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Minimal Separators

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Abstract. A separator of a connected graph G is a set of vertices whose removal disconnects G. In this paper we give various conditions for a separator to contain a minimal one. In particular we prove that every separator of a connected graph that has no thick end, or which is of bounded degree, contains a minimal separator.

Introduction

A set S of vertices of a graph G is a separator of G if $G - S$ has at least two components. Obviously every finite separator contains a minimal one; the case is different, however, with infinite separators. In [2] Sabidussi proved that a separator (*isthmoid* in [2]) contains a minimal one if it contains a separator S such that $G - S$ has only finitely many components and S is equal to its boundary with one of them. In this paper we continue this study by characterizing those separators which contain a minimal one, and by showing in particular that if a separator S of a graph G contains no minimal separator, then S has an infinite intersection with some ray which belongs to a thick end of G. An immediate consequence of this result is that, if a graph G has no thick end, thus *a fortiori* if G is rayless, then every separator of G contains a minimal one.

1 Preliminaries

The graphs we consider are undirected, without loops and multiple edges. If $x \in V(G)$, the set $V(x; G) := \{y \in V(G) : \{x, y\} \in E(G)\}$ is the *neighborhood* of x, and its cardinality $d(x; G)$ is the *degree* of x. A graph is *locally finite* if all its vertices have finite degrees. For $A \subseteq V(G)$ we denote by $G[A]$ the subgraph of G induced by A, and we set $G - A := G[V(G) - A]$. The *union* of a family $(G_i)_{i \in I}$ of graphs is the graph $\bigcup_{i \in I} G_i$ given by $V(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} V(G_i)$ and $E(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} E(G_i)$. The *intersection* is defined similarly. If $(G_i)_{i \in I}$ is a family of subgraphs of a graph G, the subgraph of G induced by the union of this family will be denoted by $\bigvee_{i \in I} G_i$.

A *path* $P = \langle x_0, \dots, x_n \rangle$ is a graph with $V(P) = \{x_0, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{\{x_i, x_{i+1}\} : 0 \leq i < n\}$. A *ray* or *one-way infinite path* $\langle x_0, x_1, \dots \rangle$, and a *double ray* or *two-way infinite path* $\langle \dots, x_{-1}, x_0, x_1, \dots \rangle$ are defined similarly. A graph is *rayless* it if contains no ray. A path $P := \langle x_0, \dots, x_n \rangle$ is called an (x_0, x_n)-*path*, x_0 and x_n are its *endpoints*, while the other vertices are called its *internal* vertices. For $A, B \subseteq V(G)$, an (A, B)-*path* of G is an (x, y)-path of G such that $V(P) \cap A = \{x\}$ and $V(P) \cap B = \{y\}$; an (A, B)-*linkage* of G is a set of pairwise disjoint (A, B)-paths of G. For $x \in V(G)$ and $A \subseteq V(G)$, an (x, A)-*linkage* of G is a set of $(\{x\}, A)$-paths of G which have pairwise only

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x in common. If there exists an infinite (A, B)-linkage (resp. (x, A)-linkage) in G, then we say that A and B (resp. x and A) are *infinitely linked in G*.

2 Separators

2.1

The set of components of a graph G is denoted by $\mathcal{C}_G$, and if x is a vertex of G, then $\mathcal{C}_G(x)$ is the component of G containing x. If S is a subset of $V(G)$ and X a subgraph of $G - S$, the *boundary of S with X in G* is the set $\mathcal{B}_G(S, X) := \{x \in S : V(x; G) \cap V(X) \neq \varnothing\}$. The set $\mathcal{B}_G(S) := \mathcal{B}_G(S, G - S)$ is the *boundary* of S in G. Finally we define

$$\mathbb{F}_G(S) := \{\mathcal{B}_G(S, X) : X \in \mathcal{C}_{G-S}\}.$$

If no confusion is likely, then we will write $\mathcal{B}(S, X)$, $\mathcal{B}(S)$ and $\mathbb{F}(S)$ for $\mathcal{B}_G(S, X)$, $\mathcal{B}_G(S)$ and $\mathbb{F}_G(S)$, respectively.

2.2

A subset S of $V(G)$ is a *separator* of G if $|\mathcal{C}_{G-S}| \geq 2$. A separator S of G is *minimal* if no proper subset of S is a separator of G. Clearly a separator S is minimal if and only if $\mathbb{F}(S) = \{S\}$; moreover we then have $\mathcal{B}(S) = S$.

