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BERNSTEIN AND JACKSON THEOREMS FOR THE HEISENBERG GROUP

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Abstract

We describe on the Heisenberg group H_n a family of spaces $M(h, X)$ of functions which play a role analogous to the trigonometric polynomials in T^n or the functions of exponential type in $\mathbf{R}^n$. In particular we prove that for the space $M(h, X)$, Jackson's theorem holds in the classical form while Bernstein's inequality hold in a modified form. We end the paper with a characterization of the functions of the Lipschitz space Λ^r_X by the behavior of their best approximations by functions in the space $M(h, X)$.

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Introduction

Let G be $\mathbf{R}^n$ or the n-dimensional torus T^n and $X = L^p(G)$ or $C_0(G)$. For every positive real number h, we consider the subspace of X:

$$M(h, X) = \{f \in X : \hat{f}(\lambda) = 0 \text{ if } \lambda \in \hat{G} \text{ and } |\lambda| > h\}.$$

More explicitly for $G = T^n$, $M(h, X)$ is the space of all trigonometric polynomials of degree equal or less than h, while, if $G = \mathbf{R}^n$, $M(h, X)$ consists of all entire functions of exponential type h in $\mathbf{C}^n$ which, as functions of the variable $x \in \mathbf{R}^n$, lie in X.

It is well known that the following inequalities hold:

a) (Jackson's theorem) *for every integer $N > 0$, there exists a constant C_N such that*

$$\operatorname*{Inf}_{g \in M(h, X)} \|f - g\|_X \leqslant C_N \omega_N(1/h, f, X)$$

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for every $f \in X$, *where* ω_N *is the N-th modulus of smoothness*;

b) (Bernstein's theorem) *for every multiindex* I *and every* $f \in M(h, X)$

$$\|D^I f\|_X \leqslant h^{|I|}\|f\|_X.$$

Our goal is to find for G the Heisenberg group H_n a family of spaces $M(h, X)$ ($h \in \mathbf{R}^+$) which satisfy conditions a) and b). For this purpose we consider the non-trivial representations π_λ of the Heisenberg group H_n and we suppose that π_λ acts on the Bargmann space $\mathscr{H}_\lambda$. If $\hat{f}(\lambda)$ is the Fourier transform of a function $f \in X$, we denote by $\{\hat{f}(\lambda)\}_{\alpha,\beta}$ ($\alpha, \beta \in \mathbf{N}^n$) the matrix entries of $\hat{f}(\lambda)$ with respect to the canonical orthonormal basis in $\mathscr{H}_\lambda$.

We define the space $M(h, X)$ in the following way:

$$M(h, X) = \left\{f \in X \colon \{\hat{f}(\lambda)\}_{\alpha,\beta} = 0 \text{ if } (2|\beta| + n)|\lambda| > h^2\right\}.$$

We prove that for these spaces Jackson's theorem holds in the classical form, while Bernstein's inequality holds in a modified form, with a constant greater than one. Finally we give a characterization of the functions of the Lipschitz spaces $\Lambda^r{}_X$ by the behavior of their best approximations by functions of the classes $M(h, X)$.

Notation

The Heisenberg group H_n is the Lie group whose underlying manifold is $\mathbf{R} \times \mathbf{C}^n$ and whose composition law is given by

$$(t, z) \cdot (t', z') = (t + t' + 2\operatorname{Im} z \cdot \bar{z}', z + z')$$

where $t, t' \in \mathbf{R}$, $z = (z_1, \ldots, z_n)$, $z' = (z'_1, \ldots, z'_n) \in \mathbf{C}^n$ and $z \cdot z' = \sum_{j=1}^n z_j \bar{z}'_j$. The complexified Heisenberg Lie algebra $(\mathfrak{h}_n)_\mathbf{C}$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t}, \quad \overline{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j\frac{\partial}{\partial t} \qquad (j = 1, \ldots, n).$$

We denote by V the vector space spanned by $Z_j, \overline{Z}_j$ ($j = 1, \ldots, n$). Since the only non-trivial commutation rule is $[Z_j, \overline{Z}_j] = -2iT$, V generates $(\mathfrak{h}_n)_\mathbf{C}$ as an algebra (that is H_n is stratified). The natural dilations on $(\mathfrak{h}_n)_\mathbf{C}$ are given by

$$\delta_\varepsilon(T + Z) = \varepsilon^2 T + \varepsilon Z \qquad (Z \in V, \varepsilon > 0).$$

We shall denote also by δ_ε the corresponding dilations on H_n:

$$\delta_\varepsilon(t, z) = (\varepsilon^2 t, \varepsilon z) \qquad (t, z) \in H_n.$$

We define on H_n a homogeneous norm

$$\rho(t, z) = \left(t^2 + |z|^4\right)^{1/4}.$$

If D is an invariant differential operator on H_n, we say that D is homogeneous of degree N if

$$D(f \circ \delta_\varepsilon) = \varepsilon^N (Df \circ \delta_\varepsilon). \tag{1}$$

In particular $N = 1$ if and only if $D \in V$.

