-1})) \ldots).$$

Now in G_m any right-normed product will reduce to 0 if one of the entries is 0 or if two adjacent entries are equal. Since in any substitution of elements of G_m into $w(x_1, \ldots, x_n)$ for $m<n$ either some x_i is 0 or $x_i = x_j$ for some $i \neq j$, we have that $w(x_1, \ldots, x_n) = 0$ is a law in G_m for $m<n$. However, in G_n the substitution $x_i = g_i$ gives $w(x_1, \ldots, x_n) = g_n \neq 0$. Hence G_n is critical, as required. (Park (1976) shows that this variety contains infinitely many subdirectly irreducible algebras, these are precisely the H_n/ρ of the above proof.)

4.2 COROLLARY. Var(M) *does not satisfy Max.*

PROOF. If $\mathfrak{U}_n = \text{var}(G_n)$ then Theorem 4.1 shows that

$$\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset \ldots \subset \mathfrak{U}_n \subset \mathfrak{U}_{n+1} \subset \ldots$$

is an infinite ascending chain of subvarieties of var(M).

It remains, of course, to prove that var(M) does satisfy Min: at present the best we can do is leave that as a conjecture.

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