UNIFORM AND EQUICONTINUOUS SCHAUDER BASES OF SUBSPACES

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1. Introduction. A sequence $\{M_i\}$ of non-trivial subspaces of a linear topological space X is a *basis of subspaces* for X if and only if corresponding to each $x \in X$ there is a unique sequence $\{x_i\}$, $x_i \in M_i$, such that

$$x = \lim_n \sum_1^n x_i.$$

Corresponding to a basis of subspaces $\{M_i\}$ for X is a sequence of orthogonal projections $\{E_i\}$ ($E_i^2 = E_i$ and $E_i E_j = 0$ if $i \neq j$) defined by $E_i(x) = x_i$ if

$$x = \sum_{j=1}^{\infty} x_j, \qquad x_j \in M_j.$$

If each E_i is continuous, the basis of subspaces is a *Schauder basis of subspaces* (SBOS). A SBOS $\{M_i, E_i\}$ for X is an e-*Schauder basis of subspaces* if and only if O is a point of equicontinuity of the sequence of projections $\{S_n\}$, where

$$S_n(x) = \sum_1^n E_i(s) \qquad \text{for each } x \in X.$$

Every SBOS of a barrelled (*tonnelé*) space is an e-SBOS. The class of second-category linear topological spaces is a proper subclass of the class of barrelled spaces. We show, on the other hand, that no Schauder basis of vectors with respect to the weak or weak* topology of a Banach space is an e-Schauder basis. Theorem 1 gives a necessary and sufficient condition for a subset of a linear topological space with an e-SBOS to be totally bounded. It is essentially a generalization of a theorem of Mazur (**1**, p. 237) stated by Banach, without proof, for Banach spaces.

By a *norm-SBOS*, a *w-SBOS*, or a *w*-SBOS* we mean a SBOS for a Banach space relative to its norm topology, its weak topology, or its weak* topology, respectively.

Let X be a Banach space which is also a linear topological space with a topology T, where T is not necessarily the norm topology. A SBOS $\{M_i, E_i\}$ of X is T-*uniform* if and only if

$$x = \lim_n \sum_1^n E_i(x)$$

uniformly for $||x|| \leqslant 1$, where the limit is relative to the topology T. Karlin (**2**, Theorem 4) has shown that an infinite-dimensional Banach space does not

Received July 11, 1963. This research was supported by the Air Force Office of Scientific Research and by National Science Foundation Grant No. NSF GP-2179.

admit a norm-uniform norm-Schauder basis. More generally, we observe that a Banach space does not admit a norm-uniform norm-SBOS. In contrast, we show that every w*-SBOS for a Banach space is w*-uniform. On the other hand, the existence of non-w-uniform w-SBOS is a consequence of Theorem 3, which gives a necessary and sufficient condition for a w-SBOS to be w-uniform. For Schauder bases of vectors this condition reduces to the familiar "shrinking" property enjoyed by numerous bases.

Indeed, if X is a space with a Schauder basis $\{x_i, f_i\}$ ($\{f_i\}$ the biorthogonal coefficient functionals) and M_i is the linear span of x_i, $E_i(x) = f_i(x)x_i$, then clearly $\{M_i, E_i\}$ is a SBOS for X. Thus all of the results of this paper are valid for bases of vectors.

2. Total boundedness and e-Schauder bases.

REMARK 1. *No w*-Schauder basis of vectors for the conjugate of a Banach space is an e-Schauder basis.*

Proof. Let X be a Banach space, X^* its conjugate, and $\{f_i\}$ a w*-Schauder basis for X^*. Then **(6**, Theorem 1**)** there is a sequence $\{x_i\}$ which is a norm basis for X and $f_i(x_j) = \delta_{ij}$. Let

$$x_0 = \sum_1^\infty (1/2^i||x_i||)x_i.$$

Then $f_i(x_0) = 1/2^i||x_i||$. Consider the w*-neighbourhood $U(0; x_0; 1)$. Let $V(0; x^{(1)}, \ldots, x^{(m)}; \epsilon)$ be an arbitrary w*-neighbourhood of 0. Let L denote the linear span (in the norm topology) of the elements $x^{(1)}, \ldots, x^{(m)}$. Since $L \neq X$ there is an $f \in X^*$ such that $f(x) = 0$ for each $x \in L$ and $f \neq 0$. Thus f is the w*-limit of the series $\sum_1^\infty f(x_i)f_i$ in which not all of the coefficients $f(x_i)$ are zero. Let n_0 be the least positive integer for which $f(x_{n_0}) \neq 0$. Since $f(L) = 0$ it follows that for an arbitrary scalar a, $af \in V$. If $a \neq 0$,

$$|S_{n_0}(af)(x_0)| = \left| \sum_1^{n_0} af(x_i)f_i(x_0) \right| = |a|\ |f(x_{n_0})|\ |f_{n_0}(x_0)| \neq 0,$$

and the last expression is greater than 1 for $|a|$ sufficiently large. For such an a, $S_{n_0}(af) \notin U$.

