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GRADIENT ESTIMATES ON R^d

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ABSTRACT. This paper uses both the maximum principle and coupling method to study gradient estimates of positive solutions to $Lu = 0$ on $\mathbf{R}^d$, where

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

with (a_{ij}) uniformly positive definite and $a_{ij}, b_i \in C^1(\mathbf{R}^d)$. We obtain some upper bounds of $|\nabla u|/u$ and $\|\nabla u\|_\infty / \|u\|_\infty$, which imply a Harnack inequality and improve the corresponding results proved in Cranston [4]. Besides, two examples show that our estimates can be sharp.

1. **Introduction.** Gradient estimates are a fundamental subject in the study of Riemannian manifolds since they can be used to obtain the Harnack inequality, heat kernel estimates, and so on. Estimates of $|\nabla u|/u$ for a harmonic function u on a Riemannian manifold have been studied by Yau ([10]) and Cranston and Zhao ([5]). In the past few years, Cranston ([3], [4]) estimated $\|\nabla u\|_\infty / \|u\|_\infty$ for bounded positive u solution to $(\Delta + Z)u = 0$ with smooth vector field Z, and the estimates presented in [3] are improved by the author ([9]). Instead of functions on general Riemannian manifolds, this paper deals with positive solutions to $Lu = 0$ on $\mathbf{R}^d$ with

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i},$$

where (a_{ij}) is uniformly positive definite and $a_{ij}, b_i \in C^1(\mathbf{R}^d)$, $i, j \leq d$.

It is well known that, L can be rewritten as $\Delta + Z$ referring to some Riemannian metric and C^1-vector field Z on $\mathbf{R}^d$. However, it is not possible for us to compute the lower bound of Ricci curvature for general (a_{ij}). So we may obtain nothing from the known estimates on Riemannian manifolds. For this reason, it is interesting to give some gradient estimates of u depending on (a_{ij}) and (b_i). Since $Lu = 0$ is an ordinary differential equation for $d = 1$, we consider the case $d > 1$ only. Set

$$\alpha(x) = \inf\Big\{ \sum_{i,j} a_{ij}(x) \xi_i \xi_j : \xi \in \mathbf{R}^d, |\xi| = 1 \Big\},$$

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$$\beta(x) = \sup\Big\{\sum_{i,j} a_{ij}(x)\xi_i\xi_j : \xi \in \mathbf{R}^d, |\xi| = 1\Big\},$$

$$\gamma(x) = \frac{\alpha(x)}{\beta(x)}, \quad c_1^2 = \sup_x \sum_{i,j,k}\Big(\frac{\partial}{\partial x_k}a_{ij}(x)\Big)^2,$$

$$c_2^2 = \sup_x \sum_{i,j}\Big(\frac{\partial}{\partial x_i}b_j(x)\Big)^2, \quad c_3^2 = \sup_x \sum_i b_i(x)^2.$$

Throughout this paper, we assume that $\inf \alpha > 0$, $\sup \beta < \infty$ and $u \in C^2(\mathbf{R}^d)$, $u > 0$. For the estimate of $|\nabla u|/u$, assume in addition that $u \in C^3(\mathbf{R}^d)$ and $c_i < \infty$, $i \leq 3$. The main results are the following.

THEOREM 1.1. *Let $D \subset \mathbf{R}^d$ be a connected open domain, $\delta_x = \operatorname{dist}(x, \partial D)$ for $x \in D$. If $Lu = 0$ and $u > 0$ in D, then there exists a constant C depending only on* $\inf \alpha$, $\sup \beta$, *d and $c_i (i \leq 3)$ such that*

$$\frac{|\nabla u(x)|}{u(x)} \leq C\Big(1 + \frac{1}{\delta_x}\Big), \quad x \in D.$$

In particular, if $(a_{ij}) = I$ and $b = 0$, then

$$\frac{|\nabla u(x)|}{u(x)} \leq \frac{2d + \sqrt{2d(3d-1)}}{\delta_x}, \quad x \in D.$$

The following Harnack inequality is a direct consequence of Theorem 1.1.

COROLLARY 1.2. *Suppose that $Lu = 0$ and $u > 0$ in D. Let $Q_\delta = \{x \in D :$* $\operatorname{dist}(x, \partial D) \geq \delta\}$, *$\delta > 0$. For $B(x_0, \delta') \subset Q_\delta$, there exists a constant C depending only on* $\inf \alpha, \sup \beta, \delta, \delta'$ *and $c_i (i \leq 3)$ such that*

$$\sup_{B(x_0,\delta')} u \leq C \inf_{B(x_0,\delta')} u.$$

THEOREM 1.3. *Suppose that $Lu = 0$ and $u > 0$ on $\mathbf{R}^d$. Let $k = 2c_3\alpha + \sqrt{c_1(d-1)\big(\gamma\alpha^2 + (d-1)\alpha\beta\big)}$. We have*

$$\frac{|\nabla u(x)|}{u(x)} \leq \sup_{\mathbf{R}^d} \frac{k + \sqrt{k^2 + 4c_2(d-1)\gamma\alpha^3 - 4c_3^2\gamma^2\alpha^2}}{2\gamma\alpha^2}, \quad x \in \mathbf{R}^d.$$

