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ON DERIVATIONS INDUCED BY p-ADIC FIELDS

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1. Introduction. This paper is concerned with a question which occurs in [**6**, p. 346] and uses the notation of that article. Thus $K \supset K_0$ are p-adic fields ($p \neq 2$) with residue fields $k \supset k_0$ and having respective rings of integers $R \supset R_0$, $G_0 = G_0(K/K_0)$ is the group of inertial automorphisms of K over K_0, $I(K/K_0)$ is the R module of integral derivations on K over K_0 and $\bar{I}(K/K_0)$ is the k space of derivations on k induced by $I(K/K_0)$. The question here dealt with is the following. Given fields $k \supset k_0$ of characteristic $p(\neq 0, 2)$ with k/k_0 finitely generated, which subspaces of the k space, $\mathrm{Der}(k/k_0)$, of derivations on k over k_0 have the form $\bar{I}(K/K_0)$ for some pair of p-adic fields $K \supset K_0$ having $k \supset k_0$ as residue fields. We note the following connection between $\bar{I}(K/K_0)$ and $G_0(K/K_0)$.

If α is in the jth ramification group

$$G_j = \{\alpha \text{ in } G_0 | \alpha \text{ induces the identity map on } R/p^{j+1}R\}$$

and if $\alpha^* = (\alpha - \mathrm{Id})|_R$, Id being the identity map on K, then

$$\ln \alpha = \sum \{(-1)^{i+1}(\alpha^*)^i/i | i = 1, 2, \ldots\}$$

is a derivation on R [**4**; p. 817, Theorem 2.1]. Also ϕ_j; $G_j \rightarrow \bar{I}(K/K_0)$, where $\phi_j(\alpha)$ is the map induced by $\mathrm{p}^{-(j+1)} \ln \alpha$, is a group homomorphism. For $j = 0, 1, \ldots$ the sequence

$$0 \rightarrow G_{j+1} \xrightarrow{\iota} G_j \xrightarrow{\phi_j} \bar{I}(K/K_0) \rightarrow 0$$

is exact where ι is the natural injection.

The following basic result of p-adic Galois Theory is the starting point for this study.

THEOREM [**6**, p. 342, Theorem 3]. *An R-module I of integral derivations on K constant on K_0 is the full module $I(K/K_0)$ if and only if there are derivations $d_1, \ldots, d_r$ in I and integers $a_1, \ldots, a_r$ in K such that the Jacobian* det $(d_i(a_j))$ *is a unit where r is the transendency degree of k/k_0.*

It is readily seen that $\bar{I}(K/K_0) = \mathrm{Der}\ (k/k_0)$ if and only if k/k_0 is separable [**6**, p. 342, Corollary 2]. In general $\bar{I}(K/K_0)$ depends on K/K_0 as well as on k/k_0 (see Example 5.5).

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Our analysis is made in terms of distinguished subfields of k/k_0, a concept introduced by Dieudonne [**2**]. A field s, $k \supset s \supset k_0$ is distinguished if s/k_0 is separable and if for some $n \geqq 0$, $k \subset k_0^{p^{-n}}(s)$. There are a number of reasons for this approach in addition to the fact that the theory in question is simple in the separable case and that distinguished subfields are precisely the separable intermediate fields having minimal codegree in k. Distinguished subfields can be characterized among maximal separable intermediate fields in terms of extension properties of higher derivations [**8**]. The property of the embedding of s in k, for s distinguished, which is responsible for the higher derivation extension property also has implication for derivation inertia on related p-adic fields, a basic concept in this paper.

We obtain a complete characterization of those subspaces of Der (k/k_0) having the form $\bar{I}(K/K_0)$, save for one case, under the assumption that k is a simple extension of some distinguished subfield. This is Theorem 5.2. The result gains significance from the fact that if k is a simple extension of one distinguished subfield it is a simple extension of every distinguished subfield [**8**].

It is also shown that given a distinguished subfield s there are p-adic fields K/K_0 such that $\bar{I}(K/K_0)|_s$ consists of all derivations of s/k_0 into k. This is Theorem 3.3.

Section 2 is concerned with derivation inertia in $I(K/K_0)$ and generalizes results of [**12**, p. 497, Theorem 1].

2. Derivation inertia. Let $k \supset s \supset k_0$ be fields of characteristic p, $\neq 0$, and assume s/k_0 separable. We do not assume k/k_0 finitely generated in this section. Let $K \supset S \supset K_0$ be p-adic fields having $k \supset s \supset k_0$ as respective residue fields and having rings of integers $R \supset R_S \supset R_0$. We state without proof the following well known result.

(2.1) PROPOSITION. *If $\bar{C}$ is a p-basis for s over k_0 and m is a positive integer then $\bar{C}^{p^m} = \{\bar{c}^{p^m} | \bar{c}$ in $\bar{C}\}$ is a p-basis for $k_0(s^{p^m})$ over k_0.*

Henceforth $\bar{C}$ denotes a fixed p-basis for s over k_0 and C is a set of representatives c in S of the elements $\bar{c}$ in $\bar{C}$. For future reference we note that each $\bar{a}$ in $k_0(s^{p^m})$ has a representation

$$(2.2)\qquad \bar{a} = \sum\{\bar{a}_{i_1,\ldots,i_r}(\bar{c}_1^{\,i_1}\ldots\bar{c}_r^{\,i_r})^{p^m} | 0 \leqq i_j < p; j = 1,\ldots,r,\ \bar{c}_j \in \bar{C}, \bar{a}_{i_1,\ldots,i_r} \in k_0(s^{p^{m+1}})\}.$$

Let $I(S/K_0, K)$ be the R module of integral derivations d from S into K such that $d(a) = 0$ for a in K_0. A derivation of S into K is integral if for a in R_S $d(a)$ is in R.

$R_0[R_S^{p^m}]_{U_m}$ is the ring of quotients of the subring $R_0[R_S^{p^m}]$ generated by $R_S^{p^m}$ over R_0 with respect to the set U_m of units of R_S contained in

$R_0[R_S^{p^m}]$. The significant properties of $R_0[R_S^{p^m}]_{U_m}$ for our purposes are the following.

(2.3) PROPOSITION. *$R_0[R_S^{p^m}]_{U_m}$ is a subring of R_S which under the canonical map of R_S onto s maps onto $k_0(s^{p^m})$. If d is in $I(S/K_0, K)$ and a is in $R_0[R_S^{p^m}]_{U_m}$ then $d(a)$ is in p^mR.*

Proof. The first assertion is a direct consequence of the definition of $R_0[R_S^{p^m}]_{U_m}$. If a is in R_S and d is in $I(S/K_0, K)$ then $d(a^{p^m}) = p^m a^{p^m-1} d(a)$ is in p^mR. Thus d maps $R_0[R_S^{p^m}]$ into p^mR. The last assertion of (2.3) now follows from the quotient rule for derivations.

