F. Gross, C.-C. Yang and C. Osgood
Nagoya Math. J.
Vol. 51 (1973), 123–130

PRIMEABLE ENTIRE FUNCTIONS

FRED GROSS[1], CHUNG-CHUN YANG[2]
AND CHARLES OSGOOD[2]

I. Introduction.

An entire function $F(z) = f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is meromorphic and $g(z)$ is entire (g may be meromorphic when f is rational). $F(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that one of the functions f and g is bilinear (a rational function). F is said to be E-prime (E-pseudo prime) if every factorization of the above form into entire factors implies that one of the functions f and g is linear (a polynomial). We recall here that an entire non-periodic function f is prime if and only if it is E-prime [5]. This fact will be useful in the sequel.

In this paper we consider the following question: Given an entire function $f(\not\equiv 0)$ can one find an entire function g such that $g(z)f(z)$ is prime? We shall call an entire function f primeable if and only if there exists an entire function g such that $g(z)f(z)$ is prime. We prove that given an entire function f there always exists an entire function g (of zero order) such that $g(z)f(z)$ does not possess any nonlinear polynomial factor. We then apply this result to show that certain classes of entire functions are primeable.

II. Preliminary Results.

We shall first prove some theorems which are interesting themselves.

THEOREM 1. *Given $F(z)$, an entire transcendental function, there exists an entire function $h(z)$ of order zero such that $h(z)F(z)$ can not*

Received May 26, 1972.
[1] Mathematics Research Center, Naval Research Laboratory, and University of Maryland, Baltimore County.
[2] Mathematics Research Center, Naval Research Laboratory.

be written as either $p(g)$ or $g(p)$, where p is a polynomial of degree larger than one.

As a preliminary we now show except for a countable set of $a \in C$, $z(z-a)F(z)$ may *not* be written as $g(p)$, where p is a polynomial of degree larger than one. Suppose the assertion is false. Without loss of generality we may assume that $p(0) = 0$ and that $p(z)$ is monic. Then $p(z) = z \prod_{j=1}^{n} (z - e_j)$ for some $n \geq 1$. Since $g(p(0)) = 0$ we see that the e_j are each either equal to a or equal to a zero of $zF(z)$. Then $p(z) = z(z-a)^{\alpha} \prod_{j=1}^{m} (z - f_j)$, for nonnegative integers α and m, where each f_j is a zero of $zF(z)$. Also, if a is not a root of $zF(z)$, α equals either 0 or 1, since if $\alpha > 1$ it would follow that $z = a$ is a multiple root of $z(z-a)F(z)$. There are at most countably many polynomials $q(z) = z \prod_{j=1}^{m} (z - f_j)$. Since we are assuming the existence of a noncountable number of values of a for which there exists an entire function $g(z)$ with $z(z-a)F(z) = g(p(z))$, one sees that there exist two distinct complex numbers a_1 and a_2, neither of which is a zero of $zF(z)$, such that for the *same* polynomial $q(z)$ and the same choice of α (equal to zero or one)

$$z(z-a_1)F(z) = g_1((z-a_1)^{\alpha} q(z))$$

and

$$z(z-a_2)F(z) = g_2((z-a_2)^{\alpha} q(z)) .$$

If above $\alpha = 0$ we shall show, shortly, that we have a contradiction. If we *never* have the above situation with $\alpha = 0$ we have instead that

$$z(z-a_j)F(z) = g_j((z-a_j)q(z))$$

for a *noncountable* set of different a_j's.

If $\alpha = 0$ above we have that

$$(z-a_1)(z-a_2)^{-1} = g_1(q(z))(g_2(q(z)))^{-1} ,$$

where $q(z) = z \prod_{j=1}^{m} (z - f_j)$. If $m > 0$ then, counting multiplicities, we have that should the right hand side above have any poles it has more than one. Since the left hand side has *exactly* one pole, $m = 0$ and we are through in this case.

If we can not assume the above case then consider each $z(z-a_j)F(z) = g_j(z(z-a_j)q(z))$. Since $g_j(0) = 0$ we see that

$$g_j(z(z-a_j)q(z)) = z(z-a_j)q(z)h_j(z(z-a_j)q(z)) .$$

Also $H(z) = F(z)(q(z))^{-1}$ must be an entire function. We have then that $H(z) = h_j(z(z - a_j)q(z))$. Let $z_j = z_j(w)$ denote *any* solution to the equation

$$z(z - a_j)q(z) = w .$$

Then each $H(z_j(w)) = h_j(w)$ is entire. Now $z_j'(w)$ must become infinite when $z_j(w)$ equals a zero of $(z(z - a_j)q(z))'$. Thus, at each zero of $(z(z - a_j)q(z))'$, $H'(z)$ must be zero. Given any such zero z_j we see that

