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A MINIMAX INEQUALITY WITH APPLICATIONS TO EXISTENCE OF EQUILIBRIUM POINTS

KOK-KEONG TAN AND ZIAN-ZHI YUAN

A new minimax inequality is first proved. As a consequence, five equivalent fixed point theorems are formulated. Next a theorem concerning the existence of maximal elements for an L_C-majorised correspondence is obtained. By the maximal element theorem, existence theorems of equilibrium points for a non-compact one-person game and for a non-compact qualitative game with L_C-majorised correspondences are given. Using the latter result and employing an "approximation" technique used by Tulcea, we deduce equilibrium existence theorems for a non-compact generalised game with L_C correspondences in topological vector spaces and in locally convex topological vector spaces. Our results generalise the corresponding results due to Border, Borglin-Keiding, Chang, Ding-Kim-Tan, Ding-Tan, Shafer-Sonnenschein, Shih-Tan, Toussaint, Tulcea and Yannelis-Prabhakar.

1. INTRODUCTION

In [**22, 23**], Tuclea proved some very general equilibrium existence theorems for generalised games (abstract economies) with correspondences defined on a compact strategy (choice) set of players (agents). These theorems generalised most known equilibrium theorems on compact generalised games due to Borglin and Keiding [**3**], Shafer and Sonnenschein [**16**], Toussaint [**21**] and Yannelis and Prabhakar [**26**].

In this paper, we shall first introduce the notions of correspondence of class L_C, L_C-majorant of ϕ at x and L_C-majorised correspondences which generalise the corresponding definitions of Ding and Tan [**7**]. Next, a new minimax inequality is proved which generalises the corresponding result of Shih and Tan [**18**]. As a consequence, five equivalent fixed point theorems are formulated which generalise the corresponding results of Ben-El-Mechaiekh, Deguire and Granas [**1**], Border [**2**], Ding and Tan [**7, 8**], Mehta and Tarafdar [**15**], Shih and Tan [**18**] and Tarafdar [**19**]. An existence theorem of maximal elements for an L_C-majorised correspondence is obtained which generalises the corresponding results of Borglin and Keiding [**3**], Ding and Tan [**7**], Toussaint [**21**], Tulcea [**22**] and Yannelis and Prabhakar [**26**]. By applying earlier results, we prove equilibrium existence theorems for a non-compact one-person game and for a non-compact qualitative game with an infinite number of players and with L_C-majorised

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correspondences. The latter result is applied to obtain an equilibrium existence theorem for a non-compact generalised game with an infinite number of players and with L_C correspondences. Finally, by employing an "approximation" technique used by Tulcea [22], we also give some equilibrium existence theorems for a one-person game and for a generalised game in locally convex spaces.

Now we give some notation. The set of all real numbers is denoted by $\mathbf{R}$. Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A, by $\mathcal{F}(A)$ the family of all non-empty finite subsets of A, by $\text{int}_X(A)$ the interior of A in X and by $cl_X(A)$ the closure of A in X. A is said to be compactly open in X if for each non-empty compact subset C of X, $A \cap C$ is open in C. If A is a subset of a vector space, we shall denote by coA the convex hull of A. If A is a non-empty subset of a topological vector space E and $S, T : A \to 2^E$ are correspondences, then coT, $T \cap S : A \to 2^E$ are correspondences defined by $(coT)(x) = coT(x)$ and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively. If X and Y are topological spaces and $T : X \to 2^Y$ is a correspondence, the Graph of T, denoted by Graph T, is the set $\{(x,y) \in X \times Y : y \in T(x)\}$ and the correspondence $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x,y) \in cl_{X\times Y}\text{Graph } T\}$ (the set $cl_{X\times Y}$ Graph T is called the adherence of the graph of T), and $clT : X \to 2^Y$ is defined by $clT(x) = cl_Y(T(x))$ for each $x \in X$. It is east to see that $clT(x) \subset \overline{T}(x)$ for each $x \in X$.

Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then (1) ϕ is said to be of class $L_{\theta,C}$ if (a) for each $x \in X$, $co\phi(x) \subset Y$ and $\theta(x) \notin co\phi(x)$ and (b) there exists a correspondence $\psi : X \to 2^Y$ such that for each $x \in X$, $\psi(x) \subset \phi(x)$ and for each $y \in Y$, $\psi^{-1}(y)$ is compactly open in X and $\{x \in X : \phi(x) \neq \emptyset\} = \{x \in X : \psi(x) \neq \emptyset\}$; (2) $(\phi_x, \psi_x; N_x)$ is an $L_{\theta,C}$-majorant of ϕ at x if $\phi_x, \psi_x : X \to 2^Y$ and N_x is an open neighbourhood of x in X such that (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin co\phi_x(z)$, (b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $co\phi_x(z) \subset Y$ and (c) for each $y \in Y$, $\psi_x^{-1}(y)$ is compactly open in X; (3) ϕ is said to be $L_{\theta,C}$-majorised if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $L_{\theta,C}$-majorant (ϕ_x, ψ_x, N_x) of ϕ at x such that for any non-empty finite subset A of the set $\{x \in X : \phi(x) \neq \emptyset\}$, we have $\{z \in \bigcap_{x\in A} N_x : \bigcap_{x\in A} co\phi_x(z) \neq \emptyset\} = \{z \in \bigcap_{x\in A} N_x : \bigcap_{x\in A} co\psi_x(z) \neq \emptyset\}$.

It is clear that every correspondence of class $L_{\theta,C}$ is $L_{\theta,C}$-majorised. We note that our notions of the correspondence ϕ being of class $L_{\theta,C}$ and $L_{\theta,C}$-majorised correspondence generalise the notions of correspondence of class $L_{\theta,F}$ and $L_{\theta,F}$-majorised correspondences and $\mathcal{L}_\theta^*$ and $\mathcal{L}_\theta^*$-majorised correspondence respectively introduced by Ding and Tan [7] and Ding, Kim and Tan in [8] which in turn generalise the notions of $\phi \in C(X,Y,\theta)$ and C-majorised correspondence respectively introduced by Tulcea in [22]. In this paper, we shall deal mainly with either the case (I) $X = Y$ and X is a non-empty convex subset of the topological vector space E and $\theta = I_X$, the identity map on X, or the case (II) $X = \prod_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a non-empty convex subset of a topological vector space. In both cases (I) and (II), we shall write L_C in place of $L_{\theta,C}$.

2. A New minimax inequality

The proof of Lemma 1 of Fan in [10] actually produces the following slight improvement which is observed in [6, Lemma 3].

Lemma 2.1. *Let X and Y be non-empty sets in a topological vector space E and $F\colon X \to 2^Y$ be such that*

(i) *for each $x \in X$, $F(x)$ is closed in Y;*

(ii) *for each $A \in \mathcal{F}(X)$, $co(A) \subset \bigcup_{x\in A} F(x)$;*

(iii) *there exists an $x_0 \in X$ such that $F(x_0)$ is compact.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

We remark here that even though all topological vector spaces are assumed to be "Hausdorff" in [10], in proving Lemma 1 in [10], "Hausdorff" is never needed. The above lemma differs from Lemma 1 of Fan [10] in the following ways: (a) E is not assumed to be Hausdorff and (b) Y need not be the whole space E.

Theorem 2.2. *Let X be a non-empty convex subset of a topological vector space and $\phi, \psi : X \times X \to \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

(a) *$\phi(x,y) \leqslant \psi(x,y)$ for each $(x,y) \in X \times X$;*

(b) *for each fixed $x \in X$, $y \to \phi(x,y)$ is a lower semi-continuous function of y on each non-empty compact subset C of X;*

(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x\in A} \psi(x,y) \leqslant 0$;*

(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $\psi(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $y \in K$ such that $\phi(x,y) \leqslant 0$ for all $x \in X$.