An element $\bar{a}$ in $k_0(s^{p^m})$, not in $k_0(s^{p^{m+1}})$, has the form (2.2) and thus has a representative

$$(2.4) \qquad a^{(m)} = \sum\{a_{i_1,\dots,i_r}(c_1^{p^m})^{i_1}\dots(c_r^{p^m})^{i_r} | a_{i_1,\dots,i_r} \in R_0[R_S^{p^{m+1}}]_{U_m}, c_j \in C, j \leqq r\}.$$

Such an element $a^{(m)}$ is called an *inertial representative* of $\bar{a}$ with respect to $\bar{C}$ or simply an inertial representative of $\bar{a}$. Noting that $\cap\{k_0(s^{p^m})| m \geqq 1\}$ is the algebraic closure k_0^c of k_0 in s [**7**, p. 273, Corollary 7.3] if $\bar{a}$ is in $\cap\{k_0(s^{p^m})|m \geqq 1\}$ it is separable algebraic over k_0 and by Hensels Lemma [**9**, p. 230] $\bar{a}$ has a representative $a^{(\infty)}$ in R_S which is algebraic over R_0; $a^{(\infty)}$ is an inertial representative of $\bar{a}$ in this case.

By a straightforward approximation process it is seen that any a in R_S has a representation

$$a = \sum p^{n_i}a^{(m_i)} + b$$

where the n_i are increasing with i, the m_i are finite, b is algebraic over R_0 and the sum is, in general, infinite. The representative $\sum p^{n_i}a^{(m_i)} + b$ is called an *inertial form* of a with respect to $\bar{C}$ or simply an inertial form of a and is so named because it exhibits the derivation inertia of a as indicated in Theorem 2.6 below.

For a in K let $V(a) = n$ where $a = p^na_0$ and a_0 is a unit. The following generalizes a definition due to Neggers [**12**, p. 496].

(2.5) *Definition.* The *relative derivation inertia* $\Delta_{S/K_0}(a)$ or simply $\Delta(a)$, of a in R_S is given by

$$\Delta(a) = \min\{V(d(a))|d \text{ in } I(S/K_0, K)\}.$$

The following result was first proved by Neggers [**12**, p. 497, Theorem 1] in the case in which k_0 is contained in the maximal perfect subfield of k, though the published proof is in error.

(2.6) THEOREM. *If $\Sigma p^{n_i}a^{(m_i)} + b$ is an inertial form of a in R_S then*

$$\Delta(a) = \min_i\{n_i + m_i\} = \min\{V(d(a))|d \text{ in } I(S/K_0)\}$$

or, if $a = b$, $\Delta(a) = \infty$.

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Proof. If d is in $I(S/K_0, K)$ then by Proposition 2.3 and the definition of inertial representative $d(p^{n_i}a^{(m_i)})$ is in $p^{n_i+m_i}R$. Thus

$$\Delta(a) \geqq m = \min \{n_i + m_i\}.$$

We write $a = a_1 + a_2$ where

$$a_1 = \sum\{p^{n_i}a^{(m_i)}|n_i + m_i = m\}.$$

Since $\Delta(a_2) > m$, $\Delta(a) = m$ if $\Delta(a_1) = m$. The q terms in a_1 are indexed so that $n_1 < n_2 < \ldots < n_q$. Assume that

$$a^{(m_i)} = \sum a_{i,j_1,\ldots j_r}(c_1^{p^{m_i}})^{j_1} \ldots (c_r^{p^{m_i}})^{j_r}$$

as in (2.4) for $i = 1, \ldots, q$, and that $c_1^{p^{m_q}}$ occurs non-trivially in $a^{(m_q)}$. We define d in $I(S/K_0)$ by $d|_{K_0} = 0$; $d(c_1) = c_1$ and $d(c) = 0$ for c in C, $c \neq c_1$ [**5**, p. 38, Theorem 4]. Then

$$d(p^{n_q}a^{(m_q)}) = p^m j_1 a_{q,j_1\ldots j_r}(c_1^{p^{m_q}})^{j_1} \ldots (c_r^{p^{m_q}})^{j_r},$$

modulo $p^{m+1}R$. Noting that $m_q < m_i$ for $i < q$ we conclude that the residue of $p^{-m}d(a_1)$ is not zero. Thus $V(d(a_1)) = m$. Since d is in $I(S/K_0)$ the proof is complete.

Let a be in R_S with $\Delta(a) = m$. Then, for d in $I(S/K_0, K)$, $V(d(a)) = m$ if and only if the residue of $p^{-m}d(a)$ is not zero. This residue has the form

(2.7) $$g(\bar{d}) = \sum\{\bar{a}_{i,j}\bar{b}_{i,j}{}^{p^j-1}\bar{d}(\bar{b}_{i,j})|j = 0, \ldots, m; i = 1, \ldots, m_j;$$
$\bar{a}_{i,j} \in k_0(s^{p^{j+1}})$ for all i and j, and $\bar{b}_{1,j}, \ldots, \bar{b}_{m_j,j}$ are distinct non-trivial monomials of the form $\bar{c}_1{}^{i_1} \ldots \bar{c}_r{}^{i_r}$ with $\bar{c}_t$ in $\bar{C}$ and $0 \leqq i_t < p$ for $t = 1, \ldots, r\}$

where $\bar{d}$ is the map in Der $(s/k_0, k)$ induced by d.

(2.8) *Definition.* The map g in Der $(s/k_0, k)^*$ (asterisk denotes dual space) given by $\delta| \rightarrow g(\delta)$ where $g(\delta)$ is an expression of the form (2.7) is called a *simple lifting form*. If g is obtained from a in R_S in the manner described above we say g is a lifting form of a. The degree of g is the largest j to occur non-trivially in g. Thus if in (2.7) $\bar{a}_{i,m} \neq 0$ for some i then g has degree m.

(2.9) PROPOSITION. *If g is a simple lifting form of degree m and $t \geqq m$ then there is an integer a in R having lifting form g for which $\Delta(a) = t$.*

Proof. Let g be as in (2.7) and assume that $\bar{a}_j \neq 0$ where

$$\bar{a}_j = \sum\{\bar{a}_{i,j}\bar{b}_{i,j}{}^{p^j}|i = 1, \ldots, m_j\}.$$

This sum has the form (2.2). Let $a^{(j)}$ be an inertial representative of $\bar{a}_j$. Then

$$a = \sum\{p^{t-j}a^{(j)}|j = 0, \ldots, m\}$$

will have lifting form g and, by Theorem 2.6, $\Delta(a) = t$.

For future use we note that if a is an integer in S having lifting form of degree q then

(2.10) $\Delta(a) \geqq V(a) + q.$

3. Jacobian distinguished fields. Let $K \supset K_0$ be p-adic fields having residue fields $k \supset k_0$ and assume that k is finitely generated over k_0.