The Haar measure on H_n is Lebesgue measure on $\mathbf{R} \times \mathbf{C}^n$. Let X denote either $L^p(H_n)$ or $C_0(H_n)$. The subspace $Y \subset X$ is defined as the space of all infinitely differentiable f such that $Df \in X$ for every invariant differential operator D. We set

$$f_{(\varepsilon)} = \varepsilon^{-2(n+1)} f \circ \delta_{1/\varepsilon} \qquad (\varepsilon > 0). \tag{2}$$

Clearly, if $f \in L^1(H_n)$ one has

$$\int_{H_n} f_{(\varepsilon)} = \int_{H_n} f,$$

and, if $f \in Y$ and D is an invariant differential operator homogeneous of degree N,

$$Df_{(\varepsilon)} = \varepsilon^{-N} (Df)_{(\varepsilon)}. \tag{3}$$

In defining Fourier transforms for H_n (see [4]) we are concerned only with the infinite-dimensional irreducible unitary representations of H_n. These representations can be considered as acting on the Bargmann space $\mathscr{H}_\lambda$ ($\lambda > 0$) which consist of all holomorphic functions F in $\mathbf{C}^n$ such that

$$\|F\|^2 = (2\lambda/\pi)^n \int_{\mathbf{C}^n} |F(w)|^2 \exp\left(-2\lambda |w|^2\right) dw < +\infty.$$

The monomials

$$F_{\alpha,\lambda}(w) = (\sqrt{2\lambda}\, w)^\alpha / \sqrt{\alpha!}, \qquad \alpha \in \mathbf{N}^n,$$

form an orthonormal basis for the Hilbert space $\mathscr{H}_\lambda$. For $\lambda \in \mathbf{R}^* = \mathbf{R} - \{0\}$ the representations π_λ on $\mathscr{H}_{|\lambda|}$ are given by

$$(\pi_\lambda(t, z)F)(w) = F(w - \bar{z}) \exp\left(i\lambda t + 2\lambda\left(w \cdot z - |z|^2/2\right)\right)$$

if $\lambda > 0$, and $\pi_\lambda(t, z) = \pi_{|\lambda|}(-t, -\bar{z})$ if $\lambda < 0$; these exhaust all non-trivial irreducible unitary representations of H_n. The Fourier transform of a L^1-function f is the operator valued function

$$\hat{f}(\lambda) = \int_{H_n} f(u) \pi_\lambda(u)\, du.$$

Let $\mathscr{F}_\lambda$ be the linear span of $\{F_{\alpha,\lambda}\}_{\alpha \in \mathbf{N}^n}$ and let $\mathscr{R}$ be the set of all families $S = \{S(\lambda)\}_{\lambda \in \mathbf{R}^*}$ of linear operators $S(\lambda)$: $\mathscr{F}_\lambda \to \mathscr{H}_\lambda$. The matrix of $S(\lambda)$ is defined

$$\{S(\lambda)\}_{\alpha,\beta} = \left(S(\lambda) F_{\alpha,\lambda}, F_{\beta,\lambda}\right)_{\mathscr{H}_\lambda}.$$

If $f \in S(H_n)$, the Schwartz space, the Plancherel formula

$$\|f\|_2^2 = c_n \int_{\mathbf{R}^*} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n \, d\lambda$$

holds, where $\|\cdot\|_{HS}$ is the Hilbert Schmidt norm and $c_n = 2^{n-1}/\pi^{n+1}$. Then we can extend the Fourier transformation to an isometry from $L^2(H_n)$ onto the Hilbert space $\mathscr{L}^2$, where

$$\mathscr{L}^2 = \Big\{ S = \{S(\lambda)\} \in \mathscr{R} \colon \|S(\lambda)\|_{HS} < +\infty \text{ for almost all } \lambda \text{ and } \int_{\mathbf{R}^*} \|S(\lambda)\|_{HS}^n |\lambda|^n \, d\lambda < +\infty \Big\}.$$

More generally let $S'(H_n)$ be the conjugate dual of $S(H_n)$; we define (see [4])

$$S(h_n) = \{ S \in \mathscr{R} \colon S = \hat{f} \text{ for some } f \in S(H_n) \},$$

and we topologize it to be homeomorphic to $S(H_n)$. Let $S'(h_n)$ be the conjugate dual of $S(h_n)$. By polarization of the Plancherel formula we can extend the Fourier transformation to an isomorphism between $S'(H_n)$ and $S'(h_n)$ which we also denote by $\hat{}$. If $\{R(\lambda)\} \in \mathscr{R}$, we shall say that $\{R(\lambda)\} \in S'(h_n)$ if the map

$$S \to \int_{\mathbf{R}^*} \sum_{\alpha \in \mathbf{N}^n} \left| \left(R(\lambda) F_{\alpha,\lambda}, S(\lambda) F_{\alpha,\lambda} \right)_{\mathscr{H}_\lambda} \right| |\lambda|^n \, d\lambda$$

is defined and continuous from $S(h_n)$ to $\mathbf{C}$ (see [5], Chapter 2).