Proof: For each $x \in X$, let $K(x) = \{y \in K : \phi(x,y) \leqslant 0\}$. We shall show that the family $\{K(x) : x \in X\}$ has the finite intersection property. Indeed, let $\{x_1, \cdots, x_n\}$ be any finite subset of X. Set $C = co\{x_1, x_2, \cdots, x_n\}$, then C is non-empty and compact. Define $F : C \to 2^C$ by $F(x) = \{y \in C : \psi(x,y) \leqslant 0\}$ for all $x \in C$. Then we have (i) if $\{z_1, z_2, \cdots, z_m\}$ is any finite subset of C, then $co\{z_1, \cdots, z_m\} \subset \bigcup_{i=1}^{m} F(z_i)$. For if this were false, there exist $\{z_1, \cdots, z_m\} \subset C$ and $z \in co\{z_1, \cdots, z_m\}$ with $z \notin \bigcup_{i=1}^{m} F(z_i)$ so that $\psi(z_i, z) > 0$ for all $i = 1, \cdots, m$ which contradicts (c). (ii)

$F(x_0) \subset K$ by (d) so that $cl_C F(x_0) \subset cl_C(K) = K$ and $cl_C F(x_0)$ is compact. By Lemma 2.1, $\bigcap_{x \in C} cl_C(F(x)) \neq \phi$. Take any $\overline{y} \in \bigcap_{x \in C} cl_C(F(x))$, then $\overline{y} \in cl_C F(x_0) \subset K$ and $\overline{y} \in \bigcap_{i=1}^{n} cl_C(F(x_i))$. But for each $i = 1, \cdots, n$, $cl_C(F(x_i)) = cl_X\{y \in C : \psi(x_i, y) \leqslant 0\} \subset cl_C\{y \in C : \phi(x_i, y) \leqslant 0\} = \{y \in C : \phi(x_i, y) \leqslant 0\}$ by (a) and (b). It follows that $\phi(x_i, \overline{y}) \leqslant 0$ for all $i = 1, \cdots, n$. So that $\overline{y} \in \bigcap_{i=1}^{n} K(x_i)$.

Hence the family $\{K(x) : x \in X\}$ has the finite intersection property. By (b) again, each K(x) is a closed subset of K. Therefore $\bigcap_{x \in X} K(x) \neq \emptyset$. Take any $\widehat{y} \in \bigcap_{x \in X} K(x)$, then $\widehat{y} \in K$ and $\phi(x, \widehat{y}) \leqslant 0$ for all $x \in X$. ∎

The following are equivalent formulations of Theorem 2.2.

Theorem 2.2′. *(First Geometric Form) Let X be a non-empty convex subset of a topological vector space and $B, D \subset X \times X$ be such that*

(a) *$B \subset D$;*
(b) *for each fixed $x \in X$ and for each non-empty compact subset C of X, the set $\{y \in C : (x, y) \in B\}$ is open in C;*
(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there exists $x \in A$ such that $(x, y) \notin D$;*
(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $(x_0, y) \in D$ for all $y \in X \setminus K$.*

Then there exists $\widehat{y} \in K$ such that $\{x \in X : (x, \widehat{y}) \in B\} = \emptyset$.

Theorem 2.2″. *(Second Geometric Form) Let X be a non-empty convex subset of a topological vector space and $M, N \subset X \times X$ be such that*

(a) *$N \subset M$;*
(b) *for each fixed $x \in X$ and for each non-empty compact subset C of X, the set $\{y \in C : (x, y) \in M\}$ is closed in C;*
(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there exists $x \in A$ such that $(x, y) \in N$;*
(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $(x_0, y) \notin N$ for all $y \in X \setminus K$.*

Then there exists $\widehat{y} \in K$ such that $X \times \{\widehat{y}\} \subset M$.

Theorem 2.2‴. *(Maximal Element Version) Let X be a non-empty convex subset of a topological vector space and $P, Q : X \to 2^X$ be such that*

(a) *for each $x \in X$, $P(x) \subset Q(x)$;*
(b) *for each $x \in X$, $P^{-1}(x)$ is compactly open in X;*
(c) *for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, there exists $x \in A$ such that*

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$x \notin Q(y)$;

(d) *there exist a non-empty closed and compact subset* K *of* X *and* $x_0 \in X$ *such that* $X \setminus K \subset Q^{-1}(x_0)$.

Then there exists $\widehat{y} \in K$ *such that* $P(\widehat{y}) = \emptyset$.

SKETCH OF PROOFS:

(1) Theorem 2.2 $\Longrightarrow$ Theorem 2.2$'$: Let $\phi, \psi : X \times X \to \mathbf{R}$ be the characteristic function of B, D respectively. □

(2) Theorem 2.2$'$ $\Longrightarrow$ Theorem 2.2: Define $B = \{(x,y) \in X \times X : \phi(x,y) > 0\}$ and $D = \{(x,y) \in X \times X : \psi(x,y) > 0\}$. □

(3) Theorem 2.2$'$ $\Longrightarrow$ Theorem 2.1$''$: Let $B = X \times X \setminus M$ and $D = X \times X \setminus N$. □

(4) Theorem 2.2$''$ $\Longrightarrow$ Theorem 2.2$'$: Let $M = X \times X \setminus B$ and $N = X \times X \setminus D$. □

(5) Theorem 2.2$''$ $\Longrightarrow$ Theorem 2.1$'''$: Let $N = \{(x,y) \in X \times X : x \notin Q(y)\}$ and $M = \{(x,y) \in X \times X : x \notin P(y)\}$. □

(6) Theorem 2.2$'''$ $\Longrightarrow$ Theorem 2.2$''$: Define $P, Q : X \to 2^X$ by $P(y) = \{x \in X : (x,y) \notin M\}$, and $Q(y) = \{x \in X : (x,y) \notin N\}$ for each $y \in X$ respectively. □

Theorem 2.2$'$ (respectively, Theorem 2.2$''$) generalises Theorem 3 (respectively, Theorem 4) of Shih and Tan [17].

LEMMA 2.3. *Let* X *be a non-empty convex subset of a topological vector space and* $\phi, \psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ *be such that*

(i) $\psi(x,x) \leqslant 0$ *for each* $x \in X$;

(ii) *for each* $y \in X$, *the set* $\{x \in X : \psi(x,y) > 0\}$ *is convex.*

Then for each $A \in \mathcal{F}(X)$ *and for each* $y \in co(A)$, $\min_{x \in A} \psi(x,y) \leqslant 0$.

PROOF: Suppose the conclusion were false, then there exist $A \in \mathcal{F}(X)$ and $y \in co(A)$ such that $\min_{x \in A} \psi(x,y) > 0$. It follows that $A \subset \{x \in X : \psi(x,y) > 0\}$ so that $y \in co(A) \subset \{x \in X : \psi(x,y) > 0\}$ by (ii), so that $\psi(y,y) > 0$ which contradicts (i). Therefore the conclusion must hold. □

In view of Lemma 2.3, Theorem 2.2 implies the following

THEOREM 2.4. *Let* X *be a non-empty convex subset of a topological vector space and* $\phi, \psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ *be such that*

(a) $\phi(x,y) \leqslant \psi(x,y)$ *for each* $(x,y) \in X \times X$ *and* $\psi(x,x) \leqslant 0$ *for each* $x \in X$;

(b) *for each fixed* $x \in X$, $y \mapsto \phi(x,y)$ *is a lower semicontinuous function of* y *on each non-empty compact subset* C *of* X;

(c) *for each fixed* $y \in X$, *the set* $\{x \in X : \psi(x,y) > 0\}$ *is convex;*

(d) *there exist a non-empty closed and compact subset* K *of* X *and a point* $x_0 \in X$ *such that* $\psi(x_0, y) > 0$ *for all* $y \in X \setminus K$.

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Then there exists $\widehat{y} \in K$ *such that* $\phi(x, \widehat{y}) \leqslant 0$ *for all* $x \in X$.

COROLLARY 2.5. *Let* X *be a non-empty convex subset of a topological vector space and* $\phi, \psi : X \times X \to \mathbf{R} \cup \{-\infty, +\infty\}$ *be such that*

(a) $\phi(x,y) \leqslant \psi(x,y)$ *for each* $(x,y) \in X \times X$;
(b) *for each fixed* $x \in X$, $y \mapsto \phi(x,y)$ *is a lower semicontinuous function of* y *on* C *for each non-empty compact subset* C *of* X;
(c) *for each fixed* $y \in X$, *the set* $\{x \in X : \psi(x,y) > \sup_{x \in X} \psi(x,x)\}$ *is convex;*
(d) *there exist a non-empty closed and compact subset* K *of* X *and a point* $x_0 \in X$ *such that* $\psi(x_0, y) > \sup_{x \in X} \psi(x,x)$ *for all* $y \in X \setminus K$.

Then there exists $\widehat{y} \in K$ *such that* $\phi(x, \overline{y}) \leqslant \sup_{x \in X} \psi(x,x)$ *for all* $x \in X$.

Theorem 2.4 improves Theorem 1 of Shih and Tan [**18**] in the following ways: (1) the given topological vector space need not be Husdorff, (2) for each $x \in X$, $y \mapsto \phi(x,y)$ is lower semicontinuous on each compact subset of X instead of on X and (3) $\widehat{y} \in K$ instead of $\widehat{y} \in X$. When X is a compact convex set, by taking $K = X$, Corollary 2.5 is essentially Yen's generalisation [**27**, Theorem 1, p.479] of Fan's minimax inequality [**11**, Theorem 1, p.103].