(3.1) *Definition.* A distinguished subfield s of k/k_0 is K/K_0 *Jacobian*, or simply Jacobian, if

$$\bar{I}(K/K_0)|_s = \text{Der}\,(s/k_0, k).$$

The following result is due to James K. Deveney [**1**].

(3.2) THEOREM. *Let $K \supset K_0$ be p-adic fields with residue fields $k \supset k_0$ and assume k/k_0 finitely generated. There is a distinguished subfield of k/k_0 which is Jacobian.*

In this section we shall prove the following complimentary result.

(3.3) THEOREM. *For any given distinguished subfield s of the finitely generated extension k/k_0 there are p-adic fields $K \supset K_0$ having residue fields $k \supset k_0$ such that s is K/K_0 Jacobian. If k_0 is separably algebraically closed in k, K can be constructed so K_0 is algebraically closed in K.*

Proof. We prove the claim of the last sentence. The rest then follows by replacing k_0 with its separable algebraic closure $k_0{}^c$ in k and using the facts that

$$\text{Der}\,(s/k_0, k) = \text{Der}\,(s/k_0{}^c, k)$$

and s is a distinguished subfield of $k/k_0{}^c$. Proof consists of an adaptation of the construction of K found in the proof of a related theorem of [**12**, p. 284, 285; proof of Theorem 3.6]. We will generally adopt the notation of the referenced proof, henceforth denoted T&H. Thus, let $U = \{u_1, \ldots, u_n\}$ be a p-basis for $k_0{}^{p^{-1}} \cap k/k_0$. Note that $s(U)$ is a distinguished subfield of $k\backslash k_0(U)$.

If $K_0 \subset K$ are p-adic fields having $k_0 \subset s$ as residue fields then K_0 is algebraically closed in K, since k_0 is algebraically closed in s (we are assuming that k_0 is separably algebraically closed in k). Choose t_1 in s and not in k_0 and let t in K_1 be a representative of t_1. We replace K_0', k_1, t and t_1 in T&H by K_0, s, t^{pe} and $t_1{}^{pe}$, respectively, where the exponent $e > 0$ will be selected later. By T&H there is a p-adic field $K_2 = K_1$-$(\omega_1, \ldots, \omega_n)$ with residue field $s(U)$ and K_0 is algebraically closed in K_2. We note that ω_i has minimal polynomial

$$X^p - v_i(1 + pt^{pe(n-+i2)!})$$

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over $K_1(\omega_1, \ldots, \omega_{i-1})$ and v_i is a representative in K_0 of u_i^p. Thus ω_i has residue u_i.

In T&H the exponent $(n - i + 2)!$ in the minimal function of ω_i ($p^e(n - i + 2)!$ in this paper) is chosen to insure that $1 + pt$ will not have the form ab^p with a in K_0 and b in K. The argument is obscured a bit by a typographical error on the first line of page 285 (read $\phi[k_0 \cap k_2^p]$ for $[k_0 \cap k_2^p]$ etc). Thus we can assume that $t_1^{p^e} \notin \phi[k_0 \cap k^p]$ and hence that $1 + pt^{p^e}$ does not have the form of ab^p with a in K_0 and b in K [**12**, p. 284, Lemma 3.7 and proof]. We will use this fact as in T&H.

(3.3) *Observation*. If the restriction to $K_1(\omega_1, \ldots, \omega_{i-1})$ of d in Der $(K_1(\omega_1, \ldots, \omega_i)/K_0)$ is integral then $p^{-e}d(\omega_i)$ is an integer. In particular d is integral.

Proof. Apply d to both sides of $\omega_i^p = v_i(1 + pt^{p^e(n-i-2)!})$.

Each δ in Der (s/k_0) lifts to a derivation d (necessarily integral) on K_1/K_0 since s/k_0 is separable [**12**, p. 286, Theorem 4.1]. By (3.3) the extension of d to K_2 is integral. Thus δ extends to a derivation on $s(U)$ which is induced.

Since $s(U)$ is a distinguished subfield of $k/k_0(U)$ there are elements $x_1, \ldots, x_m$ in k for which $k = s(U)(x_1, \ldots, x_m)$ and x_i has minimum function $X^{p^{e_i}} - a_i$ over $s(U)(x_1, \ldots, x_{i-1})$ where a_i is in $k_0(U)((s(x_1, \ldots, x_{i-1}))^{p^{e_i}}$ [**10**, p. 115, Folgerung]. We now choose $e = \max\{e_i | i = 1, \ldots, m\}$.

Assume that a p-adic field $K_{i,0} \supset K_2$ has been constructed having residue field $s(U)(x_1, \ldots, x_{i-1})$ so that 1) K_0 is algebraically closed in $K_{i,0}$ and 2) every d in Der $(K_{i,0}/K_0)$ whose restriction to K_1 is integral is itself integral. We have observed that $K_2 = K_{1,0}$ satisfies conditions 1) and 2). Since a_i is in $k_0(U)((s(x_1, \ldots, x_{i-1}))_p^{e_i})$, ω_i is a representative of u_i and, in view of (3.3), we can choose a representative y_i in $K_{i,0}$ of a_i with the property that if d in Der $(K_{i,0}/K_0)$ is integral then $V(d(y_i)) \geqq e_i$. We need the following.

(3.4) LEMMA. [**12**, p. 284, Lemma 3.7 and proof]. *If $K_0 \subset K$ are p-adic fields with K_0 algebraically closed in K then K_0 is also algebraically closed in $K(x)$ where x is a root of $X^p - c$ and c is a unit in K which does not have the form ab^p with a in K_0 and b in K.*

Let $K_{i,1} = K_{i,0}(z_{i,1})$ where $z_{i,1}$ is a root of $X^p - y_i$ unless y_i has the form ab^p as above in which case we choose $z_{i,1}$ to be a root of $X^p - y_i(1 + pt^{p^e})$. By (3.4) and the fact that $(1 + pt^{p^e})$ does not have the form ab^p, a in K_0 and b in $K_{i,0}$, it follows that K_0 is algebraically closed in $K_{i,1}$. Also, by (3.3) if d in Der $(K_{i,1}/K_0)$ has an integral restriction to $K_{i,0}$ then

$$V(d(z_{i,1})) \geqq e_i - 1.$$

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Suppose that $K_{i,j} = K_{i,0}(z_{i,1}, \ldots, z_{i,j})$, $1 \leqq j \leqq e_i - 1$ has been constructed so that 1) the residue field of $K_{i,j}$ is

$$s(U)(x_1, \ldots, x_{i-1}, x_i^{p^{e_i-j}})$$

and $z_{i,j}$ has residue $x^{p^{e_i-1}}$ 2) K_0 is algebraically closed in $K_{i,j}$ and 3) every integral derivation on $K_{i,j}$ over K_0 maps $z_{i,j}$ into $p^{e_i-j}R_{i,j}$. Let

$$K_{i,j+1} = K_{i,j}(z_{i,j+1})$$

where $z_{i,j+1}$ is a root of $X^p - z_{i,j}$ if $z_{i,j}$ does not have the form ab^p, a in K_0 and b in $K_{i,j}$. Otherwise $z_{i,j+1}$ is chosen to be a root of $X^p - z_{i,j}(1 + pt^{p^e})$. In either case properties 1), 2) and 3) above hold with $j + 1$ replacing j. Let $K_{i,e_i} = K_{i+1,0}$.