Since the intersection of the null spaces of a finite number of non-trivial linear functionals on a Banach space X contains a non-zero element, an argument similar to the preceding proves the following:

REMARK 2. *An infinite-dimensional Banach space X does not admit an e-Schauder basis of vectors relative to its weak topology.*

REMARK 3. *Every SBOS $\{M_i, E_i\}$ for a barrelled space X is an e-SBOS.*

Since each of the projections

$$S_n(x) = \sum_1^n E_i(x)$$

https://doi.org/10.4153/CJM-1965-020-5 Published online by Cambridge University Press

is continuous and since the sequence $\{S_n\}$ is pointwise bounded, it follows **(3**, Theorem 12.3, p. 104**)** that O is a point of equicontinuity of S_n.

LEMMA 1. *If X is a linear topological space with a SBOS $\{M_i, E_i\}$, and if A is a subset of X such that*

(i) *$E_i(A)$ is totally bounded for each i and*

(ii) $\lim_n \sum_1^n E_i(x) = x$ *uniformly for $x \in A$,*

then A is totally bounded.

Proof. Let U be a neighbourhood of O. Then there is a symmetric neighbourhood V of O such that $V + V \subset U$. Also there exists n_0 such that

$$\sum_{i=n_0+1}^{\infty} E_i(a) \in V \qquad \text{for all } a \in A.$$

Now

$$B \equiv E_1(A) + \ldots + E_{n_0}(A)$$

is totally bounded. Thus there exists $x_1, x_2, \ldots, x_p$ in X such that

$$B \subset \bigcup_{i=1}^{p} (x_i + V).$$

Then

$$A \subset B + V \subset \left[\bigcup_{i=1}^{p} (x_i + V)\right] + V \subset \bigcup_{i=1}^{p} (x_i + U).$$

Thus A is totally bounded.

THEOREM 1. *If X is a linear topological space with an e-SBOS $\{M_i, E_i\}$, then $A \subset X$ is totally bounded if and only if*

(i) *$E_i(A)$ is totally bounded for each i and*

(ii) $\lim_n \sum_1^n E_i(x) = x$ *uniformly for $x \in A$.*

Proof. The sufficiency of (i) and (ii) is given by the lemma. It remains to demonstrate their necessity. We assume that A is totally bounded. Then $E_i(A)$ is totally bounded for each i. By hypothesis O is a point of equicontinuity of the sequence

$$S_n(x) = \sum_1^n E_i(x).$$

Let U be a neighbourhood of O. There exists a symmetric neighbourhood V of O such that $V + V + V \subset U$ and, because of the equicontinuity, there is a neighbourhood W of O such that $W \subset V$ and if $x \in W$ then $S_n(x) \in V$ for all n. Also there exists $a_1, \ldots, a_p \in A$ such that

$$A \subset \bigcup_{i=1}^{p} (a_i + W).$$

Because of the convergence of $S_n(a_i)$ to a_i, there is an n_0 such that $n \geqslant n_0$ implies $S_n(a_i) - a_i \in V$, $i = 1, \ldots, p$. Suppose $a \in A$. Then for some i, $a \in a_i + W$. Thus if $n \geqslant n_0$ we have

$$S_n(a) - a = S_n(a - a_i) + [S_n(a_i) - a_i] + [a_i - a],$$

where the first term on the right is in V because of equicontinuity, the second is in V because of convergence, and the third is in $W \subset V$. Thus

$$S_n(a) - a \in V + V + V \subset U \qquad \text{if } n \geqslant n_0.$$

In a complete linear topological space a closed set is compact if and only if it is totally bounded. This, together with Remark 3 and Theorem 1, yields the following corollary.

COROLLARY 1. *If X is a complete barrelled space with a SBOS $\{M_i, E_i\}$, then a closed set $A \subset X$ is compact if and only if $E_i(A)$ is compact for each i and*

$$\lim_n \sum_1^n E_i(x) = x$$

uniformly for $x \in A$.

3. Uniform bases of subspaces.

REMARK 4. *A Banach space X does not admit a norm-uniform norm-SBOS.*

Proof. It is impossible to have

$$\left\| x - \sum_{i=1}^n x_i \right\| < \frac{1}{2}$$

whenever

$$x = \sum_{i=1}^\infty x_i, \qquad x_i \in M_i, \qquad \text{and } ||x|| \leqslant 1,$$

since this is not satisfied if $||x|| = 1$ and $x \in M_k$ with $k > n$.