The following are fixed point versions of Theorem 2.4:

THEOREM 2.4′. *Let* X *be a non-empty convex subset of a topological vector space and* $P, Q : X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $P(x) \subset Q(x)$;
(b) *for each* $x \in X$, $P^{-1}(x)$ *is compactly open in X;*
(c) *for each* $y \in X$, $Q(y)$ *is convex;*
(d) *there exists a non-empty closed and compact subset* K *of* X *and* $x_0 \in X$ *such that* $X \setminus K \subset Q^{-1}(x_0)$;
(e) *for each* $y \in K$, $P(y) \neq \emptyset$.

Then there exists a point $x \in X$ *such that* $x \in Q(x)$.

THEOREM 2.4″. *Let* X *be a non-empty convex subset of a topological vector space and* $P, Q : X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $P(x) \subset Q(x)$;
(b) *for each* $x \in X$, $P^{-1}(x)$ *is compactly open in X;*
(c) *there exist a non-empty closed and compact subset* K *of* X *and* $x_0 \in X$ *such that* $X \setminus K \subset (coQ)^{-1}(x_0)$;
(d) *for each* $y \in K$, $P(y) \neq \emptyset$.

Then there exists $x \in X$ *such that* $x \in coQ(x)$.

THEOREM 2.4‴. *Let* X *be a non-empty convex subset of a topological vector*

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space and $P, Q : X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $P(x) \subset Q(x)$;

(b) *for each* $x \in X$, $P^{-1}(x)$ *is compactly open in* X;

(c) *there exist a non-empty closed and compact subset* K *of* X *and* $x_0 \in X$ *such that* $X \setminus K \subset Q^{-1}(x_0)$;

(d) *for each* $y \in K$, $P(y) \neq \emptyset$.

Then there exists $x \in X$ *such that* $x \in coQ(x)$.

THEOREM 2.4''''. *Let* X *be a non-empty convex subset of a topological vector space and* $Q : X \to 2^X$ *be such that*

(1) *for each* $y \in X$, $Q^{-1}(y)$ *contains a subset* O_y *(which may be empty) of* X *which is compactly open in* X;

(2) *there exist a non-empty closed and compact subset* K *of* X *and a point* $x_0 \in X$ *such that* $x_0 \in coQ(y)$ *for all* $y \in X \setminus K$ *and* $K \subset \bigcup_{y \in X} O_y$.

Then there exists a point $\widehat{x} \in X$ *such that* $\widehat{x} \in co(Q(\widehat{x}))$.

THEOREM 2.4'''''. *Let* X *be a non-empty convex subset of a topological vector space and* $Q : X \to 2^X$ *be such that*

(1) *for each* $x \in X$, $Q(x)$ *is convex;*

(2) *for each* $y \in X$, $Q^{-1}(y)$ *contains a subset* O_y *(which may be empty) of* X *which is compactly open in* X;

(3) *there exist a non-empty closed and compact subset* K *of* X *and a point* $x_0 \in X$ *such that* $x_0 \in Q(y)$ *for all* $y \in X \setminus K$ *and* $K \subset \bigcup_{y \in X} O_y$.

Then there exists a point $\widehat{x} \in X$ *such that* $\widehat{x} \in Q(\widehat{x})$.

SKETCH OF PROOFS:

(1) Theorem 2.4 $\Longrightarrow$ Theorem 2.4': Define $\phi, \psi : X \times X \to R$ by

$$\phi(x,y) = \begin{cases} 1, & \text{if } x \in P(y), \\ 0, & \text{if } x \notin P(y); \end{cases} \qquad \psi(x,y) = \begin{cases} 1, & \text{if } x \in Q(y), \\ 0, & \text{if } x \notin Q(y). \end{cases}$$

(2) Theorem 2.4' $\Longrightarrow$ Theorem 2.4: Define $P, Q : X \to 2^X$ by $P(y) = \{x \in X : \phi(x,y) > 0\}$ and $Q(y) = \{x \in X : \psi(x,y) > 0\}$ for each $y \in X$ respectively. □

(3) Theorem 2.4' $\Longleftrightarrow$ Theorem 2.4'' $\Longleftrightarrow$ Theorem 2.4''' is obvious. □

(4) Theorem 2.4'' $\Longrightarrow$ Theorem 2.4'''': Define $P : X \to 2^X$ by $P(x) = \{y \in X : x \in O_y\}$ for each $x \in X$. □

(5) Theorem 2.4'''' $\Longrightarrow$ Theorem 2.4'': For each $y \in X$, let $O_y = P^{-1}(y)$. □

(6) Theorem 2.4'''' $\Longleftrightarrow$ Theorem2.4''''': Obvious. □

Theorem 2.4'''' generalises Theorem 1 of Browder [4] and of Tarafdar [19] in several aspects. Theorem 2.4'''' also improves Theorems 2, 3 and 4 of Metha-Tarafdar in [15] which are due to Ben-El-Mechaiekh, Deguire and Granas [1] and Border [2]. Theorem 2.4''''' generalises Theorem 1 of Yannelis [25] in the following ways: (1) the convex set X need not be closed, (2) the given topological vector space need not be Hausdorff and (3) for each $y \in X$, the set $Q^{-1}(y)$ need not be open in X.

A subset of a topological space X is called a k-test set if its intersection with each non-empty compact set C in X is closed in C. A topological space X is called a k-space if each k-test set is closed (or equivalently, a subset B of X is open in X if and only if B is compactly open in X, for example see Wilansky, [**24**, p.142] or Dugundji [**9**, p.248]). In Theorem 2.4$'$, for each $y \in X$, the set $P^{-1}(y)$ is required to be compactly open in X, while in Border [2], Browder [4], Ding and Tan [6], Ben-El-Mechaiekh, Deguire and Granas [1], Metha-Tarafar [15], Tarafdar [19], Yannelis [25], the set $P^{-1}(y)$ is required to be open in X. This generalisation would be vacuous if every topological vector space is a k-space. However, this is not the case: the topological vector space $\mathbf{R}^{\mathbf{R}}$ is not a k-space (for example, see Kelley [**13**, p.240] or Wilansky [**24**, p.143]). Therefore Theorem 2.4$'$ is a true generalisation of Theorem 2 of Ding and Tan [7].

3. Existence of maximal elements

Lemma 3.1. *Let X be a regular topological vector space and Y be a non-empty subset of a vector space E. Let* $\theta : X \to E$ *and* $P : X \to 2^Y$ *be* $L_{\theta,C}$*-majorised. If each open subset of X containing the set* $B = \{x \in X : P(x) \neq \emptyset\}$ *is paracompact, then there exists a correspondence* $\phi : X \to 2^Y$ *of class* $L_{\theta,C}$ *such that* $P(x) \subset \phi(x)$ *for each* $x \in X$.

Proof: Since P is $L_{\theta,C}$-majorised, for each $x \in B$, let N_x be an open neighbourhood of x in X and $\psi_x, \phi_x : X \to 2^Y$ be such that (1) for each $z \in N_x$, $P(z) \subset \phi_x(z)$ and $\theta(z) \notin co(\phi_x(z))$; (2) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $co(\phi_x(z)) \subset Y$; (3) for each $y \in Y$, $\psi_x^{-1}(y)$ is compactly open in X and (4) for each finite subset A of B, $\{z \in \bigcap_{x\in A} N_x : \bigcap_{x\in A} co(\phi_x(z)) \neq \emptyset\} = \{z \in \bigcap_{x\in A} N_x : \bigcap_{x\in A} co(\psi_x(z)) \neq \emptyset\}$. Since X is regular, for each $x \in B$ there exists an open neighbourhood G_x of x in X such that $cl_X G_x \subset N_x$. Let $G = \bigcup_{x\in B} G_x$, then G is an open subset of X which contains $B = \{x \in X : P(x) \neq \emptyset\}$ so that G is paracompact by assumption. By Theorem VIII.1.4 of Dugundji [**9**, p.162], the open covering $\{G_x\}$ of G has an open precise neighbourhood-finite refinement $\{G'_x\}$. Given any $x \in B$, we define $\psi'_x, \phi'_x : G \to 2^Y$ by

$$\psi'_x(z) = \begin{cases} co\psi_x(z), & \text{if } z \in G \cap cl_X G'_x, \\ Y, & \text{if } z \in G \setminus cl_X G'_x, \end{cases} \quad \phi'_x(z) = \begin{cases} co\phi_x(z), & \text{if } z \in G \cap cl_X G'_x, \\ Y, & \text{if } z \in G \setminus cl_X G'_x, \end{cases}$$

then we have

(i) by (2), for each $z \in G$, $\psi'_x(z) \subset \phi'_x(z)$,

(ii) by (4), $\{z \in G : \psi'_x(z) \neq \emptyset\} = \{z \in G : \phi'_x(z) \neq \emptyset\}$ and

(iii) for each $y \in Y$, $(\psi'_x)^{-1}(y) = \{z \in G : y \in \psi'_x(z)\} = \{z \in G \cap cl_X G'_x : y \in \psi'_x(z)\} \cup \{z \in G \setminus cl_X G'_x : y \in \psi'_x(z)\} = \{z \in G \cap cl_X G'_x : y \in co\psi_x(z)\} \cup \{z \in G \setminus cl_X G'_x : y \in Y\} = [(G \cap cl_X G'_x) \cap (co\psi_x^{-1}(y))] \cup (G \setminus cl_X G'_x) = \left(G \cap (co\psi_x)^{-1}(y)\right) \cup (G \setminus cl_X G'_x)$.