By repeating the above process we construct $K = K_{m,e_m}$ with residue field k and with the property

$$\bar{I}(K/K_0)|_s \supset \text{Der } (s/k_0).$$

Since $\bar{I}(K/K_0)|_s$ is a k space and

$$\dim_k(\bar{I}(K/K_0)|_s) \leqq \dim_k (\text{Der } (s/k_0, \text{k}))$$

it follows that

$$\bar{I}(K/K_0)|_s = \text{Der } (s/k_0, k).$$

Given p-adic fields $K \supset K_0$ with residue fields $k \supset k_0$, a K/K_0 Jacobian basis for k/k_0 is a transcendency basis $\{\bar{x}_1, \ldots, \bar{x}_r\}$ for k/k_0 with the property det $(\bar{d}_i(x_j)) \neq 0$ for some set of derivations $\{\bar{d}_1, \ldots, \bar{d}_r\}$ in $\bar{I}(K/K_0)$. Clearly, a given distinguished subfield s is Jacobian if and only if s possesses a separating transcendency basis over k_0 which is a Jacobian basis. If one separating transcendency basis of s is Jacobian then all are.

(3.5) *Example.* We construct p-adic fields $K \supset K_0$ with residue fields $k \supset k_0$, k/k_0 finitely generated, and exibit a Jacobian basis which is not a separating transcendency basis for any distinguished subfield.

Let P be a perfect field with $k_0 = P(\bar{x})$, $s = k_0(\bar{y})$ and $k = s(\bar{x}^{p^{-1}})$ where $\bar{x}$ and $\bar{y}$ are indeterminates. Let $K_0 \subset S$ be p-adic fields having $k_0 \subset s$ as residue fields. K_0 is algebraically closed in S since k_0 is algebraically closed in s. Choose x in K_0 and y in S representatives of $\bar{x}$ and $\bar{y}$ respectively and let $K = S(\theta)$ where θ is a root of $X^p - x(1 + py)$. We refer to T&H as follows to establish that K/K_0 is algebraically closed. Since s/k_0 is algebraically closed $\phi_{K_0,S}$ is trivial on $k_0 \cap s^p$ [**12**, p. 283] so $\bar{y}$ is not in $\phi_{K_0,S}(k_0 \cap s^p)$. Hence K_0 is algebraically closed in K [**12**, p. 284, Lemma 3.7].

Select d in $I(S/K_0)$ so that $d(y)$ is a unit [**5**, p. 38, Theorem 4] and let d' be the extension of d to Der (K/K_0). Then

$$d'(\theta) = xd(y)/\theta^{p-1}$$

so d' is in $I(K/K_0)$. Also $d'(\theta y^p)$ is a unit so $\{\bar{\theta}\bar{y}^p\}$ is a Jacobian basis for k/k_0. Since $(\bar{\theta}\bar{y}^p)^p$ is in $k_0(k^{p^2})$, $\{\bar{\theta}\bar{y}^p\}$ cannot be a separating transcendency basis for a distinguished subfield. For if $\{u\}$ is a separating transcendency basis for a distinguished subfield s then, by Proposition 2.1,

$$u^{p^m} \notin k_0(s^{p^{m+1}}) \quad \text{for } m \geqq 0.$$

However, for m large $k_0(s^{p^m}) = k_0(k^{p^m})$ [**3**, p. 288, Proposition 1]. Thus, for m large u^{p^m} is not in $k_0(k^{p^{m+1}})$. Thus $\{\bar{\theta}\bar{y}^p\}$ is not a separating transcendency basis for a distinguished subfield of k/k_0.

4. Lifting forms. As in Section 3, we assume k/k_0 finitely generated. Let $K \supset S \supset K_0$ be p-adic fields with residue fields $k \supset s \supset k_0$, s being a distinguished subfield of k/k_0. Assume that $k = k_0(\bar{\theta})$. Then $K = S(\theta)$ where θ has residue $\bar{\theta}$. Let

$$(4.1) \quad f(x) = X^{p^n} + a_{p^n-1}X^{p^n-1} + \ldots + a_0$$

be the minimum function of θ over S and let

$$m = \min \{\Delta(a_i) | 0 \leqq i \leqq p^n - 1\}.$$

We use the convention

$$f^d(\theta) = d(a_{p^n-1})\theta^{p^n-1} + \ldots + d(a_0).$$

(4.2) LEMMA. Min $\{V(f^d(\theta)) | d$ *in* $I(S/K_0)\} = m$.

Proof. Clearly $V(f^d(\theta)) \geqq m$ for d in $I(S/K_0)$. Choose d in $I(S/K_0)$ and a_j so that $V(d(a_j)) = m$. Then

$$f^d(\theta) = \sum d(a_i)\theta^i = p^m g(\theta)$$

and, by choice of d, $g(\theta)$ is a unit.

(4.3) LEMMA. *If s is not Jacobian then* i) $V(f'(\theta)) > m$ *and* ii) δ *is in* Der $(s/k_0, k) \cap \bar{I}(K/K_0)|_s$ *if and only if some d in $I(S/K_0, K)$ which induces δ has the property $V(f^d(\theta)) > m$. If one d which induces δ has the property all do.*

Proof. If s is not Jacobian there is a d in Der (K/K_0) which is not integral whereas $d|_S$ is integral. Thus $d(\theta) \notin R$. Since $d(\theta) = -f^d(\theta)/f'(\theta)$ and, by Lemma 4.2, $V(f^d(\theta)) \geqq m$, it follows that $V(f'(\theta)) > m$.

If δ in Der $(s/k_0, k)$ lifts to d in $I(S/K_0, K)$ and d extends integrally to K then

$$V(f^d(\theta)) \geqq V(f'(\theta)) > m.$$

Conversely, if d in $I(S/K_0, K)$ induces δ and $V(f^d(\theta)) > m$ then, by Lemma 4.2,

$$V(f^d(\theta)) > V(f^{d_1}(\theta)) = m$$

for some d_1 in $I(S/K_0, K)$. Let $f^{d}(\theta) = p^{t}u$ and $f^{d_1}(\theta) = p^{m}v$ where u and v are units in R. Then for $d_2 = d - p^{t-m}uv^{-1}d_1$ we have

$$f^{d_2}(\theta) = f^{d}(\theta) - p^{t-m}uv^{-1}f^{d_1}(\theta) = 0.$$

Since $t > m$, d_2 induces δ and extends integrally to K.