The condition $c_i < \infty$ in Theorem 1.3 is necessary since $\frac{|\nabla u|}{u}$ may be unbounded for the case that $c_i = \infty$ (see Example 1.7 below). But $\frac{\|\nabla u\|_\infty}{\|u\|_\infty}$ is always finite for bounded u under some general assumptions; this leads us to study the estimate of $\frac{\|\nabla u\|_\infty}{\|u\|_\infty}$.

To state the result, we need some notation. Suppose that $a(x) = \big(a_{ij}(x)\big) = \sigma(x)\sigma(x)^*$ for a Lipschitz continuous matrix-valued function $\sigma(x) = \big(\sigma_{ij}(x)\big)$, satisfying

$$\lambda := \inf_{x,y} \inf_{|\xi|=1} \xi^*\sigma(y)^*\sigma(x)\xi > 0.$$

Choose $g \in C(\mathbf{R}^+)$ such that $\limsup_{r \to 0} g(r)/r < \infty$ and

$$g(r) \geq (4\lambda)^{-1} \sup_{|x-y|=r} \Big\{ \|\sigma(x) - \sigma(y)\|^2 - \big|\big(\sigma(x) - \sigma(y)\big)v\big|^2 + \langle b(x) - b(y), x - y\rangle \Big\},$$

where $v = (x-y)/|x-y|$ and $\|A\|^2 = \sum_{i,j} A_{ij}^2$ for $A = (A_{ij})$. Define

$$C(r) = \exp\Big[\int_0^r \frac{g(s)}{s}\, ds\Big], \; f(r) = \int_0^r C(s)^{-1}\, ds, \quad r \geq 0.$$

THEOREM 1.4. *Suppose that $Lu = 0$ on $\mathbf{R}^d$. If u is bounded and positive, then*

$$\|\nabla u\|_\infty \leq \frac{\|u\|_\infty}{f(\infty)},$$

where $f(\infty) = \lim_{r\to\infty} f(r)$. In particular, if $f(\infty) = \infty$ then u is constant.

COROLLARY 1.5. *Suppose that $a = \frac{1}{2}I$ and $b_i(x) = \sum_j b_{ij}x_j, i \leq d$. Let λ_d be the biggest eigenvalue of $\big(\frac{1}{2}(b_{ij} + b_{ji})\big)$. We have*

$$\|\nabla u\|_\infty \leq \|u\|_\infty \sqrt{\lambda_d^+} \Big/ \sqrt{\pi},$$

where $\lambda_d^+ = \max\{0, \lambda_d\}$.

Corollary 1.5 improves the corresponding estimate in [4]: $\|\nabla u\|_\infty \leq \|u\|_\infty \sqrt{2\lambda_d^+}$. Besides, the following two examples show that both estimates in Theorem 1.3 and Theorem 1.4 can be sharp.

EXAMPLE 1.6. Take $a = I$, $b_i = c$, $c > 0$, $i \leq d$. Then $\alpha = \beta = 1$, $c_1 = c_2 = 0$ and $k = 2c_3 = 2\sqrt{d}c$. By Theorem 1.3 we have $|\nabla u| \leq \sqrt{d}cu$. On the other hand, take $u(x) = \exp[-c\sum_i x_i]$, then $u > 0$, $Lu = 0$ and $|\nabla u| = \sqrt{d}cu$.

EXAMPLE 1.7. Take $a = \frac{1}{2}I$, $b_1(x) = cx_1$ $(c > 0)$ and let b_i $(i \geq 2)$ be constants. Let

$$u(x) = \int_0^\infty e^{-cr^2}\, dr + \int_0^{x_1} e^{-cr^2}\, dr, \quad x \in \mathbf{R}^d.$$

Then $u > 0$, $Lu = 0$ and

$$\begin{aligned}
\sup_x \frac{|\nabla u(x)|}{u(x)} &= \sup_{x_1} e^{-cx_1^2} \Big\{ \int_0^\infty e^{-cr^2}\, dr + \int_0^{x_1} e^{-cr^2}\, dr \Big\}^{-1} \\
&\geq \lim_{x_1 \to -\infty} e^{-cx_1^2} \Big\{ \int_{-x_1}^\infty e^{-cr^2}\, dr \Big\}^{-1} \\
&= \lim_{x_1 \to -\infty} \frac{-2cx_1 e^{-cx_1^2}}{e^{-cx_1^2}} \\
&= \infty.
\end{aligned}$$

Hence $\frac{|\nabla u|}{u}$ is unbounded, but we can compute

$$\frac{\|\nabla u\|_\infty}{\|u\|_\infty} = \Big\{ 2\int_0^\infty e^{-cr^2}\, dr \Big\}^{-1} = \frac{\sqrt{c}}{\sqrt{\pi}}.$$

This is just the upper bound given by Corollary 1.5.