Not every infinite separator contains a minimal one. We will recall two classic examples in order to illustrate our main results.

Example 2.3 Let $R = \langle 0, 1, \dots \rangle$ be a ray, and let $(R_n)_{n\in\mathbb{N}}$ be a family of pairwise disjoint rays which are disjoint from R and such that $R_n = \langle x_0^n, x_1^n, \dots \rangle$ for every non-negative integer n. Finally let $G := R \cup \bigcup_{n\in\mathbb{N}} (R_n \cup \bigcup_{p\in\mathbb{N}} \langle x_p^n, n + p\rangle)$. Then the set $\mathbb{N} = V(R)$ is a separator of G, but it contains no minimal separator since, for every non-negative integer n, the set $\{p : p \geq n\}$ is also a separator of G.

***Example 2.4* (Sabidussi [2, Example 1])** *Let $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$ be two disjoint countable sets. Define a graph G by $V(G) = A \cup B$ and $E(G) = \{\{a_i, b_j\} : i \geq j\}$. Then the set A is a separator of G, but it contains no minimal separator since, for every non-negative integer n, the set $\{a_i : i \geq n\}$ is also a separator of G.*

2.5

There exist other graphs which contain no minimal separator. The following example is due to Sabidussi. For every $i \in \mathbb{N}$, let $A_i = (a_n^i)_{n\in\mathbb{N}}$ and $B_i = (b_n^i)_{n\in\mathbb{N}}$ be two sequences of pairwise distinct elements such that $A_i \neq B_j$ for every $i, j \in \mathbb{N}$. Put $A := \bigcup_{i\in\mathbb{N}} A_i$ and $B := \bigcup_{i\in\mathbb{N}} B_i$. Define a graph G by $V(G) = A \cup B$, and $E(G) = \{\{a_h^i, b_k^j\} : i \geq j \text{ and } h \leq k\}$. One can prove that no separator of G contains a minimal separator; in fact any separator of G contains the neighborhood of some vertex.

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2.6

Let $B \subseteq A \subseteq V(G)$. We denote by ϕ_{AB} the function from $\mathcal{C}_{G-A}$ into $\mathcal{C}_{G-B}$ which maps every component X of $G - A$ to the unique component of $G - B$ containing X. To ϕ_{AB} is associated the map $f_{AB}\colon \mathbb{F}(A) \to \mathbb{F}(B)$ such that $f_{AB}\big(\mathcal{B}(A, X)\big) = \mathcal{B}\big(B, \phi_{AB}(X)\big)$ for every $X \in \mathcal{C}_{G-A}$. By Sabidussi [2, Lemma 3] ϕ_{AB}, and thus f_{AB}, are onto if $\mathcal{B}(A) = A$.

Lemma 2.7 *If S is a separator of G, then every element A of $\mathbb{F}(S)$ is a separator of G such that $\mathcal{B}(A) = A$, and which satisfies the inequality $|\mathbb{F}(A)| \leq |\mathbb{F}(S)|$ whenever $\mathcal{B}(S) = S$.*

Proof Since $A \in \mathbb{F}(S)$, $A = \mathcal{B}(S, X)$ for some $X \in \mathcal{C}_{G-S}$. Due to the fact that S is a separator of G, there exists a $Y \in \mathcal{C}_{G-S}$ such that $Y \neq X$. Let $x \in V(X)$ and $y \in V(Y)$. Then every (x, y)-path of G meets S, and thus A since $A \in \mathcal{B}(S, X)$. This proves that X and $\phi_{SA}(Y)$ are distinct components of $G - A$, and hence that A is a separator of G.

Furthermore $\mathcal{B}(A) = \mathcal{B}\big(\mathcal{B}(S, X)\big) = \mathcal{B}(S, X) = A$, and if $\mathcal{B}(S) = S$, then, by 2.6, f_{SA} is onto, thus $|\mathbb{F}(A)| \leq |\mathbb{F}(S)|$. ■

Remark 2.8 Note that a separator S can contain a minimal separator, while no element of $\mathbb{F}(S)$ contains a minimal separator, as is shown by the following example. Take the graph G defined in Example 2.4. Let x be a new vertex, and let $H := G \cup \langle b_0, x\rangle$. Then $A' := A \cup \{b_0, x\}$ is a separator of H such that $\mathbb{F}(A') = \{\{a_i : i \geq n\} : n \geq 1\}$. Furthermore $\{b_0\}$ is the only minimal separator of H contained in A'.