Finally, if d and d_1 in $I(S/K_0, K)$ both induce δ then $d_1 - d = pd_2$ and d_2 is integral. Thus

$$f^{d_1}(\theta) = f^{d}(\theta) + pf^{d_2}(\theta)$$

and if $V(f^d(\theta)) > m$ then $V(f^{d_1}(\theta)) > m$ as well since $V(f^{d_2}(\theta)) \geqq m$.

Our immediate objective is the characterization of those subspaces of Der $(s/k_0, k)$ of the form $\bar{I}(K/K_0)|_s$. Lemma 4.3 and the following observation suggest the characterization provided in Theorem 4.4. If d is in $I(S/K_0, K)$ then $V(f^d(\theta)) > m$ if and only if the residue of $p^{-m}f^d(\theta)$ is zero. This residue is

$$\{\sum \bar{\theta}^i g_{(r_i)}(\bar{d}) | i = i_1, \ldots, i_q\}$$

where $g_{(r_i)}$ is the lifting form of a_i (see (2.7)) and $\{a_{i_1}, \ldots, a_{i_q}\}$ are the coefficients of $f(X)$ having minimum inertial index m. Thus we have the following definition under the continuing assumption that $k = s(\bar{\theta})$ and $[k:s] = p^n$. Let $g_{(r)}$ be a simple lifting form of degree r and let J be a non-empty subset of the non-negative integers $< p^n$. Given

$$\{g_{(r_i)} | i \in J, r_i < n - 1 \text{ and } r_i < V(i) \text{ for all } i \text{ in } J\}$$

the map

$$L = \sum \{\bar{\theta}^i g_{(r_i)} | i \in J\}$$

is a lifting form of s/k_0 into k or simply a lifting form. The zero map of Der $(s/k_0, k)^*$ is the trivial lifting form. The set of all lifting forms is $\mathcal{L}(s/k_0, k)$. Note that if $n = 1$ there are no non-trivial lifting forms.

(4.4) THEOREM. *If k/k_0 has a cosimple distinguished subfield and s is any distinguished subfield of k/k_0 then a k subspace M of* Der $(s/k_0, k)$ *has the form $\bar{I}(K/K_0)|_s$ for some pair of p-adic fields $K \supset K_0$ with residue fields $k \supset k_0$ if and only if $M =$* kernel (L) *for some L in $\mathcal{L}(s/k_0, k)$.*

Proof. If k/k_0 has a cosimple distinguished subfield then every distinguished subfield is cosimple [**8**]. Thus k is a simple extension of s. Suppose that $M = \bar{I}(K/K_0)|_s$ for some pair of p-adic fields $K \supset K_0$. Let S be an intermediate p-adic field with residue field s [**11**, p. 434, Theorem 12]. Then $K = S(\theta)$ and $k = s(\bar{\theta})$ for some unit θ in K having residue $\bar{\theta}$. If s is Jacobian then M is the kernel of the trivial form. Assume s not Jacobian and let (4.1) be the minimum function of θ over S. Thus a_i is in pR_S for $i \neq 0$ and a_0 in R_S has residue $\bar{a}_0$, $X^{p^n} - \bar{a}_0$ being the minimum polynomial of $\bar{\theta}$ over s. Let $A = \{a_{i_1}, \ldots, a_{i_q}\}$ be the set of those

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coefficients of $f(X)$ having minimum relative derivation inertia m. By Lemma 4.3 if d is in $I(S/K_0, K)$ there is a d_1 in $I(S/K_0, K)$ which extends integrally to K and has the same induced derivation $\bar{d}$ if and only if $V(f^d(\theta)) > m$, or, if and only if

$$L(\bar{d}) = \sum\{\bar{\theta}^i g_{(r_i)}(\bar{d})|i = i_1, \ldots, i_q\} = 0$$

where $g_{(r_i)}$ is the lifting form of a_i (see (2.7)). We refer to L as the lifting form of $f(X)$. It is shown below that L is in $\mathscr{L}(s/k_0, k)$.

Note that

$$V(f'(\theta)) = \min\{V(p^n), V(a_{p^n-1}) + V(p^n - 1), \ldots, V(a_1) + V(1)\}.$$

Let $t = V(f'(\theta))$. Thus, $V(a_i) + V(i) \geqq t$ for $i > 0$. By Lemma 4.3, $t > m$. By (2.10) $m \geqq V(a_i) + r_i$ for a_i in A. Thus $V(i) > r_i$ for each term $\theta^i g_{(r_i)}$ in L with $i \neq 0$. It follows from the last two inequalities that if a_i is in A and $i \neq 0$ then $n - 2 \geqq r_i$ since $V(a_i) > 0$ and $n \geqq t > m$. Since $\bar{a}_0$ is in $k_0(s^{p^n})$, $a_0 = a_0' + pa_0''$ where a_0' is an inertial representative of $\bar{a}_0$, and $\Delta(a_0') \geqq n$. It follows, since $\Delta(a_0) \geqq m$, that $\Delta(a_0'') \geqq m - 1$ and if $\Delta(a_0) = m$ then $\Delta(a_0'') = m - 1$ and, by (2.10), the lifting form of a_0'' has degree $r_0 \leqq m - 1 < n - 1$. Thus L is in $\mathscr{L}(s/k_0, k)$.

Conversely, let L be in $\mathscr{L}(s/k_0, k)$. In view of the above discussion we need $f(X)$ monic with coefficients in R_S, $f(X)$ induces the minimum polynomial $X^{p^n} - \bar{a}_0$ of $\bar{\theta}$ over s, has lifting form L, and has the property $m < V(f'(\theta))$, m being as above the minimum of the derivation inertias of the coefficients of $f(X)$. Let

$$L = \sum\{\bar{\theta}^i g_{(r_i)}|i = 0, \ldots, p^n - 1\}$$

and let $i = q \neq 0$. If $g_{(r_q)}$ is non-trivial and has the form (2.7) we choose an inertial representative $a_q^{(j)}$ for each summand

$$\sum\{\bar{a}_{i,j}\bar{b}_{i,j}^{p^j}|i = 1, \ldots, m_j\}$$

and let

$$a_q = p^t \sum p^{n-t-j-1} a_q^{(j)}$$

where $t = n - V(q) > 0$. Now

$$n - t - j - 1 \geqq n - t - r_q - 1 = V(q) - r_q - 1$$

and, by definition of $\mathscr{L}(s/k_0, k)$,

$$V(q) - r_q - 1 \geqq 0.$$

Thus $V(a_q) > 0$. Also, $V(a_q) + r_q = n - 1$, since r_q is the maximum value of j occurring in the definition of a_q. Hence

$$V(a_q) + V(q) > n - 1.$$

Thus, choosing $a_q = 0$ if $g_{(r_q)}$ is the trivial form, we conclude, in particular, that $V(f'(\theta)) = n$.