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2. **Some lemmas.** For convenience, we simply denote $u^{(i)} = \frac{\partial}{\partial x_i} u$, $u^{(ij)} = \frac{\partial^2}{\partial x_i \partial x_j} u$ for $u \in C^2(\mathbf{R}^d)$. Then

$$|\nabla u|^2 = \sum_i u^{(i)2}, \quad |\nabla |\nabla u||^2 = \frac{1}{|\nabla u|^2} \sum_j \Big(\sum_i u^{(i)} u^{(ij)} \Big)^2. \tag{2.1}$$

LEMMA 2.1. *Suppose that $Lu = 0, u > 0$ in D. There exists a nonnegative function $h \leq c_3$ such that*

$$|\nabla u| L |\nabla u| \geq \Big(\frac{(1-s)\gamma\alpha}{d-1} - s\beta \Big) |\nabla|\nabla u||^2 - \Big(\frac{c_1}{4\alpha s} + c_2 - \frac{h^2}{(d-1)\beta} \Big) |\nabla u|^2$$
$$- \frac{2h}{d-1} |\nabla |\nabla u|| \cdot |\nabla u|$$

holds for $s \in [0,1]$ and points in D with $|\nabla u| > 0$.

PROOF. Fix $p \in D$ with $|\nabla u|(p) > 0$; the proof consists of two parts.

a) Suppose that $a(p) = \big(a_{ij}(p)\big) = \operatorname{diag}\{\lambda_1, \ldots, \lambda_d\}$ with $\alpha(p) = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d = \beta(p)$. Then at p,

$$\frac{1}{2} L |\nabla u|^2 = \sum_{i,j} \lambda_i u^{(ij)2} + \sum_i u^{(i)} (L u^{(i)}).$$

On the other hand,

$$\frac{1}{2} L |\nabla u|^2 = |\nabla u| L |\nabla u| + \sum_i \lambda_i (|\nabla u|^{(i)})^2$$
$$= |\nabla u| L |\nabla u| + \frac{1}{|\nabla u|^2} \sum_i \lambda_i \Big(\sum_j u^{(j)} u^{(ij)} \Big)^2.$$

Hence

$$|\nabla u| L |\nabla u| = \sum_i u^{(i)} L u^{(i)} + \frac{1}{|\nabla u|^2} \Big(|\nabla u|^2 \sum_{i,j} \lambda_i u^{(ij)2} - \sum_i \lambda_i \Big(\sum_j u^{(j)} u^{(ij)} \Big)^2 \Big). \tag{2.2}$$

Since $Lu^{(k)} = (Lu)^{(k)} - \sum_{i,j} a_{ij}^{(k)} u^{(ij)} - \sum_i b_i^{(k)} u^{(i)}$ and $Lu = 0$,

$$\sum_k u^{(k)} (L u^{(k)}) \geq -\frac{|\nabla u|}{\sqrt{\alpha}} \sqrt{\sum_{i,j,k} a_{ij}^{(k)2}} \sqrt{\sum_{i,j} \lambda_i u^{(ij)2}} - c_2 |\nabla u|^2$$
$$\geq -\frac{c_1 |\nabla u|}{\sqrt{\alpha}} \sqrt{\sum_{i,j} \lambda_i u^{(ij)2}} - c_2 |\nabla u|^2$$
$$\geq -\frac{c_1 |\nabla u|^2}{4 s \alpha} - s \sum_{i,j} \lambda_i u^{(ij)2} - c_2 |\nabla u|^2.$$

Here in the last step, we have used the fact that $r^2 + s^2 \geq 2rs$. Combining this with (2.2) we have

$$|\nabla u| L |\nabla u| \geq -\frac{c_1 |\nabla u|^2}{4 s \alpha} - c_2 |\nabla u|^2 + (1-s) \sum_{i,j} \lambda_i u^{(ij)2} \tag{2.3}$$
$$- \frac{1}{|\nabla u|^2} \sum_i \lambda_i \Big(\sum_j u^{(j)} u^{(ij)} \Big)^2.$$