Lemma 2.9 **(Sabidussi [2, Theorem 1])** *Let S be a separator of a connected graph G. If S contains a separator S_0 such that $\mathcal{B}(S_0) = S_0$ and $\mathcal{C}_{G-S_0}$ is finite, then S contains a minimal separator.*

We get the following result immediately which we will generalize later (Theorem 4.8).

Corollary 2.10 *Let G be a locally finite connected graph. Then a separator S of G contains a minimal separator if and only if S contains a separator S_0 such that $\mathcal{B}(S_0) = S_0$ and $\mathcal{C}_{G-S_0}$ is finite.*

Theorem 2.11 *Let S be a separator of a connected graph G. The following statements are equivalent:*

(i) S contains a minimal separator;
(ii) S contains a separator S_0 such that $\mathcal{B}(S_0) = S_0$ and such that, for every $x \in S_0$, there are only finitely many elements of $\mathbb{F}(S_0)$ which do not contain x;
(iii) S contains a separator S_0 such that $\mathcal{B}(S_0) = S_0$ and $\mathbb{F}(S_0)$ is finite.

Proof (i) ⇒ (ii). This is obvious since, if S_0 is a minimal separator that is contained in S, then $\mathcal{B}(S_0) = S_0$ and $\mathbb{F}(S_0) = \{S_0\}$.

(ii) ⇒ (iii). Let $S' \subseteq S$ be a separator of G satisfying the properties of (ii). We are done if S' is a minimal separator. Suppose that S' is not minimal. Then there exist $x \in S'$ and $S_0 \in \mathbb{F}(S')$ such that $x \notin S_0$. Put $\mathbb{F}_x := \{F \in \mathbb{F}(S') : x \notin F\}$. For every $F \in \mathbb{F}(S') - \mathbb{F}_x$,

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$f_{S'S_0}(F) = \mathcal{B}\big(S_0, \mathcal{C}_{G-S_0}(x)\big)$. Hence $\mathbb{F}(S_0) = \{\mathcal{B}\big(S_0, \mathcal{C}_{G-S_0}(x)\big)\} \cup \{f_{S'S_0}(F) : F \in \mathbb{F}_x\}$. Therefore, as $\mathbb{F}_x$ is finite by (ii), and as $f_{S'S_0}$ is onto by 2.6, $\mathbb{F}(S_0)$ is also finite.

(iii) $\Rightarrow$ (i). Let $S_0 \subseteq S$ be a separator of G satisfying the properties of (iii). For each $S' \in \mathbb{F}(S_0)$ let $X(S')$ be a component of $G - S_0$, such that $S' = \mathcal{B}\big(S_0, X(S')\big)$. Then S_0 is a separator of the connected graph $H := G[S_0] \vee \bigcup_{S' \in \mathbb{F}(S_0)} X(S')$ such that $\mathcal{B}_H(S_0) = S_0$ and $H - S_0$ has only finitely many components. By Lemma 2.9, S_0 contains a minimal separator S_1 of H. If X and X' are components of $G - S_0$ such that $\mathcal{B}(S_0, X) = \mathcal{B}(S_0, X')$, then clearly $\mathcal{B}(S_1, X) = \mathcal{B}(S_1, X')$. Hence S_1 is also a separator of G such that $\mathbb{F}_G(S_1) = S_1$, thus which is minimal. ■

3 Ends of a Graph

For the next results we need the concept of an end.

3.1

The *ends* of a graph G are the classes of the equivalence relation $\sim_G$ defined on the set of all rays of G by: $R \sim_G R'$ if and only if there is a ray R'' whose intersections with R and R' are infinite. We will denote by $[R]_G$ the end of G containing the ray R.

A vertex x is said to *dominate* an end τ if x is infinitely linked to the vertex set of some (hence every) ray in τ. If there exists an infinite set of pairwise disjoint rays in τ, then τ is said to be *thick*; otherwise it is said to be *thin*. By [1, Proposition 2.13], an end which is dominated by infinitely many vertices is thick.

3.2

An infinite subset A of $V(G)$ is *concentrated* (in G) if there exists an end τ such that $A - V\big(\mathcal{C}_{G-S}(\tau)\big)$ (where $\mathcal{C}_{G-S}(\tau)$ is the component of $G - S$ that contains a ray belonging to τ) is finite for every finite $S \subseteq V(G)$ (A is said to be "*concentrated in* τ").