If $g_{(r_0)}$ is non-trivial and given by (2.7) we let

$$a_0'' = p^t \sum^{n-t-j-1} a_0^{(j)}$$

as in the definition of a_q. Note that $a_0'' \in pR_S$ since $r_0 \leqq n - 1$. If $g_{(r_0)}$ is the trivial form, $a_0'' = 0$. We choose a_0' to be an inertial representative of $-\bar{\theta}^{p^n}$, the latter being in $k_0(s^{p^n})$, and let $a_0 = a_0' + a_0''$. By construction of a_q, $q \geqq 0$, if d is in $I(S/K_0K)$ the residue of $p^{-(n-1)}\theta^q d(a_q)$ is $\bar{\theta}^q g_{(r_q)}(\bar{d})$. Thus, if $g_{(r_q)}$ is non-trivial $\Delta(a_q) = n - 1$ and L is the lifting form of $f(X)$. Also, since $V(a_i) > 0$ for $i > 0$ and $\bar{a}_0 = -\bar{\theta}^{p^n}$, $f(X)$ induces the minimum polynomial of $\bar{\theta}$ over s. Note too that

$$m = n - 1 < V(f'(\bar{\theta})) = n$$

where θ is a root of $f(X)$. Thus we let $K = S(\theta)$. The residue field of K is k and by Lemma 4.3 a given δ in Der $(s/k_0, k)$ is in $\bar{I}(S/K_0, K)|_s$ if and only if $L(\delta) = 0$.

(4.5) COROLLARY. *If* $[k:s] = p$ *every distinguished subfield is Jacobian.*

Proof. This is easily shown directly. It is also a consequence of Theorem 4.4 since, if $n = 1$, there are no non-trivial lifting forms.

5. Characterization of $\bar{I}(K/K_0)$. Throughout this section it is assumed that k is a finitely generated extension of k_0.

(5.1) PROPOSITION. *A distinguished subfield s is Jacobian if and only if the restriction map $\rho:\delta \rightarrow \delta|_s$ of $\bar{I}(K/K_0)$ to $\bar{I}(K/K_0)|_s$ is bijective. If s is cosimple then* Der $(k/s) \subset \bar{I}(K/K_0)$ *if and only if s is not Jacobian in which case the following is split exact.*

$$0 \rightarrow \text{Der }(k/s) \xrightarrow{\iota} \bar{I}(K/K_0) \xrightarrow{\rho} \bar{I}(K/K_0)|_s \rightarrow 0.$$

Proof. By definition s is Jacobian if and only if

$$\bar{I}(K/K_0)|_s = \text{Der }(s/k_0, k)$$

which, since

$$\dim_k \bar{I}(K/K_0) = \dim_k \text{Der }(s/k_0, k),$$

is equivalent to ρ being bijective. If s is cosimple

$$\dim_k (\text{Der }(k/s)) = 1.$$

Clearly,

$$\text{kernel } \rho = \bar{I}(K/K_0) \cap \text{Der }(k/s).$$

Hence s is Jacobian if and only if Der $(k/s) \not\subset \bar{I}(K/K_0)$ and if Der $(k/s) \subset \bar{I}(K/K_0)$ then

$$\text{kernel } \rho = \text{Der } (k/s).$$

Proposition 5.1 and Theorem 4.4 are combined to obtain the following.

(5.2) THEOREM. *If k is a simple extension of some distinguished subfield of k/k_0 and M is a subspace of* Der (k/k_0) *containing* Der (k/s) *for a distinguished subfield s then M has the form $\bar{I}(K/K_0)$ for some pair of p-adic fields $K \supset K_0$ having $k \supset k_0$ as residue fields if and only if $M|_s$ is the kernel of a non-trivial lifting form.*

Proof. If $M = \bar{I}(K/K_0)$ then by Theorem 4.4 $M|_s$ is the kernel of a lifting form. By Proposition 5.1 s is not Jacobian so the lifting form is non-trivial.

To prove the converse let $M_0 = \{d \in M | d(\bar{\theta}) = 0\}$ where $k = s(\theta)$. If d is in M and not in M_0 then for $d_1 \neq 0$ in Der (k/s)

$$d_2 = d - d(\bar{\theta})d_1(\bar{\theta})^{-1}d_1$$

is in M_0 and $d_2|_s = d|_s$. Thus $M_0|_s = M|_s$. Also, since k/s is simple and Der $(k/s) \subset M$ if follows that

$$\dim_k M = \dim_k (M|_s) + 1$$

and so

$$M = M_0 + \text{Der } (k/s).$$

By Theorem 4.4 there are p-adic fields $K \supset K_0$ having $k \supset k_0$ as residue fields such that $\bar{I}(K/K_0)|_s = M|_s$. The kernel of a non-trivial lifting form is a proper subspace of Der $(s/k_0, k)$ by Theorem 2.6, Proposition 2.9 and the remarks following the proof of Theorem 2.6. Hence s is not Jacobian. Thus

$$\text{Der } (k/s) \subset \bar{I}(K/K_0).$$

It follows that

$$M_0 \subset \bar{I}(K/K_0) \quad \text{or} \quad M \subset \bar{I}(K/K_0).$$

Since M and $\bar{I}(K/K_0)$ have the same dimension, $M = \bar{I}(K/K_0)$.

The following facts relate to our next result which addresses the case not covered in Theorem 5.2. The largest subfield of k/k_0 in which $k_0(s^{p^i})$ is distinguished, where s is a distinguished subfield of k/k_0, is

$$k_0(k^{(i)}) = \{a \in k | a^{p^m} \in k_0(k^{p^{m+i}}) \text{ for some } m \geqq 0\}$$

[**3**, p. 288, Theorem 2]. We shall use a connection between $k_0(k^{(1)})$ and separating transcendency bases of distinguished subfields called distinguished transcendency bases.

(5.3) PROPOSITION [8]. *There is a distinguished transcendency basis containing a if and only if a is not in $k_0(k^{(1)})$. Every distinguished transcendency basis is p-independent over $k_0(k^{(1)})$.*

Assume p-adic fields $K \supset K_0$ with residue fields $k \supset k_0$ as given and let $k_{\bar{I}}$ be the field of constants of $\bar{I}(K/K_0)$.