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Next,

$$
\begin{aligned}
|\nabla u|^2 \sum_{i,j} \lambda_i u^{(ij)2} - \sum_i \lambda_i \Big(\sum_j u^{(j)} u^{(ij)}\Big)^2 &= \sum_{i,j,k} \lambda_i (u^{(k)2} u^{(ij)2} - u^{(j)} u^{(ij)} u^{(k)} u^{(ik)}) \\
&= \frac{1}{2} \sum_i \lambda_i \sum_{j,k} (u^{(k)} u^{(ij)} - u^{(j)} u^{(ik)})^2 \\
&\geq \sum_i \lambda_i \sum_j (u^{(j)} u^{(ii)} - u^{(i)} u^{(ij)})^2 \\
&\geq \frac{1}{\beta} \sum_j \sum_i (\lambda_i u^{(j)} u^{(ii)} - \lambda_i u^{(i)} u^{(ij)})^2 \\
&\geq \frac{1}{(d-1)\beta} \sum_j \Big(\sum_i \lambda_i u^{(j)} u^{(ii)} - \sum_i \lambda_i u^{(i)} u^{(ij)}\Big)^2.
\end{aligned}
$$

Let $h = |\sum_i b_i u^{(i)}|/|\nabla u|$; then $h \in [0, c_3]$. Since $\sum_i \lambda_i u^{(ii)} = -\sum_i b_i u^{(i)}$, by (2.1) we have

(2.4)

$$
\begin{aligned}
&|\nabla u|^2 \sum_{i,j} \lambda_i u^{(ij)2} - \sum_i \lambda_i \Big(\sum_j u^{(j)} u^{(ij)}\Big)^2 \\
&\qquad \geq \frac{1}{(d-1)\beta} \sum_j \Big(\sum_i \lambda_i u^{(i)} u^{(ij)} + u^{(j)} \sum_i b_i u^{(i)}\Big)^2 \\
&\qquad \geq \frac{\alpha^2 |\nabla u|^2}{(d-1)\beta} \big|\nabla |\nabla u|\big|^2 + \frac{|\nabla u|^2 |\sum_i b_i u^{(i)}|^2}{(d-1)\beta} \\
&\qquad\quad - \frac{2}{d-1} \sum_j |u^{(j)}| \cdot \Big|\sum_i u^{(i)} u^{(ij)}\Big| \cdot \Big|\sum_i b_i u^{(i)}\Big| \\
&\qquad \geq \frac{\alpha^2 |\nabla u|^2}{(d-1)\beta} \big|\nabla |\nabla u|\big|^2 - \frac{2h |\nabla u|^3}{d-1} \big|\nabla |\nabla u|\big| + \frac{h^2 |\nabla u|^4}{(d-1)\beta}.
\end{aligned}
$$

By this and (2.3) we obtain the needed inequality.

b) In general, there exists an orthonormal matrix σ such that $\sigma a(p) \sigma^* = \mathrm{diag}\{\lambda_1, \ldots, \lambda_d\}$. Take $x = \sigma y$. Under the new coordinate system $\{y_1, \ldots, y_d\}$,

$$
L = \sum_{i,j} \bar{a}_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_i \bar{b}_i \frac{\partial}{\partial y_i}
$$

with

$$
\bar{a}_{ij}(y) = \sum_{s,t} \sigma_{is} a_{st}(\sigma y) \sigma_{jt}, \quad \bar{b}_i(y) = \sum_j b_j(\sigma y) \sigma_{ji}.
$$

Then $\bar{a}(\sigma^* p) = \mathrm{diag}\{\lambda_1, \ldots, \lambda_d\}$. On the other hand, it is easy to check that

$$
\sum_{i,j,k} \Big(\frac{\partial}{\partial y_k} \bar{a}_{ij}\Big)^2 \leq c_1^2, \quad \sum_{i,k} \Big(\frac{\partial}{\partial y_k} \bar{b}_i\Big)^2 \leq c_2^2, \quad \sum_i \bar{b}_i^2 \leq c_3^2
$$

and

$$|\nabla u|^2(p) = \sum_i \Big(\frac{\partial}{\partial y_i}\bar{u}(\sigma^* p)\Big)^2, \quad L\bar{u} = 0,$$

$$|\nabla u|^2(p)\big|\nabla|\nabla u|\big|^2(p) = \sum_j \Big(\sum_i \Big(\frac{\partial}{\partial y_i}\bar{u}(\sigma^* p)\Big)\Big(\frac{\partial^2}{\partial y_i \partial y_j}\bar{u}(\sigma^* p)\Big)\Big)^2,$$

where $\bar{u}(y) = u(\sigma y)$. By a) the proof is completed. ∎

LEMMA 2.2. *Suppose that $Lu = 0$, $u > 0$ in D. Let $\phi = \frac{|\nabla u|}{u}$. If $d_1 := \frac{(1-s)\alpha\gamma}{d-1} - s\beta > 0$, then*

$$L\phi \geq d_1\Big(\frac{|\nabla \phi|^2}{\phi} + \phi^3 - 2\phi|\nabla\phi|\Big) - \frac{2h}{d-1}(|\nabla\phi| + \phi^2)$$
$$- \Big(\frac{c_1}{4s\alpha} + c_2 - \frac{h^2}{(d-1)\beta}\Big)\phi - 2\beta\phi|\nabla\phi|$$

for points in D with $\phi > 0$.