It follows that for each non-empty compact subset C of X, $(\psi'_x)^{-1}(y) \cap C = \left(G \cap (co\psi_x)^{-1}(y) \cap C\right) \cup ((G \setminus cl_X G'_x) \cap C)$ is open in C by (3) and Lemma 5.1 of Yannelis and Prabhakar in [26]. Now define $\psi, \phi : X \to$ by

$$\psi(z) = \begin{cases} \bigcap_{x \in B} \psi'_x(z), & \text{if } z \in G, \\ \emptyset, & \text{if } z \in X \setminus G; \end{cases} \qquad \phi(z) = \begin{cases} \bigcap_{x \in B} \phi'_x(z), & \text{if } z \in G, \\ \emptyset, & \text{if } z \in X \setminus G. \end{cases}$$

Let $z \in X$ be given, Clearly (2) implies $\psi(z) \subset \phi(z)$ and $co\phi(z) \subset Y$. If $z \in X \setminus G$, then $\phi(z) = \emptyset$ so that $\theta(z) \notin co\phi(z)$; if $z \in G$, then $z \in G \cap cl_X G'_x$ for some $x \in B$ so that $\phi'_x(z) = co\phi_x(z)$ and hence $\phi(z) \subset co\phi_x(z)$. As $\theta(z) \notin co\phi'_x(z)$ by (1) we must have $\theta(z) \notin co\phi(z)$. Therefore $\theta(z) \notin co\phi(z)$ for all $z \in X$. Now we show that for each $y \in Y$, $\psi^{-1}(y)$ is compactly open in X. Indeed, let $y \in Y$ be such that $\psi^{-1}(y) \neq \emptyset$ and C be a compact subset of X; fix an arbitrary $u \in \psi^{-1}(y) \cap C = \{z \in X : y \in \psi(z)\} \cap C = \{z \in G : y \in \psi(z)\} \cap C$. Since $\{G'_x\}$ is a neighbourhood-finite refinement, there exists an open neighbourhood M_u of u in G such that $\{x \in B : M_u \cap G'_x \neq \emptyset\} = \{x_1, \cdots, x_n\}$. Note that for each $x \in B$ with $x \notin \{x_1, \cdots, x_n\}$, $\emptyset = M_u \cap G'_x = M_u \cap cl_X G'_x$ so that $\psi'(z) = Y$ for all $z \in M_u$. Thus we have $\psi(z) = \bigcap_{x \in B} \psi'_x(z) = \bigcap_{i=1}^{n} \psi'_{x_i}(z)$ for all $z \in M_u$. It follows that $\psi^{-1}(y) = \{z \in X : y \in \psi(z)\} = \{z \in G : y \in \bigcap_{x \in B} \psi'_x(z)\} \supset \{z \in M_u : y \in \bigcap_{x \in B} \psi'_x(z)\} = \{z \in M_u : y \in \bigcap_{i=1}^{n} \psi'_{x_i}(z)\} = M_u \cap \{z \in G : y \in \bigcap_{i=1}^{n} \psi'_{x_i}(z)\} = M_u \cap [\bigcap_{i=1}^{n} (\psi'_{x_i})^{-1}(y)]$. But $M'_u = M_u \cap [\bigcap_{i=1}^{n} (\psi'_{x_i})^{-1}(y)] \cap C$ is an open neighbourhood of u in C such that $M'_u \subset \psi^{-1}(y) \cap C$ since $(\psi'_{x_i})^{-1}(y)$ is compactly open in X. This shows that for each $y \in Y$, $\psi^{-1}(y)$ is compactly open in X. Next we claim $\{z \in X : \phi(z) \neq \emptyset\} = \{z \in X : \psi(z) \neq \emptyset\}$. Indeed, for each $w \in X$ with $\phi(w) \neq \emptyset$, we must have $w \in G$. Since $\{G'_x\}$ is neighbourhood-finite, the set $\{x \in B : w \in cl_X G'_x\}$ is finite, say, $= \{x'_1, \cdots, x'_m\}$ so that if $x \notin \{x'_1, \cdots, x'_m\}$, then $w \notin cl_X G'_x$ and $\phi'_x(w) = \psi'_x(w) = Y$. Thus we have $\phi(w) = \bigcap_{x \in B} \phi'_x(w) = \bigcap_{i=1}^{m} co\phi_{x'_i}(w)$ and $\psi(w) = \bigcap_{x \in B} \psi'_x(w) = \bigcap_{i=1}^{m} co\psi_{x'_i}(w)$. Since $w \in \bigcap_{i=1}^{m} cl_X G_{x'_i} \subset \bigcap_{i=1}^{m} N_{x'_i}$, it follows

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from (4) that $\psi(w) \neq \emptyset$. Hence $\{z \in X : \phi(z) \neq \emptyset\} \subset \{z \in X : \psi(z) \neq \emptyset\}$. Conversely, (2) implies that $\{z \in X : \psi(z) \neq \emptyset\} \subset \{z \in X : \phi(z) \neq \emptyset\}$. Therefore $\{z \in X : \psi(z) \neq \emptyset\} = \{z \in X : \phi(z) \neq \emptyset\}$. This shows that ϕ is class of $L_{\theta,C}$. To complete the proof, it remains to show that $P(z) \subset \phi(z)$ for each $z \in X$. Indeed, let $z \in X$ with $P(z) \neq \emptyset$. Note then $z \in G$. For each $x \in B$, if $z \in G \setminus cl_X G'_x$, then $\phi'_x(z) = Y \supset P(z)$ and if $z \in G \cap cl_X G'_x$, we have $z \in cl_X G'_x \subset cl_X G_x \subset N_x$ so that by (1), $P(z) \subset \phi_x(z) \subset \phi'_x(z)$. It follows that $P(z) \subset \phi'_x(z)$ for each $x \in B$ so that $P(z) \subset \bigcap_{x \in B} \phi'_x(z) = \phi(z)$. ∎

Lemma 3.1 generalises Lemma 2 of Ding and Tan [**7**] which in turn generalises Lemma 2 of Ding, Kim and Tan [**8**] and Proposition 1 of Tuleca [**22**].

Theorem 3.2. *Let X be a non-empty convex subset of a topological vector space and $Q : X \to 2^X$ be of class $L_{I_X,C}$. Suppose that there exist a non-empty closed and compact subset K of X and a point $x_0 \in X$ such that $x_0 \in coQ(y)$ for all $y \in X \setminus K$. Then there exists a point $x \in K$ such that $Q(x) = \emptyset$.*

Proof: If the conclusion were false, then for each $x \in K$, $Q(x) \neq \emptyset$. Since Q is of class $L_{I_X,C}$, let $P : X \to 2^X$ be a correspondence such that (a) for each $x \in X$, $P(x) \subset Q(x)$, (b) for each $y \in X$, $p^{-1}(y)$ is compactly open in X and (c) $\{x \in X : p(x) \neq \emptyset\} = \{x \in X : Q(x) \neq \emptyset\}$. By Theorem 2.4″, there exists a point $x \in X$ such that $x \in coQ(x)$ which contradicts that Q is of class $L_{I_X,C}$. Therefore the conclusion must hold. ∎

Theorem 3.3. *Let X be a non-empty paracompact convex subset of a topological vector space and $P : X \to 2^X$ be $L_{I_X,C}$-majorised. Suppose that there exist a non-empty closed and compact subset K of X and a point $x_0 \in X$ such that $x_0 \in coP(y)$ for each $y \in X \setminus K$. Then there exists a point $x \in K$ such that $P(x) = \emptyset$.*

Proof: Suppose that the conclusion does not hold, then $P(x) \neq \emptyset$ for all $x \in X$ and hence the set $\{x \in X : P(x) \neq \emptyset\} = X$ is paracompact. By Lemma 3.1, there exists a correspondence $\phi : X \to 2^X$ of class $L_{I_X,C}$ such that for each $x \in X$, $P(x) \subset \phi(x)$. Note that $x_0 \in coP(y) \subset co\phi(y)$ for all $y \in X \setminus K$. By Theorem 3.2, there exists a point $x \in K$ such that $\phi(x) = \emptyset$ so that $P(x) = \emptyset$ which is a contradiction. Therefore, there exists a point $x \in K$, such that $P(x) = \emptyset$. ∎

Theorem 3 generalises Theorem 5 of Ding and Tan [**7**] which in turn generalises Corollary 1 of Borglin and Keiding [**3**], Theorem 2.2 of Toussaint [**21**], Theorem 2 of Tulcea [**22**] and Corollary 5.1 of Yannelis and Prabhakar [**26**].