$$a_j = (2z_jq(z_j) + z_j^2q'(z_j))(q(z_j) + z_jq'(z_j))^{-1} ,$$

unless the denominator vanishes. If there always exists a z_j, say z_j^*, which leaves the denominator nonzero then we may define a one to one mapping from a_j to z_j^*. It would follow that $H'(z)$ equals zero on a noncountable set, so $H(z)$ is a constant and $F(z)$ is a polynomial contrary to assumption. This would conclude our present proof. If the denominator vanishes then the denominator, $2z_jq(z_j) + z_j^2q'(z_j)$, must vanish *also*. Thus, $z_jq(z_j)$ vanishes. If $z_j = 0$ is put in the equation $(z(z - a_j)q(z))' = 0$ we see that $q(0) = 0$. Therefore z_j must be a zero of $q(z) = 0$. This equation is of degree one less than $(z(z - a_j)q(z))' = 0$; hence, there exists a point z_j^*. This proves that $z(z - a)F(z) = g(p)$ is possible for at worst a countable number of different values of a.

What we wish to do next is to construct a transcendental function $h_1(z)$ of order zero such that $h_1(z)f(z)$ can not be written as $p(g)$ where p is a polynomial of degree larger than 1. We shall see that our $h_1(z)$ is such that the same assertion holds for each $(z - a)zh_1(z)F(z)$; therefore, we may choose a such that $(z - a)zh_1(z)F(z)$ may not be written as $g(p)$ either.

We wish to rely in what follows on the following result essentially due to Borel [2].

LEMMA. *Suppose that $A(r) > 0$ is monotone increasing and $B(r) > 0$ is monotone decreasing for all $r \geq r_0$, $A(r)$ and $B(r)$ are continuous, and that $\int_{r_0}^{\infty} B(r)dr < \infty$. Then $A(r + B(A(r))) < A(r) + 1$ except for, at most, a set of finite measure.*

Let $M(r) = \max_{|z| \leq r} |F(z)|$. Then $M(r) > 0$ and is monotone increasing if $r \geq 1$. Set $n(r) = (\log (M(r)))^{2/3}$. Set $f(r) = \log (M(r + 1/n(r)))$. Then we have

LEMMA 1. *For any positive constant N, $M(r)/e^{n(r)}f(r)r > N$ for all*

r except, at most, a set of finite measure.

Proof. In the lemma of Borel's set $A(r) = (\log M(r))^{1/3}$ and $B(r) = r^{-2}$. Then

$$f(r)^{1/3} = (\log (M(r + (n(r))^{-1})))^{1/3} < (1 + \log (M(r)))^{1/3}, \quad \text{so}$$

$f(r) < 2 \log (M(r))$, except on a set of finite measure. The result now follows trivially, since $M(r) \to \infty$, as $r \to \infty$, faster than r^2 say.

LEMMA 2. *Except for a set of finite measure*

$$M\left(r + \frac{1}{f(r)}\right) < e^{n(r)} M(r) .$$

Proof. This is equivalent to $\log (M(r + 1/f(r))) < n(r) + \log (M(r))$ except for a set of finite measure. Since $f(r) > \log (M(r))$ this would be implied by

$$\log (M(r + (\log (M(r)))^{-1})) < n(r) + \log (M(r)) ,$$

except for a set of finite measure. Let us drop references to the exceptional set in what follows. Setting $A(r) = (\log (M(r)))^{1/2}$ and $B(r) = r^{-2}$ we see by the Borel's Lemma that $\log (M(r + (\log (M(r)))^{-1})) < ((\log (M(r)))^{1/2} + 1)^2$. Now $\log M(r) + 2(\log M(r))^{1/2} + 1 < n(r) + \log (M(r))$. This proves Lemma 2.