PROOF. By Lemma 2.1,

$$L\phi = \frac{1}{u}L|\nabla u| + |\nabla u|L\frac{1}{u} - \frac{2}{u^2}\sum_{i,j} a_{ij}u^{(j)}|\nabla u|^{(i)}$$
$$= \frac{1}{u|\nabla u|}(|\nabla u|L|\nabla u|) + \frac{2\phi}{u^2}\sum_{i,j} a_{ij}u^{(i)}u^{(j)} - \frac{2}{u^2}\sum_{i,j} a_{ij}u^{(j)}(\phi u^{(i)} + u\phi^{(i)})$$
$$\geq \frac{1}{u|\nabla u|}\Big[d_1\big|\nabla|\nabla u|\big|^2 - \frac{2h}{d-1}\big|\nabla|\nabla u|\big| \cdot |\nabla u| - \Big(\frac{c_1}{4s\alpha} + c_2 - \frac{h^2}{(d-1)\beta}\Big)|\nabla u|^2\Big]$$
$$- 2\phi\beta|\nabla\phi|$$
$$= \frac{d_1\big|\nabla|\nabla u|\big|^2}{u|\nabla u|} - \frac{2h\big|\nabla|\nabla u|\big|}{(d-1)u} - \Big(\frac{c_1}{4s\alpha} + c_2 - \frac{h^2}{(d-1)\beta}\Big)\phi - 2\beta\phi|\nabla\phi|.$$

Note that

$$\big|\phi|\nabla u| - u|\nabla\phi|\big| \leq \big|\nabla|\nabla u|\big| = |u\nabla\phi + \phi\nabla u| \leq u|\nabla\phi| + \phi|\nabla u|,$$

which proves the lemma. ∎

LEMMA 2.3. *Suppose that $Lu = 0$ and $u > 0$ on $\mathbf{R}^d$. Let*

$$k_1 = \frac{\sqrt{c_1(d-1)}}{2\sqrt{\gamma\alpha^2 + (d-1)\alpha\beta}}.$$

If $|\nabla u| \cdot \big|\nabla|\nabla u|\big| > 0$ and $k_1|\nabla u|/\big|\nabla|\nabla u|\big| \leq 1$, then

$$L\phi \geq \frac{\gamma\alpha}{d-1}\Big(\frac{|\nabla\phi|^2}{\phi} + \phi^3 - 2\phi|\nabla\phi|\Big) - \Big(\frac{\sqrt{c_1\big(\gamma\alpha + (d-1)\beta\big)}}{\sqrt{(d-1)\alpha}} + \frac{2h}{d-1}\Big)(|\nabla\phi| + \phi^2)$$
$$- \Big(c_2 - \frac{h^2}{(d-1)\beta}\Big)\phi - 2\beta\phi|\nabla\phi|$$

PROOF. Take $s = k_1|\nabla u|/\big|\nabla|\nabla u|\big|$. By Lemma 2.1 we have

$$|\nabla u|L|\nabla u| \geq \frac{\gamma\alpha}{d-1}\big|\nabla|\nabla u|\big|^2 - \Big(c_2 - \frac{h^2}{(d-1)\beta}\Big)|\nabla u|^2 - \left(\frac{\sqrt{c_1\big(\gamma\alpha + (d-1)\beta\big)}}{\sqrt{(d-1)\alpha}} + \frac{2h}{d-1}\right)\big|\nabla|\nabla u|\big| \cdot |\nabla u|.$$

Then the remainder of the proof is the same as above. ■

3. **Proof of Theorem 1.1.** Let s small enough such that $d_1 > 0$ for all $x \in \mathbf{R}^d$. Let $d_2 = \frac{c_3}{d-1}, d_3 = \frac{c_1}{4s\alpha} + c_2$. Then Lemma 2.2 gives us

$$L\phi \geq d_1\Big(\frac{|\nabla\phi|^2}{\phi} + \phi^3\Big) - 2d_2(|\nabla\phi| + \phi^2) - d_3\phi - 2(\beta + d_1)\phi|\nabla\phi| \tag{3.1}$$

for $\phi > 0$. Fix $p \in D$ with $\phi(p) > 0$. Take $F(x) = \phi(x)(\delta_p^2 - \rho(x)^2)$, where $\rho(x) = |x - p|$. Then there exists $x_1 \in D$ such that $F(x_1) = \sup\{F(x) : |x - p| \leq \delta_p\}$. Hence

$$LF(x_1) \leq 0 \quad \text{and} \quad \nabla\phi(x_1) = \frac{2\phi(x_1)|x_1 - p|\nabla\rho(x_1)}{\delta_p^2 - |x_1 - p|^2}. \tag{3.2}$$