We observe that, if $f, g \in S'(H_n)$,

$$\text{(4)} \qquad \begin{aligned} \widehat{Z_j f}(\lambda) F_{\alpha,\lambda} &= -\left(2|\lambda|(\alpha_j + 1)\right)^{1/2} \hat{f}(\lambda) F_{\alpha+e_j,\lambda}, \\ \widehat{\bar{Z}_j f}(\lambda) F_{\alpha,\lambda} &= \left(2|\lambda|\alpha_j\right)^{1/2} \hat{f}(\lambda) F_{\alpha-e_j,\lambda}, \end{aligned} \qquad \text{if } \lambda > 0$$

(where $\{e_j\}_{j=1,\dots,n}$ is the canonical basis for $\mathbf{R}^n$, if $\lambda < 0$ we must reverse the right sides) and

$$\text{(5)} \qquad D(f * g) = f * Dg$$

(where D is any left invariant differential operator).

Moduli of smoothness in H_n

Moduli of smoothness in H_n were studied by I. R. Inglis [6]. The results of this Section are analogous to the results obtained by P. M. Soardi [7] in $\mathbf{R}^n$ in the non-isotropic case.

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For every integer N and every $0 \leqslant \theta \leqslant 1$ we define

$$\Delta^N_{u,\theta} f(u') = \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} f(u' \cdot \delta_{j+\theta} u), \qquad u, u' \in H_n, f \in X.$$

We set $\Delta^N_{u,0} f = \Delta^N_u f$ and we remark that $\Delta^{N+1}_u f = \Delta^N_{u,1} f - \Delta^N_u f$.

DEFINITION 1. For every $h \in \mathbf{R}^+$ and $N \in \mathbf{N}$, the function

$$\omega_N(h, f, X) = \operatorname*{Sup}_{\rho(u) \leqslant h} \|\Delta^N_u f\|_X$$

is called the N-th modulus of smoothness.

For ease of notation we shall write $\omega_N(h, f)$ and $\|\cdot\|$ instead of $\omega_N(h, f, X)$ and $\|\cdot\|_X$. Since H_n is stratified the space of left-invariant operators which are homogeneous of degree N is exactly the linear span of the monomials $X_1 X_2 \cdots X_N$, where $X_i = Z_j$ or $\overline{Z}_k$ ($i = 1,\dots,N$; $j, k = 1,\dots,n$). We denote by V_N the set of all differential monomials in Z_j and $\overline{Z}_k$ ($j, k = 1,\dots,n$). Obviously $V = V_1$.

LEMMA 1. *Suppose $D \in V_N$. There exists a constant $C = C(D)$ such that for every $h > 0$ and $f \in X$*

$$\|f - g\| + h^N \|Dg\| \leqslant C\omega_N(h, f)$$

for some $g \in Y$.

PROOF. We choose $\phi \in S(H_n)$ such that $\int_{H_n} \phi = 1$ and $\operatorname{supp} \phi \subseteq \{u \in H_n: \rho(u) \leqslant 1/N^2\}$. We define

$$P = \frac{1}{N!} \sum_{j=1}^{N} (-1)^{N+j} \binom{N}{j} j^N \phi_{(j)},$$

$$Q = \sum_{j=1}^{N} (-1)^{N+j} \binom{N}{j} P_{(j)}$$

(for the definition of $\phi_{(j)}$ and $P_{(j)}$ see (2)). Obviously $\int_{H_n} P = 1$, $\int_{H_n} Q = (-1)^{N+1}$ and $\operatorname{supp} Q \subseteq \{u \in H_n: \rho(u) \leqslant 1\}$. We define $g = (-1)^{N+1} f * Q_{(h)}$. By changing variables one sees that

$$g(u) - f(u) = \int_{\rho(v) \leqslant h} h^{-2(n+1)} P\left(\delta_{1/h} v^{-1}\right) \cdot \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} f(u \cdot \delta_j v)\, dv.$$

Then

$$\|f - g\| \leqslant \int_{\rho(v)\leqslant h} \left|P_{(h)}(v^{-1})\right| \left\|\Delta_v^N f\right\| dv$$
$$\leqslant \|P\|_1 \omega_N(h, f).$$