LEMMA 3. *Given $F(z)$ as above and any positive integer $N > 0$, we may choose a point a_1, with $|a_1| = r$, such that $|F(z)| > N$ and $|(1 - za_1^{-1})F(z)| > N$ if $(16e^{n(r)} f(r))^{-1} < |z - a_1| < (8e^{n(r)} f(r))^{-1}$.*

Proof: Let $r > 0$ be chosen so that the inequalities of Lemmas 1 and 2 are both satisfied (with $2^5 N$ replacing N in Lemma 1). Let c_0, c_1, and c_2 be circles with center at a_1 of respective radii, $(8e^{n(r)} f(r))^{-1}$, $(2f(r))^{-1}$, and $(f(r))^{-1}$. Now if w is inside of c_1 we see that

$$|F'(w)| = \left|(2\pi i)^{-1} \int_{c_2} F(z)(z - w)^{-2} dz\right| \le 4f(r)e^{n(r)} M(r) .$$

Thus if w is inside of c_0 we see that

$$|F(w)| \ge M(r) - (8e^{n(r)} f(r))^{-1}(4f(r)e^{n(r)} M(r)) \ge \tfrac{1}{2} M(r) > N .$$

Then if

$$(16e^{n(r)} f(r))^{-1} < |z - a_1| < (8e^{n(r)} f(r))^{-1}$$

we have that

$$\left|\left(1-\frac{z}{a_1}\right)F(z)\right| \geq \frac{|z-a_1|}{|2a_1|}M(|z|) \geq \frac{M(r)}{2^5 r f(r) e^{n(r)}} > N$$

by Lemma 2. This proves Lemma 3.

LEMMA 4. *Let c be a circular path which winds once about $z = a_1$ in the positive direction and has radius $3.2^{-5}(e^{n(r)}f(r))^{-1}$. Then, under the conditions of Lemma 3, where $H = (1 - z/a_1)F(z)$ we see that $\Gamma = H(c)$ winds once about zero in the positive direction and always stays outside of $|w| < N$.*

Proof. The second assertion is trivial. To see the first statement look at

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} &= \frac{1}{2\pi i}\int_c \frac{H'}{H}dz = \frac{1}{2\pi i}\int_c \frac{d}{dz}\left\{\ln F(z) + \ln\left(1-\frac{z}{a_1}\right)\right\}dz \\ &= \frac{1}{2\pi i}\int_c \frac{d}{dz}\left\{\ln\left(1-\frac{z}{a_1}\right)\right\}dz = 1\end{aligned}$$

(since $|F(z)| \geq N$ on and inside of c). This proves Lemma 4.

LEMMA 5. *If $H_1 = E(z)H(z)$ where $|E(z)| > \frac{1}{2}$ on and inside of c then Lemma 4 holds with H_1 substituted for H and $\frac{1}{2}N$ substituted for N.*

Proof. Trivial.

Now let us prove our Theorem. Choose two sequences of positive integers $b_1, \cdots,$ and $c_1, \cdots$ such that $\prod_{j=1}^{\infty}(1 - z/b_j)$ has zero order of growth and $\prod_{j=1}^{\infty}(1 - 1/c_j) > \frac{1}{2}$. Pick a_1 as in Lemma 3 for $F_1(z) = F(z)$ and $N = 1$, subject to the conditions that $|a_1| > b_1$ and the radius of c is less than one. Proceeding by induction apply Lemma 3 with our $M(r)$ being for the function

$$F_n(z) = \prod_{j=1}^{n-1}\left(1-\frac{z}{a_j}\right)F(z) \quad \text{and with} \quad N = n\,,$$

subject to the conditions that

$$|a_n| > b_n, \quad \text{each} \quad \left|1 - \frac{|a_j|+1}{|a_n|}\right| > \left(1 - \frac{1}{c_{n-j}}\right)$$

for every $1 \leq j \leq n-1$, and the curve c has radius less than one.

https://doi.org/10.1017/S0027763000015750 Published online by Cambridge University Press

Now set $h_1(z) = \prod_{j=1}^{\infty}(1 - z/a_j)$. We see that $h(z)$ has order zero. Also each

$$|h_1(z)F(z)/F_{n+1}(z)| > \tfrac{1}{2} ,$$

if z is on or inside of the curve c about a_n. Thus by Lemma 5 each curve $\Gamma_j = h_1(c)F(c)$, where c is about a_j, winds once about $w = 0$ in the positive direction and lies entirely inside of $\{w \,|\, |w| \geq j/2\}$.

Suppose now that

$$h_1(z)F(z) = P(g(z))$$

where P is a polynomial of degree $n > 1$. Then for all sufficiently large j we have that on the different curves c,

$$g(z) = \rho^k(h_1(z)F(z))^{1/n} + K + o(|h_1(z)F(z)|^{-1/n})$$

for some complex constant K and some positive integer k, where ρ is a primitive n-th root of unity. Continuing $g(z)$ about c we obtain

$$\rho^{k+1}(h_1(z)F(z))^{1/n} + K + o(|h_1(z)F(z)|^{-1/n}) .$$

If j is chosen sufficiently large we see that the two values must be distinct. Thus $g(z)$ is not entire. This proves that $h_1(z)F(z) = p(g)$ is impossible and completes the proof of the Theorem.