Combining this with (3.1) we have: at x_1,

$$L\phi \geq d_1\phi^3 - 2\Big(d_2 + \frac{2(\beta + d_1)\rho}{\delta_p^2 - \rho^2}\Big)\phi^2 - \Big(\frac{4d_1\rho^2}{(\delta_p^2 - \rho^2)^2} + \frac{4d_2\rho}{\delta_p^2 - \rho^2} + d_3\Big)\phi.$$

Thus

$$\begin{aligned} LF &= (\delta_p^2 - \rho^2)L\phi - 2d\phi - 2\langle\nabla\phi, 2\rho\nabla\rho\rangle \\ &\geq d_1(\delta_p^2 - \rho^2)\phi^3 - 2[d_2(\delta_p^2 - \rho^2) + 2(\beta + d_1)\rho]\phi^2 \\ &\quad - \Big(\frac{4(d_1 + 2)\rho^2}{\delta_p^2 - \rho^2} + 4d_2\rho + 2d + d_3(\delta_p^2 - \rho^2)\Big)\phi. \end{aligned}$$

By (3.2) we get

$$\begin{aligned} 0 &\geq d_1F^2(x_1) - 2[d_2\delta_p^2 + 2\delta_p(\beta + d_1)]F(x_1) - 4[\delta_p^2(d_1 + 2) + d_2\delta_p^3 + d\delta_p^2 + d_3\delta_p^4] \\ &\geq d_1F^2(x_1) - 4[d_2\delta_p^2 + \delta_p(\beta + d_1)]F(x_1) - 4\delta_p^2[d_1 + d_2 + d + 2 + (d_3 + d_2)\delta_p^2]. \end{aligned}$$

This implies

$$F(x_1) \leq C\delta_p(1 + \delta_p)$$

for some $C = C(\inf\alpha, \sup\beta, c_1, c_2, c_3, d) > 0$. Since $\phi(p) = F(p)/\delta_p^2 \leq F(x_1)/\delta_p^2$, we have

$$\phi(p) \leq C\Big(1 + \frac{1}{\delta_p}\Big).$$

Next, if $(a_{ij}) = I$ and $b = 0$, then $d_2 = d_3 = 0$ and $\alpha = \beta = 1$. By (3.1) and letting $s \longrightarrow 0$ we get

$$L\phi \geq \frac{1}{d-1}\Big(\phi^3 + \frac{|\nabla\phi|^2}{\phi}\Big) - 2\Big(1 + \frac{1}{d-1}\Big)\phi|\nabla\phi|. \tag{3.3}$$

Combining this with (3.2), we prove

$$F^2 - 4d\delta_p F - 2d(d-1)\delta_p^2 + 2\rho^2[(d-4)(d-1)+2] \le 0.$$

Since $(d-4)(d-1)+2 \ge 0$ for all $d \in \mathbf{N}$, we have

$$F^2 - 4d\delta_p F - 2d(d-1)\delta_p^2 \le 0.$$

This gives us that $F \le 2d\delta_p + \sqrt{4d^2\delta_p^2 + 2d(d-1)\delta_p^2}$ and so

$$\delta_p^2\phi(p) \le \delta_p[2d + \sqrt{2d(3d-1)}].$$

Then the proof is completed.

4. **Proof of Theorem 1.3.** Note that $k_1 \le k/2\gamma\alpha^2$, and so we need only prove the case $\psi := \sup_x \phi(x) > \sup_x k_1(x)$. For small $\varepsilon > 0$, choose $x_\varepsilon \in \mathbf{R}^d$ such that $\phi(x_\varepsilon) \ge \psi - \varepsilon > \sqrt{2\varepsilon}$ and $(\psi-\varepsilon)^2 - 2\varepsilon > \psi \sup k_1$. Take $F(x) = \phi(x) - \varepsilon\rho^2(x)$, where $\rho(x) = |x - x_\varepsilon|$. Since ϕ is bounded, there exists $y_\varepsilon \in \mathbf{R}^d$ such that $F(y_\varepsilon) = \sup F$. Then $\psi - \varepsilon\rho^2(y_\varepsilon) \ge F(y_\varepsilon) = \phi(y_\varepsilon) - \varepsilon\rho^2(y_\varepsilon) \ge F(x_\varepsilon) \ge \psi - \varepsilon$, so $\phi(y_\varepsilon) \ge \psi - \varepsilon$ and $\rho(y_\varepsilon) \le 1$. Hence at point y_ε,

$$LF \le 0 \quad \text{and} \quad |\nabla\phi| = 2\varepsilon\rho \le 2\varepsilon. \tag{4.1}$$