(5.4) PROPOSITION. *If every distinguished subfield of k/k_0 is Jacobian then $k_{\bar{I}} \subset k_0(k^{(1)})$. If transcendency degree $k/k_0 = 1$ then every distinguished subfield of k/k_0 is Jacobian if $k_{\bar{I}} \subset k_0(k^{(1)})$.*

Proof. If $k_{\bar{I}} \not\subset k_0(k^{(1)})$ there is an a in k, a not in $k_0(k^{(1)})$ such that $\delta(a) = 0$ for every δ in $\bar{I}(K/K_0)$. By Proposition 5.3 there is a distinguished transcendency basis T containing a. Clearly, the distinguished subfield containing T is not Jacobian. Let s be a distinguished subfield of k/k_0 and, assuming transcendency degree k/k_0 to be 1, let $\{a\}$ be a separating transcendency basis for s/k_0. Then a is not in $k(k^{(1)})$ by Proposition (5.3). Hence s is not in $k_{\bar{I}}$, if $k_{\bar{I}} \subset k_0(k^{(1)})$. It follows that $\{a\}$ is a Jacobian basis and s is Jacobian.

The following example illustrates the fact that in general, the property, every distinguished subfield is Jacobian, is not determined by the structure of k/k_0 alone but depends also on the p-adic over fields.

(5.5) *Example.* Let P be a perfect field. Using indeterminates $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{w}$ we define

$$k_0 = P(\bar{x}, \bar{y}), \quad s = k_0(\bar{z}\bar{y}^{p-1}, \bar{w}) \quad \text{and} \quad k = s(\bar{x}_p{}^{-2}).$$

Let p-adic fields $K_0 \subset S$ have $k_0 \subset s$ as residue fields. We note that

$$k_0(k^{(1)}) = k_0(\bar{x}^{p^{-2}}, \bar{w}^p, \bar{z}^p)$$

since $[k; k_0(\bar{x}^{p^{-2}}, \bar{w}^p, \bar{z}^p)] = p^2$,

$$k_0(k^{(1)}) \supset k_0(\bar{x}^{p^{-2}}, \bar{w}_p, \bar{z}^p),$$

and

$$[k\colon k_0(k^{(1)})] \geqq p^2$$

[3, p. 290, Theorem 11 and proof]. Let $K = S(\theta_1)$ where θ_1 is a root of $X^{p^2} - x(1 + pw^{p^2})$, where x in K_0 and w in S are representatives respectively of $\bar{x}$ and $\bar{w}$. Clearly, if d is in $I(K/K_0)$ then $d(\theta_1)$ is in pR and $\bar{\theta}_1$ is in $k_{\bar{I}}$. Hence $k_{\bar{I}} \supset k_0(k^{(1)})$. Since

$$\bar{I}(K/K_0) \subset \text{Der}\ (k/k_0(k^{(1)}))$$

and

$$\dim_k \bar{I}(K/K_0) = \dim_k \text{Der}\ (k/k_0(k^{(1)})) = 2$$

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we have

$$\bar{I}(K/K_0) = \text{Der}\,(k/k_0(k^{(1)})).$$

Thus every distinguished transcendency basis of k/k_0 is a p-basis of $k/k_0(k^{(1)})$. It follows that every distinguished subfield of k/k_0 is Jacobian.

Let θ_2 be a root of $X^{p^2} + pwX^p - x$ and let $K = S(\theta_2)$. If d in $I(S/K_0, K)$ induces δ in Der $(s/k_0, k)$ where δ is given by $\delta(\bar{w}) = 1$, $\delta(\bar{z}\bar{y}^{p-1}) = 0$, then

$$p^2(\theta_2^{p^2-1} + w\theta_2^{p-1})d(\theta_2) = -pd(w) \bmod p^2.$$

Thus $d(\theta_2)$ is not an integer and s is not Jacobian.

The next example illustrates the need for the condition transcendency degree $(k/k_0) = 1$ in the last sentence of Proposition 5.4.

(5.6) *Example.* Let P be a perfect field having characteristic $p = 3$, and let $\bar{x}$, $\bar{y}$, $\bar{z}$, be indeterminates. We define $k_0 = P(\bar{x})$, $s = k_0(\bar{y}, \bar{z})$ and $k = s(\theta)$ where $\bar{\theta}$ is a root of $X^{p^3} + \bar{x}$. Also, $K = S(\theta)$ where $K_0 \subset S$ are p-adic fields having $k_0 \subset s$ as a residue fields and θ is a root of $X^{p^3} + p^2yX^p + (1 + pz^p)x$ with x in K_0, y and z in S being respectively representative of $\bar{x}$, $\bar{y}$, and $\bar{z}$. Thus the residue field of K is k.

Note that δ in Der (k/k_0) is in $\bar{I}(K/K_0)$ if and only if

(5.7) $\quad \delta(\bar{y})\bar{\theta}^p + \bar{z}^{p-1}\bar{x}\delta(\bar{z}) = 0.$

Hence s is not Jacobian.

Let δ in Der (s/k_0) be given by $\delta(\bar{y}) = \bar{z}^2\bar{x}$, $\delta(\bar{z}) = -\bar{\theta}^p$. Choose δ_1 and δ_2 in $\bar{I}(K/K_0)$ by the conditions $\delta_1|_s = \delta$, $\delta_2|_s = 0$ and $\delta_2(\bar{\theta}) = 1$. Then $\{\delta_1, \delta_2\}$ is a basis for $\bar{I}(K/K_0)$, $k_{\bar{I}} = k_{\delta_1} \cap k_{\delta_2}$ and $k_{\delta_2} = s(\bar{\theta}^p)$. If α is in $k_{\bar{I}}$ then

$$\alpha = \sum\{a_i\bar{\theta}^{3i}|a_i \in s, i = 0, \dots, 8\}$$

since α is in k_{δ_2}. Also

$$0 = \delta_1(\alpha) = \sum\{\delta_1(a_i)\bar{\theta}^{3i}|i = 0, \dots, 8\}.$$

Writing $a_{i,y}$ for $\partial a_i/\partial\bar{y}$ and $a_{i,z}$ for $\partial a_i/\partial\bar{z}$ we then have

$$0 = \sum\{a_{i,y}\bar{\theta}^{3i}|i = 0, \dots, 8\}\bar{z}^2\bar{x} + \sum\{a_{i,z}\bar{\theta}^{3i}|i = 0, \dots, 9\}\bar{\theta}^3$$

and hence

(5.8) $\quad a_{8,z} = -\bar{z}^2a_{0,y}, \; \bar{z}^2\bar{x}a_{i,j} = -a_{i-1,z}$ for $i = 1, \dots, 8.$

To exploit (5.8) we write

$$a_i = \sum\{c_{i,j,l}\bar{y}^j\bar{z}^l|0 \leqq j, l < 3, c_{i,j,l} \in k_0(s^3)\}$$

obtaining

$$\sum\{lc_{8,j,l}\bar{y}^j\bar{z}^{l-1}|l \neq 0\} = -\bar{z}^2\sum\{lc_{i-1,j,l}\bar{y}^j\bar{z}^{l-1}|l \neq 0\}.$$

A straightforward analysis of these equations yields $c_{i,j,l} = 0$ unless $j = l = 0$ for $i = 0, \ldots, 8$. Thus, a_i is in $k_0(s^p)$ for all i or α is in $k_0(k^p)$ and

$$k_{\bar{I}} = k_0(k^p) \subset k_0(k^{(1)}).$$

If transcendency degree $(k/k_0) = r$ and $[k:s] = p^n$ for a cosimple distinguished subfield s then

$$[k:k_0(s^p)] = [k:s][s:k_0(s^p)] = p^{n+r} \text{ and}$$
$$[k:k_0(k^{(1)})] \geqq p^r$$

[**3**, p. 290, Theorem 11]. Also, since k is a simple extension of s,

$$[k:k_0(k^p)] = p^{r+1}.$$

It follows that $[k:k_0(k^{(1)})] = p^r$ or p^{r+1} since $k_0(k^{(1)}) \supset k_0(k^p)$.