THEOREM 2. *One may construct* $h(z)$ *as in Theorem* 1 *with all of the zeros of* $h(z)$ *lying asymptotically on a ray* $\arg z = \theta_0$, *unless for each positive integer* n *there exists an open sector containing* $\arg z = \theta_0$ *on which* $z^n F(z)$ *is bounded.*

Proof. Using $z^{n+2}F(z)$ for $F(z)$, where $z^nF(z)$ is not bounded on any open sector containing $\arg z = \theta_0$, we may go through the above proof using in the n-th step $M(r) \overset{def.}{\equiv} \max_{\theta} \{|F_n(re^{i\theta})| \,|\, \theta_0 - 1/n < \theta < \theta_0 + 1/n\}$ instead of $M(r)$ for $F_n(z)$. The only properties of $M(r)$ which were needed in the proofs of Lemmas 1 and 2 were monotonicity and (if r was sufficiently large) that $M(r) \geq r^2$. This proves Theorem 2.

III. Primeable Functions and Main Results.

THEOREM 3. *Let* f *be a transcendental entire function with* $\delta(0, f - h) = 1$ *for some entire function* $h(z)$ *of order less than that of* f*'s. Then* f *is primeable.*

https://doi.org/10.1017/S0027763000015750 Published online by Cambridge University Press

EXAMPLE. Let p_1, p_2, p_3 be polynomials with $p_2 \not\equiv$ constant. $p_1 \not\equiv 0$. Then $p_1 e^{p_2} + p_3$ is primeable.

Proof of the Theorem. First of all, we note that under the hypotheses of the theorem f is pseudo-prime. This is an extension of a result of Goldstein's [4], and its proof can be obtained by adopting the argument used in [6]. Thus, if $\delta(0, f-h) = 1$ for some entire function h of order less than that of f, we have, for any entire function $g(\not\equiv 0)$ of zero order, $\delta(0, gf - gh) = 1$. Therefore, gf is also pseudo-prime. Now choose g as in Theorem 1 such that gf is not a periodic function and can not be expressed as $k(q)$ or $q(k)$ where k is transcendental and q is a polynomial of degree at least two. Thus we have shown gf is E-prime and therefore is prime.

Along similar lines we have the following result.

THEOREM 4. *Let p be a nonconstant polynomial, and f be an entire function of finite order with $\delta(0, f(z) - a(z)) = 1$ for some entire function $a(z)$ of zero order. Then $p(f)$ is primeable.*

THEOREM 5. *Let F be an entire function of order ρ. Assume that ρ is finite and $\neq \frac{1}{2}$. Suppose there exists a real number α such that for any $\delta > 0$, all but a finite number of zeros of F lie in the angle $|\arg z - \alpha| < \delta$, then F is primeable.*

Proof. We deal the case $p > \frac{1}{2}$ first. According to a result of Baker's [1] F is pseudo-prime and so is $q(z)F(z)$ for any polynomial $q(z)(\not\equiv 0)$. Now as shown in the previous section that one can always find a $a \in C$ such that $F(a) \neq 0$, $a \neq 0$ and $z(z-a)F(z)$ may not be written as $g(p)$ where p is a polynomial of degree larger than one. Now suppose that $z(z-a)F(z) = p(g)$ and $P(z)$ has two distinct roots. Then the order of g is at most $1/2$ (for the proof we refer the reader to [3]). Therefore the order of F will be $\leq \frac{1}{2}$, giving a contradiction. Then p has to be a monomial. But a is a simple zero of $z(z-a)F(z)$ and hence p has to be a linear function. It follows that $z(z-a)F(z)$ is prime.

The case when $p < \frac{1}{2}$ can be proved by combining Theorem 2 and the preceding argument. Thus the theorem is proved.

References

[1] I. N. Baker, "The value distribution of composite entire functions", Acta Math. Szeged, Tom **32**, (1971).

[2] E. Borel, "Sur les zéros des fonctions entières", Acta Math., **20**, (1897) 357–396.

[3] A. Edrei, "Meromorphic functions with three radically distributed values", Trans. Amer. Math. Soc., **78**, (1955) 271–293.

[4] R. Goldstein, "On factorization of certain entire functions", J. Lond. Math. Soc., (**2**), (1970) pp. 221–224.

[5] F. Gross, "Factorization of entire functions which are periodic mod g", Indian J. of pure and applied Math., Vol. **2**, No. **3**, (1971).

[6] F. Gross and C. C. Yang, "The fix-points and factorization of meromorphic functions", Trans. Amer. Math. Soc., Vol. **168**, (1972).