Thus at y_ε,

$$2\varepsilon \ge |\nabla\phi| = \left|\frac{\nabla|\nabla u|}{u} - \frac{\phi}{u}\nabla u\right| \ge (\psi-\varepsilon)^2 - \frac{\big|\nabla|\nabla u|\big|}{u},$$

hence

$$\frac{\big|\nabla|\nabla u|\big|}{u} \ge (\psi-\varepsilon)^2 - 2\varepsilon > 0.$$

Therefore $k_1|\nabla u|/\big|\nabla|\nabla u|\big| < 1$. By Lemma 2.3 and (4.1) we have

$$\begin{aligned}
0 &\ge LF(y_\varepsilon) \ge L\phi(y_\varepsilon) - 2\varepsilon(c_3 + d\beta) \\
&\ge \frac{\gamma\alpha}{d-1}\left(\frac{4\varepsilon^2}{\psi} + (\psi-\varepsilon)^3 - 4\psi\varepsilon\right) - \left(\frac{\sqrt{c_1\big(\gamma\alpha + (d-1)\beta\big)}}{\sqrt{(d-1)\alpha}} + \frac{2h}{d-1}\right)(2\varepsilon + \psi^2) \\
&\quad - \left(c_2 - \frac{h^2}{(d-1)\beta}\right)\psi - 4\beta\psi\varepsilon - 2\varepsilon(d\beta + c_3).
\end{aligned}$$

Choose $\varepsilon_n \to 0$ such that $h_{(y_{\varepsilon_n})} \to h_0$, $\alpha_{(y_{\varepsilon_n})} \to \alpha_0$ and $\beta_{(y_{\varepsilon_n})} \to \beta_0$. Then we have

$$0 \ge \frac{\gamma_0\alpha_0}{d-1}\psi^3 - \left(\frac{\sqrt{c_1\big(\gamma_0\alpha_0 + (d-1)\beta_0\big)}}{\sqrt{(d-1)\alpha_0}} + \frac{2h_0}{d-1}\right)\psi^2 - \left(c_2 - \frac{h_0^2}{(d-1)\beta_0}\right)\psi,$$

where $\gamma_0 = \alpha_0/\beta_0$. Let $k_0 = 2c_3\alpha_0 + \sqrt{c_1(d-1)\big(\gamma_0\alpha_0^2 + (d-1)\alpha_0\beta_0\big)}$. Then

$$0 \ge \gamma_0\alpha_0^2\psi^2 - [k_0 + 2\alpha_0(h_0 - c_3)]\psi - [c_2(d-1)\alpha_0 - h_0^2\gamma_0].$$

Note that $0 \le h_0 \le c_3 \le k_0/2\alpha_0$ and $0 < \gamma_0 \le 1$. We have

$$\psi \le \frac{k_0 + 2\alpha_0(h_0 - c_3) + \sqrt{\big(k_0 + 2\alpha_0(h_0 - c_3)\big)^2 - 4\gamma_0^2\alpha_0^2 h_0^2 + 4c_2(d-1)\gamma_0\alpha_0^3}}{2\gamma_0\alpha_0^2}$$
$$\le \frac{k_0 + \sqrt{k_0^2 + 4\gamma_0\alpha_0^2\big(c_2(d-1)\alpha_0 - c_3^2\gamma_0\big)}}{2\gamma_0\alpha_0^2}.$$

5. **Proofs of Theorem 1.4 and Corollary 1.5.** The main tool we used to prove Theorem 1.4 is coupling. For the background of coupling and martingale methods, readers are urged to refer to Chen and Li ([2]). Take a second-order differential operator $\bar{L}$ on $\mathbf{R}^d \times \mathbf{R}^d$:

$$\bar{L}(x,y) = L(x) + L(y) + \sum_{i,j}\big(C_{ij}(x,y) + C_{ji}(x,y)\big)\frac{\partial^2}{\partial x_i \partial y_j},$$

where

$$C(x,y) = \sigma(x)\left(\sigma(y)^* - 2\frac{\sigma(y)^{-1}vv^*}{|\sigma(y)^{-1}v|^2}\right), \quad v = \frac{x-y}{|x-y|}.$$

Let (x_t, y_t) be the $\bar{L}$—diffusion process on $\mathbf{R}^d \times \mathbf{R}^d$ and $T = \inf\{t \ge 0 : x_t = y_t\}$. We call (x_t, y_t) the *coupling by reflection* of the L—diffusion process and T the coupling time (see [2] and [8]).