Clearly, if A is concentrated in τ, then any vertex that dominates τ is infinitely linked to A.

3.3

A set A of vertices of a graph G is *fragmented* (in G) if its elements are pairwise separated in G by a finite $S \subseteq V(G)$, *i.e.*, $\mathcal{C}_{G-S}(X) \neq \mathcal{C}_{G-S}(y)$ for every pair $\{x, y\}$ of elements of $A - S$.

In particular any finite set of vertices is fragmented, and every subset of a fragmented set is fragmented. Furthermore, if an infinite set A is fragmented in G, then there exists a vertex of G that is infinitely linked to A in G.

4 Main Results

Theorem 4.1 *If a separator S of a connected graph G contains no minimal separator, then there exists a ray whose intersection with S is infinite, and which belongs to a thick end of G.*

Proof (a) Construct a sequence $S_0, S_1, \ldots$ of subsets of S such that S_n is a separator of G,

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$\mathcal{B}(S_n) = S_n$ and $S_{n+1} \in \mathbb{F}(S_n) - \{S_n\}$. Let S_0 be some element of $\mathbb{F}(S)$. By Lemma 2.7, S_0 is a separator of G such that $\mathcal{B}(S_0) = S_0$. Suppose that $S_0, \dots, S_n$ have already been constructed. Since S_n is a separator of G and is contained in S, it contains no minimal separator. Then there exists an element S_{n+1} of $\mathbb{F}(S_n)$ which is different from S_n.

Now, for every non-negative integer n, let $a_n \in S_n - S_{n+1}$, and let $X_n \in \mathcal{C}_{G-S_{n-1}}$ (with $S_{-1} := S$) be such that $\mathcal{B}(S_{n-1}, X_n) = S_n$. The graphs X_n are clearly pairwise disjoint since, for $n < p$, $\mathcal{B}(S_{p-1}, X_n) = S_{p-1} \neq S_p = \mathcal{B}(S_{p-1}, X_p)$. Finally, for every non-negative integer n, let P_n be an (a_n, a_{n+1})-path of G such that $P_n - \{a_n, a_{n+1}\} \subseteq X_n$. Then $R := \bigcup_{n\in\mathbb{N}} P_n$ is a ray of G since the X_n are pairwise disjoint.

(b) Denote by τ the end of G which contains R, and put $A := \{a_n : n \in \mathbb{N}\}$ and $A_n := A \cap S_n$. For each n, A_n is infinite, and thus is concentrated in τ. Hence, since $S_n = \mathcal{B}(S_n, X_n)$, we have two cases:

- There exists a vertex b_n of X_n which is infinitely linked to A_n in G, thus which dominates τ.
- No vertex of X_n is infinitely linked to A_n in G. Then there exists an infinite $\big(A_n, V(X_n)\big)$-linkage L. Denote by B_n the set of endpoints in X_n of all elements of L. The set B_n is then concentrated in τ in the graph G. Furthermore, no vertex of X_n is infinitely linked to B_n; otherwise it would be infinitely linked to A_n. Hence B_n contains no infinite subset which is fragmented in X_n; thus, by [1, Theorem 3.8], B_n contains an infinite subset C_n which is concentrated in X_n. Therefore, there exists a ray R_n of X_n such that C_n is concentrated in $[R_n]_{X_n}$. This proves that $R_n \in \tau$ since B_n, thus C_n, are concentrated in τ in the graph G.

Consequently, since the graphs X_n are pairwise disjoint, we get a set of vertices that dominate τ, and a set of rays that belong to τ; and at least one of these sets is infinite. In either case, this means that the end τ is thick. ∎

Remark 4.2 The necessary condition for the non-existence of a minimal separator, given in Theorem 4.1, is manifest in Examples 2.3 and 2.4.

In Example 2.3, G is a locally finite graph whose only end is thick, since $(R_n)_{n\in\mathbb{N}}$ is a family of pairwise disjoint elements of this end. Furthermore the separator $\mathbb{N}$ is the vertex set of the ray R which also belongs to this end.

In Example 2.4, G is a bipartite graph whose only end is thick since it is dominated by all elements of the infinite set B. Furthermore the vertex set of every ray of G contains an infinite subset of the separator A.

The following result is an immediate consequence of Theorem 4.1.