4. Equilibrium existence theorems in topological vector spaces

Let I be a (possibly infinite) set of agents. For each $i \in I$, let its choice or strategy

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set X_i be a non-empty subset of a topological vector space. Let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $P_i : X \to 2^{X_i}$ be a correspondence. Following the notion of Gale and Mas-Colell in [12], the collection $\Gamma = (X_i, P_i)_{i \in I}$ will be called a qualitative game. A point $\widehat{x} \in X$ is said to be an equilibrium of the game Γ if $P_i(\widehat{x}) = \emptyset$ for all $i \in I$. For each $i \in I$. let A_i be subset of X_i. Then for each fixed $k \in I$, we define $\prod_{j \in I, j \neq k} A_j \otimes A_k = \{x = (x_i)_{i \in I} : x_i \in A_i \text{ for all } i \in I\}$.

A generalised game (abstract economy) is a family of quadruples $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ where I is a (finite or infinite) set of players (agents) such that for each $i \in I$, X_i is a non-empty subset of a topological vector space and $A_i, B_i : X = \prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and $P_i : X \to 2^{X_i}$ is a preference correspondence. When $I = \{1, \cdots, N\}$ where N is a positive integer, $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ is also called an N-person game. An equilibrium of Γ is a point $\widehat{x} \in X$ such that for each $i \in I$, $\widehat{x}_i = \pi_i(\widehat{x}) \in \overline{B_i}(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. We remark that when $\overline{B_i}(\widehat{x}) = cl_{X_i} B_i(\widehat{x})$ (which is the case when B_i has a closed graph in $X \times X_i$; in particular, when clB_i is upper semicontinuous with closed values), our definition of an equilibrium point coincides with that of Ding, Kim and Tan [8]; and if in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point coincides with the standard definition; for example in Borglin and Keiding [3], Tulcea [22] and Yannelis and Prabhakar [26].

As an application of Theorem 3.2, we have the folowing existence theorem of an equilibrium point for a one-person game.

THEOREM 4.1. *Let X be a non-empty convex subset of a topological vector space. Let $A, B, P : X \to 2^X$ be such that*

(i) *for each $x \in X$, $A(x)$ is non-empty and $co(A(x)) \subset B(x)$;*
(ii) *for each $y \in X$, $A^{-1}(y)$ is compactly open in X;*
(iii) *$A \cap P$ is of class L_C;*
(iv) *there exist a non-empty closed and compact subset K of X and a point $x_0 \in X$ such that $x_0 \in co(A(y) \cap P(y))$ for all $y \in X \setminus K$.*

Then there exists a point $\widehat{x} \in K$ such that $\widehat{x} \in \overline{B}(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$.

PROOF: Let $M = \{x \in X : x \notin \overline{B}(x)\}$, then M is open in X. Define $\phi : X \to 2^X$ by

$$\phi(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \notin M, \\ A(x), & \text{if } x \in M. \end{cases}$$

Since $A \cap P$ is of class L_C, for each $x \in X$, $x \notin co(A(x) \cap P(x))$ and there exists a correspondence $\beta : X \to 2^X$ such that (a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$, (b)

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for each $y \in X$, $\beta^{-1}(y)$ is compactly open in X and (c) $\{x \in X : \beta(x) \neq \emptyset\} = \{x \in X : A(x) \cap P(x) \neq \emptyset\}$. Now define $\psi : X \to 2^X$ by

$$\psi(x) = \begin{cases} \beta(x), & \text{if } x \notin M, \\ A(x), & \text{if } x \in M. \end{cases}$$

Then clearly for each $x \in X$, $\psi(x) \subset \phi(x)$ and $\{x \in X : \psi(x) \neq \emptyset\} = \{x \in X : \phi(x) \neq \emptyset\}$. If $y \in X$, then it is easy to see $\psi^{-1}(y) = \big(M \cup \beta^{-1}(y)\big) \cap A^{-1}(y)$ and is compactly open in X by (ii) and (b). Finally, if $x \in M$, then $x \notin \overline{B}(x)$, it follows from (i) that $x \notin coA(x) = co\phi(x)$, and if $x \notin M$, then $x \notin co(A(x) \cap P(x)) = co\phi(x)$ by (i). This shows that ϕ is of class L_C. By (iv), $x_o \in co\phi(y)$ for all $y \in X \setminus K$. Hence ϕ satisfies all hypotheses of Theorem 3.2.Thus there exists a point $\widehat{x} \in K$ such that $\phi(\widehat{x}) = \emptyset$; since for each $x \in X$, $A(x) \neq \emptyset$, we must have $\widehat{x} \in \overline{B}(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$. ∎

As an application of Theorem 3.3, we have the following

THEOREM 4.2. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

(a) *X_i is a non-empty convex subset of a topological vector space for each $i \in I$;*
(b) *$P_i : X \to 2^{X_i}$ is L_C-majorised for each $i \in I$;*
(c) *$\bigcup_{i \in I} \{x \in X : P_i(x) \neq \emptyset\} = \bigcup_{i \in I} int_X\{x \in X : P_i(x) \neq \emptyset\}$;*
(d) *there exist a non-empty closed and compact subset K of X and a point $x_0 = \big(x_i^0\big)_{i \in I} \in X$ such that $x_i^0 \in coP_i(y)$ for all $i \in I$ and all $y \in X \setminus K$.*

Then Γ has an equilibrium point in K.

PROOF: For each $x \in X$, let $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$. Define a correspondence $P : X \to 2^X$ by

$$P(x) = \begin{cases} \bigcap_{i \in I(x)} coP_i'(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}$$

where $P_i'(x) = \prod_{j \neq i, \, j \in I} X_j \otimes P_i(x)$ for each $x \in X$. Then for each $x \in X$, $P(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. We shall show that P is L_C-majorised. For each $x \in X$ with $P(x) \neq \emptyset$, by (c) let $i(x) \in I$ be such that $x \in int_X\{z \in X : P_{i(x)}(z) \neq \emptyset\}$ and by (b) let $N(x)$ be an open neighbourhood of x in X and $\phi_{i(x)}, \psi_{i(x)} : X \to 2^{X_i}$ be correspondences such that (i) for each $z \in N(x)$, $P_{i(x)}(z) \subset \phi_{i(x)}(z)$ and $z_{i(x)} \notin co\phi_{i(x)}(z)$; (ii) for each $z \in X$, $\psi_{i(x)}(z) \subset \phi_{i(x)}(z)$; (iii) for each $y \in X_{i(x)}$, $\psi_{i(x)}^{-1}(y)$ is compactly open in X; (iv) for each finite subset $\{x_1, \dots, x_n\}$ of $\{x \in X : P(x) \neq \emptyset\}$ with $i(x_1) = \dots = i(x_n)$, $\{z \in \bigcap_{j=1}^n N(x_j) : \bigcap_{j=1}^n co\psi_{i(x_j)}(z) \neq \emptyset\} = \{z \in \bigcap_{j=1}^n N(x_j) : \bigcap_{j=1}^n co\phi_{i(x_j)}(z) \neq \emptyset\}$.