Throughout the remainder of the paper we adopt the following notation: $R(E)$ denotes the range of a projection E of a Banach space X, E^* denotes the adjoint of E, and J denotes the canonical embedding of X into X^{**}.

The following lemma is proved in (**4**, Theorem 3.1).

LEMMA 2. *If $\{M_i, E_i\}$ is a SBOS for a Banach space X, then $\{R(E_i^*), E_i^*\}$ is a w*-SBOS for X^*. Conversely, if $\{N_i, P_i\}$ is a w*-SBOS for X^*, then $\{R(E_i), E_i\}$, where $E_i = J^{-1}P_i^*J$ is a SBOS for X.*

THEOREM 2. *Every w*-SBOS of the conjugate space of a Banach space is w*-uniform.*

Proof. Let X be a Banach space and $\{N_i, P_i\}$ a w*-SBOS of X^*. By Lemma 2, $\{R(E_i), E_i\}$ where $E_i = J^{-1}P_i^*J$ is a norm SBOS for X. It is easily shown that for arbitrary $f \in X^*$ and $x \in X$,

https://doi.org/10.4153/CJM-1965-020-5 Published online by Cambridge University Press

$$\sum_1^\infty [P_i(f)](x) = \sum_1^\infty f(E_i(x)).$$

Given a w* neighbourhood of O, $U(0; x^{(1)}, \ldots, x^{(m)}; \epsilon)$ there is an N such that if $n \geqslant N$, then

$$\left\| \sum_{n+1}^\infty E_i(x^{(j)}) \right\| < \epsilon, \qquad j = 1, 2, \ldots, m.$$

Thus if $||f|| \leqslant 1$ and $n \geqslant N$,

$$\left| \left[f - \sum_1^n P_i(f) \right] x^{(j)} \right| = \left| \sum_{n+1}^\infty [P_i(f)] \, x^{(j)} \right|$$
$$= \left| \sum_{n+1}^\infty f(E_i(x^{(j)})) \right| \leqslant ||f|| \left\| \sum_{n+1}^\infty E_i(x^{(j)}) \right\| < \epsilon$$

for $j = 1, 2, \ldots, m$. Thus $\{N_i, P_i\}$ is w*-uniform.

We say that a SBOS $\{M_i, E_i\}$ for a Banach space X is *shrinking* if and only if $\lim_n ||f||_n = 0$ for each $f \in X^*$, where $||f||_n$ is the norm of f on the closed linear span of

$$\bigcup_{i=n+1}^\infty M_i.$$

It is readily seen that a SBOS $\{M_i, E_i\}$ is shrinking if and only if $\{R(E_i^*), E_i^*\}$ is a SBOS for X^*.

THEOREM 3. *A w-SBOS $\{M_i, E_i\}$ for a Banach space X is w-uniform if and only if $\{M_i, E_i\}$ is shrinking.*

Proof. Let $\{M_i, E_i\}$ denote a w-uniform w-SBOS for a Banach space X. Karlin (**2**, Theorem 1) has shown that a weak basis for a Banach space is a norm basis. Ruckle (**5**, Theorem I.20) has recently observed that a w-SBOS is a norm-SBOS, which we use below. Let $f \in X^*$ and consider the weak neighbourhood $U(0; f; \epsilon/2)$ where $\epsilon > 0$ is arbitrary. The uniformity of the basis implies the existence of N such that if $n \geqslant N$ and $||x|| \leqslant 1$, then

$$\left| f(x) - f \sum_1^n E_i(x) \right| < \epsilon/2.$$

Thus

$$\left\| f - \sum_{i=1}^n E_i^*(f) \right\| \leqslant \epsilon$$

if $n \geqslant N$ and therefore

$$f = \sum_{i=1}^\infty E_i^*(f).$$

The uniqueness of this expression follows from the continuity and orthogonality of $\{E_i^*\}$.

Conversely, suppose that $\{R(E_i^*), E_i^*\}$ is a norm-SBOS of X^*. Then given a weak neighbourhood of O, $U(0; f^{(1)}, \ldots, f^{(m)}; \epsilon)$, there is an N such that $n \geqslant N$ implies

$$\left\| \sum_{n+1}^{\infty} E_i^*(f^{(j)}) \right\| < \epsilon \qquad \text{for } j = 1, 2, \ldots, m.$$

Thus if $||x|| \leqslant 1$ and $n \geqslant N$, then

$$\left| f^{(j)}\left(x - \sum_{1}^{n} E_i(x)\right) \right| = \left| \left[\sum_{n+1}^{\infty} E_i^*(f^{(j)}) \right] x \right| \leqslant \left\| \sum_{n+1}^{\infty} E_i^*(f^{(j)}) \right\| \, ||x|| < \epsilon;$$

so $\{M_i, E_i\}$ is w-uniform.

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