Case 1. $[k:k_0(k^{(1)})] = p^{n+1}$. If every distinguished subfield is Jacobian then by Proposition 5.4 $k_{\bar{I}} \subset k_0(k^{(1)})$ and since $k_0(k^p) = k_0(k^{(1)})$ it follows that

$$k_{\bar{I}} = k_0(k^p) = k_0(k^{(1)}).$$

The following example illustrates this case.

(5.9) *Example*. Let P be a perfect field with $k_0 = P(\bar{x}, \bar{y})$, $s = k_0(\bar{z})$ and $k = s(\bar{\theta})$ where $\bar{\theta}^p = \bar{x} + \bar{y}\bar{z}^p$, $\bar{x}$, $\bar{y}$, and $\bar{z}$ being indeterminates. If $K_0 \subset S \subset K$ are p-adic fields with residue fields $k_0 \subset s \subset k$ then $K = S(\theta)$ where θ has residue $\bar{\theta}$. Let $f(X)$ be the minimum polynomial of θ over S. Since the induced polynomial $\bar{f}(X) = X^p - \bar{\theta}^p$ and $\bar{\theta}^p$ is in $k_0(s^p)$ it follows that $f^d(\theta)/f'(\theta)$ is an integer for d in $I(S/K_0, K)$. Hence s is Jacobian. We have shown that if $[k:s] = p$ then s is Jacobian. Since $k_0(k^{(1)}) = k_0(k^p)$ in this case [**3**, p. 288, Contention] it follows that

$$k_0(k^p) = k_{\bar{I}} = k_0(k^{(1)}).$$

Case 2. $[k:k_0(k^{(1)})] = p^r$. In this case

$$[k_0(k^{(1)}):k_0(k^p)] = p$$

so, if every distinguished subfield is Jacobian then either (a) $k_{\bar{I}} = k_0(k^{(1)})$ in which case

$$\bar{I}(K/K_0) = \text{Der}\ (k/k_{\bar{I}})$$

since $\dim_k \text{Der}\ (k/\bar{I}) = \dim_{kI}$ or (b) $k_{\bar{I}} = k_0(k^p)$.

(5.10) *Example*. The following construction illustrates both cases (a) and (b). Let P be a perfect field, let $\bar{x}$ and $\bar{y}$ be indeterminates and define $k_0 = P(\bar{x})$, $s = P(\bar{x}, \bar{y})$ and $k = P(\bar{x}^{p^{-1}}, \bar{y})$. Since $[k:s] = p$ every dis-

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tinguished subfield is Jacobian. We note that

$$k_0(k^{(1)}) = k_0(\bar{x}^{p-1}, \bar{y}^p) \supseteq k_0(k^p)$$

since

$$[k:k_0(k^{(1)})] \geqq p \text{ [3, p. 290, Theorem 11]},$$
$$k_0(k^{(1)}) \supset p(\bar{x}^{p-1}, \bar{y}^p) \text{ and}$$
$$[k:P(\bar{x}^{p-1}, \bar{y}^p)] = p.$$

Let $K_0 \subset S$ be p-adic fields having $k_0 \subset s$ as residue fields. We construct K in two ways. Let x in K_0 and y in S be representatives of $\bar{x}$ and $\bar{y}$ respectively. In case (a) $K = S(\theta)$ where θ is a root of $X^p - x$. Then $d(\theta) = 0$ for all d in $I(K/K_0)$. Hence $\bar{\theta} = \bar{x}^{p-1}$ is in $k_{\bar{I}}$ so $k_{\bar{I}} = k_0(k^{(1)})$. For case (b) let $K = S(\theta_1)$ where θ_1 is a root of $X^p - (x + py)$. Since s is Jacobian there is a d in $I(K/K_0)$ such that $d(y)$ is a unit. Then $d(\theta_1) = d(y)/\theta_1^{p-1}$ is a unit so $\bar{x}^{p-1}$ is not in $k_{\bar{I}}$ and $k_{\bar{I}} = k_0(k^p)$.

The final example illustrates that in general $k_{\bar{I}}$ does not determine $\bar{I}$.

(5.11) *Example.* Let $k_0 \subset s \subset k$ be the fields of Example 5.9. Let $K_0 \subset S$ be p-adic fields over $k_0 \subset s$. Choose representatives x and y in K_0 and z in S of $\bar{x}$, $\bar{y}$, and $\bar{z}$ respectively. Let $K_1 = S(\theta_1)$ and $K_2 = S(\theta_2)$ where θ_1 and θ_2 are respectively roots of $X^p - (x + yz^p)$ and $X^p - pzX - (x + yz^p)$. Let δ in Der (s/k_0) be given by $\delta(\bar{z}) = 1$ and assume that d in Der (S/K_0) induces δ. If d_1 and d_2 denote the respective extensions of d to K_1 and K_2 then

$$d_1(\theta_1) = yz^{p-1}d_1(z)/\theta_1^{p-1}$$

so

$$\bar{d}_1(\bar{\theta}_1) = \bar{y}\bar{z}^{p-1}/\bar{\theta}_1^{p-1}.$$

Assume that $\bar{d}_1$ is in $\bar{I}(K_2/K_0)$. For d_2 in $I(K_2/K_0)$ we have

$$d_2(\theta_2) = (\theta_2 d_2(z) + yz^{p-1}d_2(z))/(u_2^{p-1} - z)$$

or

$$\bar{d}_2(\bar{\theta}_2) = \bar{\theta}_2 + \bar{y}\bar{z}^{p-1}/(\bar{\theta}_2^{p-1} - \bar{z}).$$

Equating $\bar{d}_1(\bar{\theta}_1)$ and $\bar{d}_2(\bar{\theta}_2)$ yields $\bar{x} = -\bar{y}\bar{z}^p$ which is false. Hence $\bar{I}(K_2/K_0) \neq \bar{I}(K/K_0)$.

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