Without loss of generality we may assume that $N(x) \subset \operatorname{int}_X\{z \in X : P_{i(x)}(z) \neq \emptyset\}$ so that $P_{i(x)}(z) \neq \emptyset$ and hence $i(x) \in I(z)$ for all $z \in N(x)$. Let $x \in X$ be such that $P(x) \neq \emptyset$; define $\phi'_{i(x)}, \psi'_{i(x)} : X \to 2^X$ by $\psi'_{i(x)}(z) = \prod_{j \in I, j \neq i(x)} X_j \otimes co\psi_{i(x)}(z)$ and $\phi'_{i(x)}(z) = \prod_{j \in I, j \neq i(x)} X_j \otimes co\phi_{i(x)}(z)$ for each $x \in X$, then we have: (a$'$) for each $z \in N(x)$, by (i), $P(z) = \bigcap_{i \in I(z)} coP'_i(z) \subset coP'_{i(x)}(z) = \prod_{j \in I, j \neq i(x)} X_j \otimes coP_{i(x)}(z) \subset \prod_{j \in I, j \neq i(x)} X_j \otimes co\phi_{i(x)}(z) = \phi'_{i(x)}(z)$ and $z_{i(x)} \notin co\phi'_{i(x)}(z)$; (b$'$) for each $z \in X$, by (ii), $\psi'_{i(x)}(z) \subset \phi'_{i(x)}(z)$; (c$'$) for each $y \in X$, $\left(\psi'_{i(x)}\right)^{-1}(y) = \left(co\psi_{i(x)}\right)^{-1}\left(y_{i(x)}\right)$ is compactly open in X by (iii) and Lemma 5.1 in [**26**]; (d$'$) for any finite subset A of $\{x \in X : P(x) \neq \emptyset\}$, let $\cup\{I(x) : x \in A\} = \{i_1, \dots, i_k\}$ where $i_1, \cdots, i_k$ are all distinct and for each $t = 1, \cdots, k$ let $A_t = \{x \in A : i(x) = i_t\}$. Note that for each $z \in X$, $\bigcap_{x \in A} co\psi'_{i(x)}(z) = \bigcap_{x \in A} \prod_{j \in I, j \neq i(x)} X_j \otimes co\psi_{i(x)}(z) = \bigcap_{t=1}^{k} \prod_{j \in I, j \neq i_t} X_j \otimes \left(\bigcap_{x \in A_t} co\psi_{i(x)}(z)\right)$, so that for each $z \in \bigcap_{x \in A} N(x)$, if $\bigcap_{x \in A} co\psi'_{i(x)}(z) = \emptyset$, then there exists $m \in \{1, \dots, k\}$ such that $\bigcap_{x \in A_m} co\psi_{i(x)}(z) = \emptyset$; it follows from (iv) that $\bigcap_{x \in A_m} co\phi_{i(x)}(z) = \emptyset$. Thus $\bigcap_{x \in A} co\phi'_{i(x)}(z) = \bigcap_{x \in A} \prod_{j \in I, j \neq i(x)} X_j \otimes co\phi_{i(x)}(z) = \bigcap_{t=1}^{k} \prod_{j \in I, j \neq i_t} X_j \otimes \left(\bigcap_{x \in A_t} co\phi_{i(x)}(z)\right) = \emptyset$. This fact together with (b$'$), we conclude that

$$\{z \in \bigcap_{x \in A} N(x) : \bigcap_{x \in A} co\psi_{i(x)}(z) \neq \emptyset\} = \{z \in \bigcap_{x \in A} N(x) : \bigcap_{x \in A} co\phi_{i(x)}(z) \neq \emptyset\}.$$

This shows that P is L_C-majorised. Moreover, by assumption, there exist a non-empty closed and compact subset K of X and $x^0 = (x^0_i)_{i \in I} \in X$ such that $x^0_i \in coP_i(y)$ for all $i \in I$ and for all $y \in X \setminus K$ so that $x^0 \in coP'_i(y)$ for all $i \in I$ and for all $y \in X \setminus K$ and hence $x^0 \in \bigcap_{i \in I(y)} coP'_i(y) = P(y)$ for all $y \in X \setminus K$. By Theorem 3.3, there exists an $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$. This implies that $I(\hat{x}) = \emptyset$ and therefore $P_i(\hat{x}) = \emptyset$ for all $i \in I$. ∎

Theorem 4.2 improves Theorem 7 of Ding and Tan [**7**]. In Theorem 4.2, if X_i is compact for each $i \in I$, then $X = \prod_{i \in I} X_i$ is also compact. By letting $K = X$, the condition (4) of Theorem 4.2 is satisfied trivially. Hence Theorem 4.2 generalises Theorem 2.4 of Toussaint in [**21**] and Proposition 3 of Tulcea in [**22**] in several aspects which in turn generalise the fixed point theorem of Gale and Mas-Colell [**12**].

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As an application of Theorem 4.2, we shall deduce the following equilibrium existence theorem for a non-compact generalised game with an infinite number of players.

THEOREM 4.3. *Let* $\Gamma = (X_i; A_i, B_i; P_i)_{i\in I}$ *be a generalised game such that* $X = \prod_{i\in I} X_i$ *is paracompact. Suppose that the following conditions are satisfied:*

(i) *for each* $i \in I$, X_i *is a non-empty convex subset of a topological vector space;*

(ii) *for each* $i \in I$ *and for each* $x \in X$, $A_i(x)$ *is non-empty,* $coA_i(x) \subset B_i(x)$;

(iii) *for each* $i \in I$ *and for each* $y \in X_i$, $A_i^{-1}(y)$ *is compacly open in* X;

(iv) *for each* $i \in I$, $A_i \cap P_i$ *is of class* L_C;

(v) *for each* $i \in I$, $E_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X;

(vi) *there exist a non-empty closed and compact subset* K *of* X *and* $x^0 = (x_i^0)_{i\in I} \in X$ *such that* $x_i^0 \in con(A_i(y) \cap P_i(y))$ *for all* $i \in I$ *and for all* $y \in X \setminus K$.

Then Γ *has an equilibrium in* K.

PROOF: For each $i \in I$, let $F_i = \{x \in X : x_i \notin \overline{B_i}(x)\}$, then F_i is open in X. If $i \in I$, define the map $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_i, \\ A_i(x), & \text{if } x \in F_i. \end{cases}$$

We shall prove that the qualitative game $\Gamma = (X_i, Q_i)_{i\in I}$ satisfies all hypotheses of Theorem 4.2. Let $i \in I$ be arbitrarily fixed. Since $A_i \cap P_i$ is of class L_C, for each $x \in X$, $x \notin co(A_i(x) \cap P_i(x))$ and there exists a correspondence $\beta_i : X \to 2^{X_i}$ such that (a) for each $x \in X$, $\beta_i(x) \subset A_i(x) \cap P_i(x)$, (b) for each $y \in X_i$, $\beta_i^{-1}(y)$ is compactly open in X and (c) $\{x \in X : \beta_i(x) \neq \emptyset\} = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$. Define $\psi_i : X \to 2^{X_i}$ by

$$\psi_i(x) = \begin{cases} \beta_i(x), & \text{if } x \notin F_i, \\ A_i(x), & \text{if } x \in F_i. \end{cases}$$

Then for each $x \in X$, $\psi_i(x) \subset Q_i(x)$ and $\{x \in X : \psi_i(x) \neq \emptyset\} = \{x \in X : Q_i(x) \neq \emptyset\}$. If $y \in X$, then $\psi_i^{-1}(y) = [F_i \cup \beta_i^{-1}(y)] \cap A_i^{-1}(y)$ is compactly open in X. Therefore Q_i is of class L_C. We also note that for each $i \in I$, $\{x \in X : Q_i(x) \neq \emptyset\} = \{x \in F_i : Q_i(x) \neq \emptyset\} \cup \{x \in X \setminus F_i : Q_i(x) \neq \emptyset\} = F_i \cap [(X \setminus F_i)] \cap \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} = E_i \cup F_i$ is open in X by (ii) and (v). Therefore we have that $\bigcup_{i\in I}\{x \in X : Q_i(x) \neq \emptyset\} = \bigcup_{i\in I} int_X\{x \in X; Q_i(x) \neq \emptyset\}$.

Finally, by (vi) there exist a non-empty closed and compact subset K of X and $x^0 = (x_i^0)_{i\in I}$ in X such that $x_i^0 \in coQ_i(x^0)$ for all $i \in I$ and for all $y \in X \setminus K$. By

Theorem 4.2, there exists an $\widehat{x} \in K$ such that $Q_i(\widehat{x}) = \emptyset$ for all $i \in I$; by (ii) this implies that for each $i \in I$, we must have $\widehat{x}_i \in \overline{B_i}(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. ☐

In theorem 4.3, if X_i is compact for each $i \in I$, then $X = \prod_{i \in I} X_i$ is also compact and hence it is paracompact. Letting $K = X$, the assumption (vi) is satisfied trivially.

As an immediate consequence of Theorem 4.3, we have the following result:

Corollary 4.4. *Let $\Gamma = (X_i; A_i; B_i; P_i)_{i \in I}$ be a generalised game such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose that the following conditions are satisfied:*

- (i) *for each $i \in I$, X_i is a non-empty convex subset of a topological vector space;*
- (ii) *for each $i \in I$ and for each $x \in X$ $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;*
- (iii) *for each $i \in I$ and for each $y \in X_i$, $A_i^{-1}(y)$ and $P_i^{-1}(y)$ are open in X;*
- (iv) *for each $i \in I$ and for each $x \in X$, $x_i \notin coP_i(x)$;*
- (v) *there exist a non-empty closed and compact subset K of X and $x^0 = (x_i^0)_{i \in I} \in X$ such that $x_i^0 \in co(A_i(y) \cap P_i(y))$ for each $i \in I$ and for all $y \in X \setminus K$.*

Then Γ has an equilibrium in K.