Furthermore

$$Q_{(h)} = \left(\frac{1}{N!}\right) \sum_{r,s=1}^{N} (-1)^{r+s} \binom{N}{r} \binom{N}{s} r^N \phi_{(hrs)},$$

and by (1),

$$DQ_{(h)} = \frac{h^{-N}}{N!} \sum_{r,s=1}^{N} (-1)^{r+s} \binom{N}{r} \binom{N}{s} (D\phi_{(s)})_{(hr)}.$$

A change of variables shows that

$$f * DQ_{(h)} = (h^{-N}/N!) \sum_{s=1}^{N} (-1)^s \binom{N}{s} \int_{\rho(v)\leqslant h} \sum_{r=1}^{N} (-1)^r \binom{N}{r}$$
$$\cdot f(u \cdot \delta_r v)(D\phi_{(s)})_{(h)}(v^{-1})\, dv.$$

Since $\int_{\rho(v)\leqslant h} (D\phi_{(s)})_{(h)}(v^{-1})\, dv = 0$, we have

$$h^N \|Dg\| = h^N \|fDQ_{(h)}\|$$
$$\leqslant \frac{1}{N!} \sum_{s=1}^{N} \binom{N}{s} \int_{\rho(v)\leqslant h} \left|(D\phi_{(s)})_{(h)}(v^{-1})\right| \left\|\Delta_v^N f\right\| dv$$
$$\leqslant \frac{1}{N!} \sum_{s=1}^{N} \binom{N}{s} \|D\phi_{(s)}\|_1 \omega_N(h, f).$$

LEMMA 2. *Let N be a positive integer and $0 \leqslant \theta \leqslant 1$. There exists a positive constant C, not depending on θ, such that for every $f \in X$, $g \in Y$ and $h > 0$*

$$\left\|\Delta_{u,\theta}^N f\right\| \leqslant C\left(\|f - g\| + \rho(u)^N \sum_{D \in V_N} \|Dg\|\right).$$

PROOF. Since

$$\left\|\Delta_{u,\theta}^N f\right\| \leqslant \left\|\Delta_{u,\theta}^N (f - g)\right\| + \left\|\Delta_{u,\theta}^N g\right\| \leqslant 2^N \|f - g\| + \left\|\Delta_{u,\theta}^N g\right\|,$$

we have to evaluate $\Delta_{u,\theta}^N g$ when $g \in Y$. Suppose $v = (0, z_1, \ldots, z_n) \in \exp V$; we observe that

$$\frac{d^m}{ds^m}(g(u'' \cdot \delta_s v))\bigg|_{s=j} = \left[\left(\sum_{k=1}^{n} (z_k Z_k + \bar{z}_k \bar{Z}_k)\right)^m g\right](u'' \cdot \delta_j v).$$

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We set

(6) $$\sum_{k=1}^{n}\left(z_k Z_k + \bar{z}_k \bar{Z}_k\right) = E(v).$$

Moreover there exist a constant $C' > 0$ and an integer M such that any $u \in H_n$ can be expressed as

(7) $u = v_1 \cdot \ldots \cdot v_M$ where $v_i = (0, w_i) \in \exp V$ and $\rho(v_i) \leqslant C'\rho(u)$ $(i = 1, \ldots, M$; see [3], Lemma 1.40).

Therefore an application of Taylor's theorem to the function $s \to g(u'' \cdot \delta_s v_i)$ $(u'' \in H; i = 1, \ldots, M)$ yields:

(8)
$$\begin{aligned} g(u' \cdot \delta_{j+\theta} u) - g(u') &= \sum_{i=1}^{M} g\left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-1}) \cdot \delta_{j+\theta} v_i\right) \\ &\quad - g\left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-1})\right) \\ &= \sum_{i=1}^{M} \sum_{m=1}^{N-1} \frac{(j+\theta)^m}{m!}\left[E(v_i)^m g\right]\left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-1})\right) \\ &\quad + \frac{(j+\theta)^N}{(N-1)!}\int_0^1 (1-s)^{N-1} \\ &\qquad \cdot \left[E(v_i)^N g\right]\left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i)\right) ds. \end{aligned}$$

By the inequality $\rho(u \cdot v) \leqslant \rho(u) + \rho(v)$ (see [2]), we have

(9) $$\rho(v_1 \cdots v_{i-1}\delta_s v_i) \leqslant \sum_{k=1}^{i} \rho(v_k) \leqslant C'M\rho(u) \qquad (i = 1, \ldots, M).$$

Furthermore

$$\begin{aligned} \left[E(v_i)^m g\right]\left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-1})\right) &= \sum_{k=0}^{N-m} \frac{(j+\theta)^k}{k!}\left[E(v_{i-1})^k\left(E(v_i)^m g\right)\right] \\ &\quad \cdot \left(u' \cdot \delta_{j+\theta}(v_1 \cdot \ldots \cdot v_{i-2})\right) \\ &\quad + \text{remainder term.} \end{aligned}$$