Theorem 4.3 *Let G be a connected graph that has only thin ends. Then every separator of G contains a minimal separator.*

Corollary 4.4 *Every separator of a rayless connected graph contains a minimal separator.*

Remark 4.5 The converse of Theorem 4.3 does not hold, that is, a graph may have a thick end, while each of its separators contain a minimal separator. In fact consider any graph H, and let $(y_x)_{x\in V(H)}$ be a family of pairwise distinct new vertices. Put $G :=$

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$H \cup \bigcup_{x \in V(H)} \langle x, y_x \rangle$. Then any separator of G must contain some vertex of H, and moreover every vertex of H is a separator of G. Hence every separator of G contains a minimal separator. So we get a counterexample G to the converse of Theorem 4.3 by taking for H any graph which has a thick end, for example, any infinite complete graph.

4.6

We will say that a set A of vertices of a graph G is *end-free* (resp. *thick-end-free*) if the intersection of A with any ray (resp. any ray belonging to a thick end) is finite.

In particular every dispersed set (*i.e.*, a set containing no concentrated subset), thus every fragmented set, and *a fortiori* every finite set, is end-free; and every subset of a (thick-) end-free set is (thick-)end-free. By Theorem 4.1, *every thick-end-free separator contains a minimal one.*

Theorem 4.7 *Let p be a positive integer, and let G be a connected graph whose set of vertices of degree $\geq p$ is thick-end-free. Then every separator of G contains a minimal one.*

Proof Assume that G has a separator S that contains no minimal separator. Construct, as in the proof of Theorem 4.1, a sequence $S_0, S_1, \ldots$ of subsets of S such that S_n is a separator of G with $\mathcal{B}(S_n) = S_n$ and $S_{n+1} \in \mathbb{F}(S_n) - \{S_n\}$. Further, as in the same proof, let $X_n \in \mathcal{C}_{G-S_{n-1}}$ (with $S_{-1} := S$) be such that $\mathcal{B}(S_{n-1}, X_n) = S_n$.

Let $i \leq j$. Every vertex x of S_j is adjacent to some vertex of X_i, since $S_j \subseteq S_i$. Hence $d(x; G) \geq j$ inasmuch as the X_n are pairwise disjoint. Therefore, in particular, $d(x; G) \geq p$ for every $x \in S_p$. Thus S_p is thick-end-free by hypothesis. Hence S_p contains a minimal separator of G by Theorem 4.1, contrary to the assumption. ■

This result settles the case of graphs of bounded degree. The more general class of graphs where only the set of vertices of infinite degrees is thick-end-free is a particular case of the following weaker result which generalizes Corollary 2.10 about locally finite graphs.

Theorem 4.8 *Let G be a connected graph such that each of its end is dominated by at most finitely many vertices. The following statements are equivalent:*

- *(i)* *S contains a minimal separator;*
- *(ii)* *S contains a separator S_0 with $\mathcal{B}(S_0) = S_0$ and which is end-free or such that $\mathcal{C}_{G-S_0}$ is finite;*
- *(iii)* *S contains a separator S_0 with $\mathcal{B}(S_0) = S_0$ and which is thick-end-free or such that $\mathcal{C}_{G-S_0}$ is finite.*

Proof (i) $\Rightarrow$ (ii). Let $S_0 \subseteq S$ be a minimal separator of G. Then $\mathbb{F}(S_0) = \{S_0\}$. Suppose that $\mathcal{C}_{G-S_0}$ is infinite. Then every element of S_0 has infinite degree. Assume that S_0 is not end-free. Thus there exists a ray R such that $A := S_0 \cap V(R)$ is infinite. Let $\tau := [R]_G$. Using arguments similar to those in part (b) of the proof of Theorem 4.1, we can prove that we have the two following cases:

- There exist infinitely many components of $G - S_0$ which contain a vertex infinitely linked to A. Thus, each of these vertices dominates τ, contrary to the properties of G.

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- There exist infinitely many components of $G - S_0$ which contain a ray belonging to the end τ. In this case, every element of A, being adjacent to some vertex of each of these components, cannot be separated from all these rays by the removal of a finite set of vertices. Hence every element of A dominates τ, once again contrary to the component of G since A is infinite by assumption.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) is a consequence of Theorem 4.1 if S_0 is thick-end-free and of Lemma 2.9 if $\mathcal{C}_{G-S_0}$ is finite. ■

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