Proof: Since $\{x \in X : (A_i \cap P_i)(x) \neq \emptyset\} = \bigcup_{y \in X_i} \left(A_i^{-1}(y) \cap P_i^{-1}(y)\right)$, by (iii), the conditions (iii) and (v), all hypotheses of Theorem 4.3 are satisfied. By Theorem 4.3 the conclusion follows. ☐

Corollary 4.4 generalises Theorem 2.5 of Toussaint in [**21**], Corollary 2 of Tulcea in [**22**] (also Corollary 2 in [**23**]) and Theorem 6.1 of Yannelis and Prabhakar in [**26**] to non-compact generalised games.

5. Approximation method

In this section, we shall employ the "approximation" technique used by Tulcea [**22**]. As an application of Theorem 3.2, we have the following existence theorem of "approximate" equilibrium point for a one-person game:

Theorem 5.1. *Let X be a non-empty convex subset of a topological vector space. Let $A, B, P : X \to 2^X$ be such that*

- (i) *A is lower semicontinuous such that for each $x \in X$, $A(x)$ is non-empty and $coA(x) \subset B(x)$;*
- (ii) *$A \cap P$ is of class L_C;*
- (iii) *there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that for each $y \in X \setminus K$, $x_0 \in co(A(y) \cap P(y))$.*

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Then for each open convex neighbourhood V of zero in E, the one person game $(X; A, \overline{B_V}; P)$ *has an equilibrium point in K, that is, there exists a point* $x_V \in K$ *such that* $x_V \in \overline{B_V}(x_V)$ *and* $A(x_V) \cap P(x_V) = \emptyset$, *where* $B_V(x) = (B(x) + V) \cap X$ *for each* $x \in X$.

PROOF: Let V be an open convex neighbourhood of zero in E. Define the correspondence $A_V, B_V : X \to 2^X$ by $A_V(x) = (A(x) + V) \cap X$, $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$. Then A_V has an open graph in $X \times X$ by (i) and Lemma 4.1 of Chang [5] (or see [**22**, p.7]) such that for each $x \in X$, $A_V(x) \subset B_V(x)$. Let $F_V = \{x \in X : x \notin \overline{B_V}(x)\}$, then F is open in X. Define $\Psi_V : X \to 2^X$ by

$$\Psi_V(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \notin F_V, \\ A_V(x), & \text{if } x \in F_V. \end{cases}$$

By (ii), since $A \cap P$ is of class L_C, for each $x \in X$, $x \notin co(A(x) \cap P(x))$ and there exists a correspondence $\beta : X \to 2^X$ such that (a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$, (b) for each $y \in X$, $\beta^{-1}(y)$ is compactly open in X and (c) $\{x \in X : \beta(x) \neq \emptyset\} = \{x \in X; A(x) \cap P(x) \neq \emptyset\}$. Define $\Phi_V : X \to 2^X$ by

$$\Phi_V(x) = \begin{cases} \beta(x), & \text{if } x \notin F_V, \\ A_V, & \text{if } x \in F_V, \end{cases}$$

Then clearly for each $x \in X$, $\Phi_V(x) \subset \Psi_V(x)$ and $\{x \in X : \Phi_V(x) \neq \emptyset\} = \{x \in X : \Psi_V(x) \neq \emptyset\}$. If $y \in X$, then it is easy to see $\Phi_V^{-1}(y) = \{x \in F_V : y \in A_V(x)\} \cup \{x \in X \setminus F_V : y \in \beta(x)\} = [F_V \cup \beta^{-1}(y)] \cap A_V^{-1}(y)$ is compactly open in X by (c) and the fact that A_V has an open graph. Therefore Ψ_V is of class L_C. Finally by (iii), there exist a non-empty closed and compact subset K of X and $x_0 \in X$ such that $x_0 \in co(A(y) \cap P(y)) \subset co(\Psi_V(y))$ for all $y \in X \setminus K$. Then by Theorem 3.2, there exists $\widehat{x} \in K$ such that $\Psi_V(\widehat{x}) = \emptyset$. Since for each $x \in X$, $A(x) \neq \emptyset$, we must have that $\widehat{x} \in \overline{B_V}(\widehat{x})$ and $A(\widehat{x}) \cap P(\widehat{x}) = \emptyset$; that is, the one person game $(X; A, \overline{B_V}; P)$ has an equilibrium point in K. ∎

As an application of Theorem 4.2, we have the following existence theorem of an "approximate" equilibrium point for an abstract economy:

THEOREM 5.2. *Let* $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ *be an abstract economy such that* $X = \prod_{i \in I} X_i$ *is paracompact. Suppose the following conditions are satisfied:*

(a) *for each* $i \in I$, X_i *is a non-empty convex subset of a topological vector space* E_i;

(b) *for each* $i \in I$, $A_i : X \to 2^{X_i}$ *is lower semicontinuous such that for each* $x \in X$, $A_i(x)$ *is non-empty and* $coA_i(x) \subset B_i(x)$;

(c) *for each* $i \in I$, $A_i \cap P_i$ *is of class* L_C;

(d) *for each* $i \in I$, *the set* $E^i = \{x \in X; (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in* X;

(e) *there exist a non-empty closed and compact subset* K *of* X *and* $x^0 \in X$ *such that for each* $y \in X \setminus K$, $x_i^0 \in co(A_i(y) \cap P_i(y))$ *for all* $i \in I$.

Then given any $V = \prod_{i \in I} V_i$ *where for each* $i \in I$, V_i *is an open convex neighbourhood of zero in* E_i, *there exists a point* $x_V = (x_{V_i})_{i \in I} \in K$ *such that* $x_{V_i} \in \overline{B_{V_i}}(x_V)$ *and* $A_i(x_V) \cap P_i(x_V) = \emptyset$ *for each* $i \in I$.

PROOF: Let $V = \prod_{i \in I} V_i$ be given where for each $i \in I$, V_i is an open convex neighbourhood of zero in E_i. Fix any $i \in I$, define the maps $A_{V_i}, B_{V_i} : X \to 2^{X_i}$ by $A_{V_i}(x) = (coA_i(x) + V_i) \cap X_i$ and $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. Then A_{V_i} has an open graph in $X \times X_i$ by (b) and Lemma 4.1 of Chang [5] (or see [**22**, p.7]), so that $coA_{V_i} : X \to 2^{X_i}$ has an open graph which in turn implies A_{V_i} is also lower semicontinuous. Let $F_{V_i} = \{x \in X : x_i \notin \overline{B_{V_i}}(x)\}$, then F_{V_i} is open in X. Define the map $Q_{V_i} : X \to 2^{X_i}$ by

$$Q_{V_i}(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_{V_i}, \\ A_{V_i}(x), & \text{if } x \in F_{V_i}. \end{cases}$$

We shall show that the qualitative game $\mathcal{T} = (X_i, Q_{V_i})_{i \in I}$ satisfies the hypotheses of Theorem 4.2. First for each $i \in I$, the set $\{x \in X : Q_{V_i}(x) \neq \emptyset\} = F_{V_i} \cup \{x \in X \setminus F_i : (A_i \cap P_i)(x) \neq \emptyset\} = F_{V_i} \cup [(F_{V_i}) \cap E^i] = F_{V_i} \cup E^i$ is open in X by (d).

Given any $i \in I$, since $A_i \cap P_i$ is of class L_C, for each $x \in X$, $x \notin co(A_i(x) \cap P_i(x))$ and there exists a correspondence $\beta_i : X \to 2^{X_i}$ such that (a) for each $x \in X$, $\beta_i(x) \subset A_i(x) \cap P_i(x)$, (b) for each $y \in X$, $\beta_i^{-1}(y)$ is compactly open in X and (c) $\{x \in X : \beta_i(x) \neq \emptyset\} = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$. Define $\Phi_{V_i} : X \to 2^{X_i}$ by

$$\Phi_{V_i}(x) = \begin{cases} \beta(x), & \text{if } x \notin F_{V_i}, \\ A_{V_i}, & \text{if } x \in F_{V_i}, \end{cases}$$

then clearly for each $x \in X$, $\Phi_{V_i}(x) \subset Q_{V_i}(x)$ and $\{x \in X : \Phi_{V_i}(x) \neq \emptyset\} = \{x \in X : Q_{V_i}(x) \neq \emptyset\}$. If $y \in X$, then it is easy to see $\Phi_{V_i}^{-1}(y) = \{x \in F_{V_i} : y \in A_{V_i}(x)\} \cup \{x \in X \setminus F_{V_i} : y \in \beta_i(x)\} = [F_{V_i} \cup \beta^{-1}(y)] \cap A_{V_i}^{-1}(y)$ is compactly open in X by (b) and the fact that A_{V_i} has an open graph. Therefore, Q_{V_i} is of class L_C. Together with (e), the qualitative game $\mathcal{T} = (X_i, Q_{V_i})_{i \in I}$ satisfies all the hypotheses of Theorem 4.2, so that by Theorem 4.2, there exists a point $\widehat{x}_V = (\widehat{x}_{V_i})_{i \in I} \in K$ such that $Q_{V_i}(\widehat{x}_V) = \emptyset$ for all $i \in I$. For each $i \in I$, since $A_i(x)$ is non-empty, we must have $\widehat{x}_{V_i} \in \overline{B_{V_i}}(\widehat{x}_V)$ and $A_i(\widehat{x}_V) \cap P_i(\widehat{x}_V) = \emptyset$. ∎