Repeating this process we obtain

$$\begin{aligned} g(u' \cdot \delta_{j+\theta} u) &= \sum_{s_M=0}^{N-1} (j+\theta)^{s_M} \sum_{s_{M-1}=0}^{s_M} \cdots \sum_{s_1=0}^{s_2} \left(s_1! \prod_{k=2}^{M}((s_k - s_{k-1})!)\right)^{-1} \\ &\quad \cdot \left[E(v_1)^{s_1} E(v_2)^{s_2 - s_1} \cdots E(v_M)^{s_M - s_{M-1}} g\right](u') \\ &\quad + \text{remainder terms} \\ &= \sum_{s_M=0}^{N-1} (j+\theta)^{s_M} Q_{s_M}(u, u') + R(u, u', j+\theta), \end{aligned}$$

where $R(u, u', j+\theta)$ consists of all remainder terms, hence it can be written as a sum of terms like

$$(10) \qquad (j+\theta)^N \int_0^1 (1-s)^{N-1-k} w^\alpha \overline{w}^\beta [Dg](u' \cdot u(s))\, ds$$

where $D \in V_N$ and $0 \leqslant k \leqslant N-1$; $w = (w_1, \ldots, w_n) \in C^{nM}$ with $v_i = (0, w_i)$ and $|w_i| \leqslant C'\rho(u)$ by (8); $\alpha, \beta \in \mathbf{N}^{nM}$ and $|\alpha| + |\beta| = N$; $\rho(u(s)) \leqslant (j+\theta)C'M\rho(u)$ by (9). Now

$$\begin{aligned}\Delta^N_{u,\theta} g(u') &= \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} g(u' \cdot \delta_{j+\theta} u) \\ &= \sum_{s_M=0}^{N-1} \sum_{j=0}^{N} (j+\theta)^{s_M} (-1)^{N+j} \binom{N}{j} Q_{s_M}(u, u') \\ &\quad + \sum_{j=0}^{N} (-1)^{N+j} \binom{N}{j} R(u, u', j+\theta).\end{aligned}$$

Since $\sum_{j=0}^N j^k (-1)^j \binom{N}{j} = 0$ if $k < N$, the first term in the previous sum is 0; it follows from (10) that

$$\|R(u, \cdot, j+\theta)\| \leqslant C''(j+\theta)^N \rho(u)^N \sum_{D \in V_N} \|Dg\|.$$

Because $0 \leqslant \theta \leqslant 1$, finally we obtain

$$(11) \qquad \|\Delta^N_{u,\theta} g\| \leqslant C''' \rho(u)^N \sum_{D \in V_N} \|Dg\|,$$

where C''' depends only on N.

We can summarize Lemmas 1 and 2 in the following statement: let N be a positive integer; there exist two constants C_1 and C_2 such that for every $f \in X$ and $h > 0$

$$(12) \qquad C_1 \omega_N(h, f) \leqslant \inf_{g \in Y} \Big(\|f - g\| + h^N \sum_{D \in V_N} \|Dg\| \Big) \leqslant C_2 \omega_N(h, f).$$

The second member of (12) is the analogue in H_n of the classical Peetre K-functional (see also [7]).

Corollary 1. *Let* $\varepsilon > 0$, *then*

$$\omega_N(\varepsilon h, f) \leqslant C(N)(1 + \varepsilon^N)\omega_N(h, f).$$

Proof. Obvious from (12).

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COROLLARY 2. *Let N be a positive integer. There exists a constant C such that*

$$\|\Delta^N_{u,\theta} f\| \leqslant C\omega_N(\rho(u), f)$$

for every $f \in X$, $u \in H_n$ *and* $0 \leqslant \theta \leqslant 1$.

LEMMA 3. *Let K and k be two positive integers such that* $K \geqslant k$. *We suppose* $g \in X$ *and* $Dg \in X$ *for every* $D \in V_i$ $(i \leqslant k)$. *Then*

$$\|\Delta^K_u g\| \leqslant C(k)\rho(u)^k \sum_{D \in V_k} \omega_{K-k}(\rho(u), Dg).$$

PROOF. If we set $\theta = 0$ and $N = 1$ in formula (8), we get

$$\begin{aligned}
\Delta^K_u g(u') &= \sum_{j=1}^{K} (-1)^{K+j}\big(g(u' \cdot \delta_j u) - g(u')\big) \\
&= \sum_{i=1}^{M} \sum_{j=1}^{K} j(-1)^{K+j}\binom{K}{j} \int_0^1 [E(v_i)g]\big(u' \cdot \delta_j(v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i)\big)\, ds \\
&= \sum_{i=1}^{M} \sum_{j=0}^{K-1} (-1)^{j+K-1}\binom{K-1}{j} \\
&\quad \cdot \int_0^1 [E(v_i)g]\big(u' \cdot \delta_{j+1}(v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i)\big)\, ds.
\end{aligned}$$

