Dear Editor,

Asymptotic sample extremes are not functions of an extremal process — a counterexample

Let $X_1, X_2, \cdots$ be independent, identically distributed (i.i.d.) random variables and let $X_{n:1} \geqq \cdots \geqq X_{n:n}$ denote the order statistics of the sample $X_1, \cdots, X_n$. Suppose that for suitable norming constants $a_n > 0$ and b_n

$$P\{(X_{n:1} - b_n)/a_n \leqq x\} \to G(x) \qquad \text{as } n \to \infty,$$

for some non-degenerate distribution function G. By a result due to Dwass (1966), for each $k \geqq 1$, as $n \to \infty$

$$((X_{n:1} - b_n)/a_n, \cdots, (X_{n:k} - b_n)/a_n) \xrightarrow{\mathscr{D}} (\xi_1, \cdots, \xi_k),$$

where $(\xi_1, \cdots, \xi_k)$ is known as the k-dimensional extremal variate. A related result holds for $Y_n(t) = (X_{[nt]:1} - b_n)/a_n$ $(t \geqq 1/n)$, which converges in distribution to a stochastic process $\{Y(t): t > 0\}$, known as the extremal process (Dwass (1964), Resnick and Rubinovitch (1973)). Let α_1 be the time of the last jump of Y in $(0, 1]$, α_2 the time of the next to last jump and so on. In a paper which appeared in this journal, Hall (1978) remarked that

$$\{Y(\alpha_k)\} \stackrel{\mathscr{D}}{=} \{\xi_k\}. \tag{1}$$

This remark seems to be incorrect on an intuitive basis for the following argument. Consider the order statistics $X_{n:1}, \cdots, X_{n:n}$. As n increases, any increase in $X_{n:1}$ implies an increase in $X_{n:i}$ $(i \geqq 2)$ (in fact, if $X_{n+1} > X_{n:1}$ then $X_{n+1:i} = X_{n:i-1}$ $(i \geqq 2)$). On the other hand, if X_{n+1} happens to lie in $(X_{n:j}, X_{n:j-1})$ for some $j \geqq 2$ then the sample maximum stands still (i.e. $X_{n+1:1} = X_{n:1}$), but the $X_{n:i}$ $(i \geqq j)$ jump to new (higher) levels. In other words, order statistics do not necessarily follow the steps of higher order statistics.

A counterexample to (1) is based on the structure of Y. Dwass (1964) shows that Y is a pure-jump Markov process. Given that $Y(t_0) = y$, the additional holding time is exponential with mean $1/\lambda(y)$, where $\lambda(y) = -\log G(y)$. The size of the jump has a distribution function $R_y(h) = 1 - \lambda(y+h)/\lambda(y)$ $(h > 0)$. If

https://doi.org/10.2307/3213504 Published online by Cambridge University Press

for instance we take the case $\lambda(y) = e^{-y}$, then the jumps are exponential (mean 1), independent of previous levels (or jumps) of Y. Thus, since Y is right-continuous, $Y(\alpha_1) = Y(1) \stackrel{\mathscr{D}}{=} \xi_1$ and

$$(2) \qquad (Y(\alpha_1), Y(\alpha_2), \cdots, Y(\alpha_k)) \stackrel{\mathscr{D}}{=} (\xi_1, \xi_1 - Z_1, \cdots, \xi_1 - (Z_1 + \cdots + Z_{k-1})),$$

where $Z_1, Z_2, \cdots$ are i.i.d. exponential random variables (mean 1). On the other hand Theorem 3 of Hall (1978) implies

$$(3) \qquad (\xi_1, \cdots, \xi_k) \stackrel{\mathscr{D}}{=} \left(\xi_1, \xi_1 - Z_1, \cdots, \xi_1 - \left(Z_1 + \frac{Z_2}{2} + \cdots + \frac{Z_{k-1}}{k-1}\right)\right).$$

Thus it is clear that (2) and (3) are not equal (in distribution). Hall's error has no effect on the other results of his paper. Incidentally, Hall (1978) gives an infinite-sum representation for the ξ_k. In David (1981), p. 266 and Weissman (1981) a finite-sum representation is given.

Technion — Israel Institute of Technology

Yours sincerely,
ISHAY WEISSMAN

References

DAVID, H. A. (1981) *Order Statistics*, 2nd edn. Wiley, New York.

DWASS, M. (1964) Extremal processes. *Ann. Math. Statist.* **35**, 1718–1725.

DWASS, M. (1966) Extremal processes II. *Illinois. J. Math.* **10**, 381–391.

HALL, P. (1978) Representations and limit theorems for extreme value distributions. *J. Appl. Prob.* **15**, 639–644.

RESNICK, S. I. AND RUBINOVITCH, M. (1973) The structure of extremal processes. *Adv. Appl. Prob.* **5**, 287–307.

WEISSMAN, I. (1981) Confidence intervals for the threshold parameter. *Commun. Statist.* **A10**(6), 549–557.