LEMMA 5.3. *Let X be a topological space, Y be a non-empty subset of a topological vector space E, $\mathcal{B}$ be a base for the zero neighbourhoods in E and $B : X \to 2^Y$. For each $V \in \mathcal{B}$, let $B_V : X \to 2^Y$ be defined by $B_V(x) = (B(x) + V) \cap Y$ for each $x \in X$. If $\widehat{x} \in X$ and $\widehat{y} \in Y$ are such that $\widehat{y} \in \bigcap_{V \in \mathcal{B}} \overline{B_V}(\widehat{x})$, then $\widehat{y} \in \overline{B}(\widehat{x})$.*

PROOF: Suppose $\widehat{y} \notin \overline{B}(\widehat{x})$, then $(\widehat{x}, \widehat{y}) \notin cl_{X \times Y} \operatorname{Graph} B$. Let U be an open neighbourhood of x in X and $V \in \mathcal{B}$ be such that

$$(*) \quad (U \times (\widehat{y} + V)) \cap \operatorname{Graph} B = \emptyset.$$

Choose $W \in \mathcal{B}$ such that $W - W \subset V$. Since $\widehat{y} \in \overline{B_W}(\widehat{x})$ by assumption, $(\widehat{x}, \widehat{y}) \in cl_{X \times Y} \operatorname{Graph} B_W$ so that $(U \times (\widehat{y} + W)) \cap \operatorname{Graph} B_W \neq \emptyset$. Take any $x \in U$ and $w_1 \in W$ with $(x, \widehat{y} + w_1) \in \operatorname{Graph} B_W$ so that $\widehat{y} + w_1 \in B_W(x) = (B(x) + W) \cap Y$. Let $z \in B(x)$ and $w_2 \in W$ be such that $\widehat{y} + w_1 = z + w_2 \in Y$. It follows that $z = \widehat{y} + w_1 - w_2 \in \widehat{y} + W - W \subset \widehat{y} + V$ so that $(\widehat{y} + V) \cap B(x) \neq \emptyset$ where $x \in U$. This contradicts $(*)$. Thus we must have $\widehat{y} \in \overline{B}(\widehat{x})$. ∎

We shall now obtain the following equilibrium existence theorem of a generalised game in locally convex topological vector spaces:

THEOREM 5.4. *Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (a) *for each $i \in I$, X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i;*
- (b) *for each $i \in I$, $A_i : X \to 2^{X_i}$ is lower semicontinuous and $B_i : X \to 2^{X_i}$ such that for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;*
- (c) *for each $i \in I$, $A_i \cap P_i$ is of class L_C;*
- (d) *for each $i \in I$, the set $E^i = \{x \in X; (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;*
- (e) *there exist a non-empty compact subset K of X and $x^0 \in X$ such that $x_i^0 \in co(A_i(y) \cap P_i(y))$ for all $i \in I$ and for all $y \in X \setminus K$.*

Then there exists an $\widehat{x} = (\widehat{x}_i)_{i \in I} \in K$ such that $\widehat{x}_i \in \overline{B_i}(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$ for each $i \in I$.

PROOF: For each $i \in I$, let $\mathcal{B}_i$ be the collection of all open convex neighbourhoods of zero in E_i and $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$. Given any $V \in \mathcal{B}$, let $V = \prod_{i \in I} V_i$ where $V_i \in \mathcal{B}_i$ for each $i \in I$. By Theorem 5.2, there exists a $\widehat{x}_V \in K$ such that $\widehat{x}_{V_i} \in \overline{B_{V_i}}(\widehat{x}_V)$ and $A_i(\widehat{x}_V) \cap P_i(\widehat{x}_V) = \emptyset$ for each $i \in I$, where $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. It follows that the set $Q_V = \{x \in K : x_i \in \overline{B_{V_i}}(x)$ and $A_i(x) \cap P_i(x) = \emptyset\}$ is a non-empty closed and hence compact subset of K by condition (d).

Now we want to prove $(Q_V)_{V \in \mathcal{B}}$ has the finite intersection property. Let $\{V_1, \cdots, V_n\}$ be any finite subset of $\mathcal{B}$. For each $i = 1, \cdots, n$, let $V_i = \prod_{j \in I} V_{ij}$ where

$V_{ij} \in \mathcal{B}_i$ for each $j \in I$; let $V = \prod_{j \in I} \left(\bigcap_{i=1}^{n} V_{ij} \right)$, then $Q_V \neq \emptyset$. Clearly $Q_V \subset \bigcap_{i=1}^{n} Q_{V_i}$ so that $\bigcap_{i=1}^{n} Q_{V_i} \neq \emptyset$. Therefore the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property. Since K is compact, $\bigcap_{V \in \mathcal{B}} Q_V \neq \emptyset$. Now take any $\widehat{x} \in \bigcap_{V \in \mathcal{B}} Q_V$, then for each $i \in I$, $\widehat{x}_i \in \overline{B_{V_i}}(\widehat{x})$ for each $V_i \in \mathcal{B}_i$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. By Lemma 5.3, for each $i \in I$, $\widehat{x}_i \in \overline{B_i}(\widehat{x})$. □

Corollary 5.5. *Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (a) *for each $i \in I$, X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space;*
- (b) *for each $i \in I$, $A_i : X \to 2^{X_i}$ has an open graph (respectively, is lower semicontinuous) and $B_i : X \to 2^{X_i}$ is such that for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;*
- (c) *for each $i \in I$, $P_i : X \to 2^{X_i}$ is lower semicontinuous (respectively, has an open graph);*
- (d) *for each $i \in I$, $A_i \cap P_i$ is of class L_C;*
- (e) *there exist a non-empty closed compact subset K of X and $x^0 \in X$ such that $x_i^0 \in co(A_i(y) \cap P_i(y))$ for all $i \in I$ and all $y \in X \setminus K$.*

Then $\mathcal{G}$ has an equilibrium point in K, that is, there exists a point $\widehat{x} \in X$ such that for each $i \in I$, $\widehat{x}_i \in \overline{B_i}(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$.

Proof: Since A_i has an open graph (respectively, is lower semicontinuous) and P_i is lower semicontinuous (respectively, has an open graph), the map $A_i \cap P_i : X \to 2^{X_i}$ is also lower semicontinuous by Lemma 4.2 of [25], so that the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is an open subset of X. Therefore all conditions of Theorem 5.4 are satisfied so that $\mathcal{G}$ has an equilibrium point in K. □

By Corolary 5.5, we have the following:

Corollary 5.6. *Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy. Suppose the following conditions are satisfied:*

- (a) *for each $i \in I$, X_i is a non-empty compact convex subset of a locally convex Hausdorff topological vector space;*
- (b) *for each $i \in I$, $A_i : X := \prod_{j \in I} X_j \to 2^{X_i}$ has an open graph (respectively, is lower semicontinuous) and $B_i : X \to 2^{X_i}$ is such that for each $x \in X$, $A_i(x)$ is non-empty and $coA_i(x) \subset B_i(x)$;*
- (c) *for each $i \in I$, $P_i : X \to 2^{X_i}$ is lower semicontinuous (respectively, has an open graph);*

(d) *for each* $i \in I$, $A_i \cap P_i$ *is of class* L_C.

Then $\mathcal{G}$ *has an equilibrium point in X, that is, there exists a point* $\widehat{x} \in X$ *such that for each* $i \in I$, $\widehat{x}_i \in \overline{B_i}(\widehat{x})$ *and* $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$.

Corollary 5.6 (and hence also Corollary 5.5 and Theorem 5.4) generalises Corollary 3 of Borglin and Keiding [**3**, p.315], Theorem 4.1 of Chang [**5**, p.247] and Theorem of Shafer and Sonnenschein [**16**, p.374] in several aspects.

Finally, we pose the following:

QUESTION. In Theorems 4.3, 5.2 and 5.4 and Corollary 5.5, can the condition "for each $i \in I$, $A_i \cap P_i$ is of class L_C" be replaced by the weaker condition "for each $i \in I$, $A_i \cap P_i$ is L_C-majorised"?

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Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5