Now

$$\begin{aligned}
\left\|\Delta^{K-1}_{v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i, 1} E(v_i) g\right\| &\leqslant C'\rho(u) \sum_{D \in V} \left\|\Delta^{K-1}_{v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i, 1} Dg\right\| \quad \text{(by (6) and (7))} \\
&\leqslant C''\rho(u) \sum_{D \in V} \omega_{K-1}(CM\rho(u), Dg) \quad \text{(by (9) and Corollary 2)} \\
&\leqslant C'''\rho(u) \sum_{D \in V} \omega_{K-1}(\rho(u), Dg) \quad \text{(by Corollary 1)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\Delta^K_u g\| &\leqslant \sum_{i=1}^{M} \int_0^1 \left\|\Delta^{K-1}_{v_1 \cdot \ldots \cdot v_{i-1} \cdot \delta_s v_i} E(v_i) g\right\| ds \\
&\leqslant MC'''\rho(u) \sum_{D \in V} \omega_{K-1}(\rho(u), Dg).
\end{aligned}$$

Repeating this process, we obtain the thesis.

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Jackson and Bernstein theorems

Before proving the main theorems we observe that given a radial Schwartz function f in H_n (in the sense that $f(t, z) = f(t, |z|)$) we have $\{\hat{f}(\lambda)\}_{\alpha,\beta} \equiv 0$ if $\alpha \neq \beta$ and

$$\{\hat{f}(\lambda)\}_{\alpha,\alpha} = \int_{\mathbf{R}\times\mathbf{C}^n} f(t, z) e^{i\lambda t} l_\alpha^0(2|\lambda|\,|z|^2)\, dt\, dz \tag{13}$$

where l_α^0 is the Laguerre function of type 0 and degree α [4].

DEFINITION 2. Let $h > 0$. We denote by $M(h, X)$ the class of all functions $f \in X$ such that

$$\{\hat{f}(\lambda)\}_{\alpha,\beta} = 0 \quad \text{if } (2|\beta| + n)|\lambda| > h^2.$$

THEOREM 1. *Let N be a positive integer. For every $f \in X$ and every $h > 0$ there exists a function $g_h \in M(h, X)$ such that*

$$\|f - g_h\| \leqslant C(N)\omega_N(1/h, f),$$

where $C(N)$ is a constant which depends only on N.

PROOF. Let $\phi: \mathbf{R} \to \mathbf{R}$ be an even C^∞-function such that $\phi(0) = 1$ and supp $\phi \subset [-1, 1]$. We consider $S \in \mathscr{R}$ such that

$$\{S(\lambda)\}_{\alpha,\beta} = \begin{cases} \phi((2|\alpha| + n)|\lambda|) & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

for every $\lambda \in \mathbf{R}^*$. Then $S(\lambda)$ is the Fourier transform of a function in $S(H_n)$ (see [4], Theorem 1). Obviously $G \in M(1, X)$. We consider

$$K = \sum_{j=1}^{N} (-1)^{N+j} \binom{N}{j} G_{(j/h)}$$

and $g_h = (-1)^{N+1} f * K$. We observe that $g_h \in M(h, X)$. Moreover

$$f * K(u) = \int_{H_n} G(v) \sum_{j=1}^{N} \binom{N}{j} f\left(u \cdot \delta_{j/h} v^{-1}\right) dv.$$

Since $\lim_{\lambda\to 0}\{\hat{G}(\lambda)\}_{\alpha,\alpha} = 1$ for all α, by (13) and the Lebesgue dominated convergence theorem we have $\int_{H_n} G = 1$. Thus

$$g_h(u) - f(u) = (-1)^{N+1} \int_{H_n} G(v) \Delta^N_{\delta_{1/h} v} f(u)\, dv$$

and

$$\|g_h - f\| \leqslant \int_{H_n} |G(v)|\, \left\|\Delta^N_{\delta_{1/h} v} f\right\| dv. \tag{14}$$

From Corollary 1 it follows that

$$\|g_h - f\| \leqslant C'(N)\omega_N(1/h, f)\int_{H_n} |G(v)|(1 + \rho(v))^N dv$$
$$= C(N)\omega_N(1/h, f).$$

THEOREM 2. *Let h be a positive number and D be a left-invariant differential operator with degree of homogeneity N. There exists a constant $C(D)$ such that $\|Df\| \leqslant C(D)h^N\|f\|$ for every $f \in M(h, X)$.*

PROOF. Let $\phi \in C_c^\infty([0, +\infty))$ such that $\phi(x) = 1$ if $x \in [0,1]$. We define S and G as in Theorem 1. Since $f \in M(h, X)$ we have $f * G_{(1/h)} = f$; hence f is a C^∞-function in H_n such that (by (3) and (5))

$$f * (DG)_{(1/h)} = h^{-N}D\big(f * G_{(1/h)}\big) = h^{-N}Df$$

and

$$\|Df\| \leqslant h^N\|DG\|_1\|f\| = C(D)h^N\|f\|.$$

REMARK. Suppose $f \in M(h, X)$; then $\hat{f}(\lambda) = 0$ if $|\lambda| > h^2/n$ and for such λ's $Df(\lambda) = 0$, for every invariant differential operator D. Therefore to avoid triviality we suppose $|\lambda| \leqslant h^2/n$. If N is the degree of homogeneity of D, by (4) we have $\{Df(\lambda)\}_{\alpha,\beta} = 0$ if $|\beta| > N + (k^2/|\lambda| - n)/2$. Namely

$$Df \in M\Big(\sqrt{(2N/n + 1)}\,,\, X\Big)$$

(while in the abelian case we have $Df \in M(h, X)$ if $f \in M(h, X)$).

Suppose $n = 1$. Let $\phi \in C_c^\infty(\mathbf{R}^*)$ and $\operatorname{supp}\phi \subset (0, h^2]$. Let $S \in \mathscr{R}$ be such that

$$\{S(\lambda)\}_{\alpha,\beta} = \begin{cases} \phi(\lambda) & \text{if } \alpha = \beta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = \hat{f}$ for some $f \in S(H_1)$ and $f \in M(h, X)$. By formula (4)

$$\big\{(\bar{Z}^N f)\hat{}(\lambda)\big\}_{0,N} = \big((2|\lambda|)^N N!\big)^{1/2}\{\hat{f}(\lambda)\}_{0,0}.$$

By the Plancherel formula we can choose a sequence of functions ϕ_k such that

$$\|f_k\|_2 = 1 \quad \text{and} \quad \|\bar{Z}^N f_k\|_2 \to (N!2^N)^{1/2}h^N \qquad \text{as } k \to \infty.$$

Hence the constant $C(D)$ of Theorem 2 is not necessarily equal to 1. This contrasts with the abelian case, and with the situation for compact Lie groups (see [1], Lemma 2).

Lipschitz spaces

We can apply the results of the previous sections to obtain a characterization of the Lipschitz spaces $\Lambda^r{}_X(H_n)$ by the behavior of their best approximation by functions of the classes $M(h, X)$.

DEFINITION 3. Suppose $r > 0$ and $N = [r] + 1$. We say that the function f belongs to the Lipschitz space $\Lambda^r{}_X(H_n)$ if $f \in X$ and there exists a constant $M = M(f)$ such that

$$\|\Delta_u^N f\| \leqslant M\rho(u)^r \tag{15}$$

for every $u \in H_n$.

The space $\Lambda^r{}_X$ becomes a Banach space if we put

$$\|f\|_{\Lambda^r{}_X} = \|f\| + M_f,$$

where M_f is the lower bound of all M's for which (16) is satisfied (compare with [3], Chapter 5-C).

THEOREM 3. *The function f belongs to $\Lambda^r{}_X(H_n)$ if and only if there exists $A > 0$ and a family of functions $g_h \in M(h, X)$, $h \geqslant 1$, such that*

$$\|f - g_h\| \leqslant A/h^r. \tag{16}$$

Moreover, if $0 \leqslant k < r$ and $D \in V_k$, there exist two constant C_1, C_2 such that

$$\|Df\| \leqslant C_1(\|f\| + A) \tag{17}$$

$$h^{k-r}\omega_N(h, Df) \leqslant C_2(\|f\| + A) \quad \textit{for every integer } N > r; \tag{18}$$

(obviously if $k = 0$ we must replace Df with f in (18)).

PROOF. We suppose $f \in \Lambda^r{}_X$. If we set $N = [r] + 1$ in Theorem 1, it follows by (14) and (15) that

$$\begin{aligned} \|f - g_h\| &\leqslant M\int_{H_n} \rho(\delta_{1/h}v)^r |G(v)|\,dv \\ &= Mh^{-r}\int_{H_n} \rho(v)^r |G(v)|\,dv = Ah^{-r}. \end{aligned} \tag{19}$$

Vice versa we consider a sequence $g_{2^j} \in M(2^j, X)$ for which inequality (16) holds and we define

$$Q_0 = g_1, \quad Q_j = g_{2^j} - g_{2^{j-1}} \qquad (j = 1, 2, \ldots).$$

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Obviously $Q_j \in M(2^j, X)$ and by definition

(20) $$\|Q_0\| = \|g_1\| < \|f\| + A,$$

$$\|Q_j\| \leqslant \|f - g_{2^j}\| + \|f - g_{2^{j-1}}\| \leqslant A(2^r + 1)/2^{rj}, \qquad j = 1, 2, \ldots.$$

If $D \in V_k$, it follows from Theorem 2 that

(21) $$\|DQ_j\| \leqslant C(D)2^{kj}\|Q_j\| \leqslant C'A2^{(k-r)j}, \qquad j = 1, 2, \ldots,$$

$$\|DQ_0\| \leqslant C'(\|f\| + A).$$

In view of (20) $f = \sum_{j=0}^{\infty} Q_j$ in the sense of X. Moreover the estimates (21) show that the series $\sum_{j=0} DQ_j$ converges in X to Df, if $D \in V_k$ $(k < r)$. Hence $Df \in X$ and (17) holds.

We consider $u \in H_n$ and we choose a positive integer K such that $2^{-(K+1)} < \rho(u) \leqslant 2^{-K}$. If $N > r - k > 0$ and $D \in V_k$ using inequalities (11) (with $\theta = 0$) and (21) we obtain

(22) $$\|\Delta_u^N DQ_j\| \leqslant C(N)\rho(u)^N \sum_{D' \in V_N} \|D'DQ_j\| \leqslant C'\rho(u)^N A2^{j(N+k-r)}.$$

Obviously

$$\|\Delta_u^N Df\| \leqslant \sum_{j=0}^{K} \|\Delta_u^N DQ_j\| + \sum_{j=K+1}^{+\infty} \|\Delta_u^N DQ_j\| = J_1 + J_2.$$

Now

$$J_1 \leqslant C''(\|f\| + A) \sum_{j=0}^{K} 2^{j(N+k-r)}\rho(u)^N$$

$$\leqslant C'''(\|f\| + A)2^{K(N+k-r)}\rho(u)^N \leqslant C'''(\|f\| + A)\rho(u)^{r-k}$$

(the first inequality follows from (22) and the third one from the choice of K); moreover

$$J_2 \leqslant 2^N \sum_{j=M+1}^{+\infty} \|Q_j\| \leqslant AC' \sum_{j=M+1}^{+\infty} 2^{j(k-r)}$$

$$= AC''2^{(M+1)(k-r)} \leqslant AC''\rho(u)^{r-k}.$$

These estimates prove (18); if we set $k = 0$, it follows that $f \in \Lambda^r{}_X$.

Remark. It follows from Theorem 3 that $f \in \Lambda^r{}_X$ implies $Df \in \Lambda^s{}_X$ where $s = r - k > 0$ if $D \in V_k$. Now, let $k \geqslant 0$ and $N > r - k > 0$. We can define on $\Lambda^r{}_X$ the following norms

$$\|f\|_{(N,k)} = \|f\| + \sup_{D \in V_k} \sup_{h>0} \omega_N(h, Df)/h^{r-k} \qquad \left(\|f\|_{([r]+1,0)} = \|f\|_{\Lambda^r_X}\right),$$

$$\|f\|_{(*)} = \|f\| + \sup_{h>0} h^r \inf_{g_h \in M(h,X)} \|f - g_h\|.$$

These norms are equivalent. In fact, let M be an integer such that $M > r > k$ and $D \in V_k$. It follows from (14) and Lemma 3 that

$$\|f - g_h\| \leqslant C'h^{-k}\int_{H_n} |G(v)|\rho(v)^k \sum_{D\in V_k} \omega_{M-k}(\rho(v)/h, Df)\, dv$$

$$\leqslant C'h^{-k} \sup_{D\in V_k} \omega_{M-k}(1/h, Df)\int_{H_n} |G(v)|\rho(v)^k(1+\rho(v))^{N-k}\, dv$$

$$\leqslant C''h^{-k} \sup_{D\in V_k} \omega_{M-k}(1/h, Df).$$

If we set $M - k = N > r - k$ we obtain

$$h^r \inf_{g_h\in M(h,X)} \|f - g_h\| \leqslant C''h^{r-k} \sup_{D\in V_k} \omega_N(1/h, Df).$$

Therefore

$$\|f\|_{(*)} \leqslant C''\|f\|_{(N,k)}$$

for every pair of integers N and k such that $k \geqslant 0$, $N > r - k > 0$. On the other hand, it follows from (18) that

$$h^{k-r}\omega_N(h, Df) \leqslant C'\Big(\|f\| + \sup_{\varepsilon>1} \varepsilon^r \inf_{g_\varepsilon\in M(\varepsilon,X)} \|f - g_\varepsilon\|\Big) \leqslant C'\|f\|_{(*)}.$$

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