Since $Lu = 0$, by the martingale property of the L-diffusion process, marginality of coupling and boundedness of u, we have

$$|u(x) - u(y)| = |E^x u(x_t) - E^y u(y_t)| \le E^{x,y}|u(x_{t\wedge T}) - u(y_{t\wedge T})|$$

for all $x, y \in \mathbf{R}^d$ and $t > 0$. If u is positive and bounded, then

$$|u(x) - u(y)| \le \|u\|_\infty P^{x,y}(T > t), \quad t > 0$$

and so

$$|u(x) - u(y)| \le \|u\|_\infty P^{x,y} \quad (T = \infty). \tag{5.1}$$

Hence, to obtain an upper bound of $\|u\|_\infty / \|u\|_\infty$, we need only to estimate $P^{x,y}(T = \infty)$. For this purpose, define

$$A(x,y) = a(x) + a(y) - C(x,y) - C(x,y)^*,$$
$$B(x,y) = b(x) - b(y),$$
$$\bar{A}(x,y) = (x-y)^* A(x,y)(x-y)/|x-y|^2, \quad x \ne y,$$
$$\bar{B}(x,y) = (x-y)^* B(x,y).$$

Then we have (see [2])

$$Lh(|x-y|) = \bar{A}(x,y)h''(|x-y|) + \frac{\operatorname{tr} A(x,y) - \bar{A}(x,y) + \bar{B}(x,y)}{|x-y|}h'(|x-y|) \tag{5.2}$$

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for all $h \in C^2(\mathbf{R})$. On the other hand,

$$\bar{A}(x,y) = v^*A(x,y)v = |\sigma(x)^*v - \sigma(y)^*v|^2 + \frac{4v^*\sigma(x)\sigma(y)^{-1}v}{|\sigma(y)^{-1}v|^2}$$
$$\geq \frac{4(\sigma(y)^{-1}v)^*\big(\sigma(y)^*\sigma(x)\big)\big(\sigma(y)^{-1}v\big)}{|\sigma(y)^{-1}v|^2} \geq 4\lambda$$

and

$$\operatorname{tr} A(x,y) - \bar{A}(x,y) = \operatorname{tr}\Big[\big(\sigma(x)-\sigma(y)\big)\big(\sigma(x)-\sigma(y)\big)^*\Big] - |\sigma(x)^*v - \sigma(y)^*v|^2$$
$$+ \frac{4}{|\sigma(y)^{-1}v|}\Big[v^*\sigma(x)\sigma(y)^{-1}v - \operatorname{tr}\big(\sigma(x)\sigma(y)^{-1}vv^*\big)\Big].$$

Note that $\operatorname{tr}\big(\sigma(x)\sigma(y)^{-1}vv^*\big) = \operatorname{tr}(v^*\sigma(x)\sigma(y)^{-1}v) = v^*\sigma(x)\sigma(y)^{-1}v$. Then

(5.3) $$g(|x-y|) \geq \big(\operatorname{tr} A(x,y) - \bar{A}(x,y) + \bar{B}(x,y)\big)\big/\bar{A}(x,y), \quad x \neq y.$$

To estimate $P^{x,y}(T=\infty)$, take

$$F(r) = \frac{1}{\lambda}\int_r^1 C(s)^{-1}\,ds \int_s^1 C(t)\,dt, \quad r \geq 0.$$

Note that $\limsup_{r\to 0} g(r)/r < \infty$, then $F(0) < \infty$. Let

$$S_N = \inf\{t \geq 0 : |x_t - y_t| \geq N\}, \quad N > |x-y|.$$

The proof of [2, Theorem 4.2] gives us that $E^{x,y}(T \wedge S_N) < \infty$ and

$$P^{x,y}(T=\infty) \leq P^{x,y}(T > S_N) \leq \frac{f(|x-y|)}{f(N)}.$$

Hence $P^{x,y}(T=\infty) \leq f(|x-y|)/f(\infty)$. By (5.1) we get

$$\frac{|u(x)-u(y)|}{|x-y|} \leq \frac{\|u\|_\infty f(|x-y|)}{f(\infty)|x-y|}, \quad x,y \in \mathbf{R}^d.$$

By letting $y \longrightarrow x$ we prove Theorem 1.4.

Finally, let $a = \frac{1}{2}I$ and $b_i(x) = \sum_j b_{ij}x_j$ $(i \leq d)$, it is easy to check that $\lambda = \frac{1}{2}$ and

$$\langle b(x)-b(y), x-y\rangle = \sum_i \big(b_i(x)-b_i(y)\big)(x_i-y_i) = \sum_{i,j} b_{ij}(x_j-y_j)(x_i-y_i)$$
$$= \frac{1}{2}\sum_{i,j}(b_{ij}+b_{ji})(x_i-y_i)(x_j-y_j) \leq \lambda_d|x-y|^2.$$

Hence we can choose $g(r) = \frac{1}{2}\lambda_d^+ r^2$ and so

$$C(r) = \exp(\lambda_d^+ r^2/4), \quad f(\infty) = \sqrt{\pi}\big/\sqrt{\lambda_d^+}.$$

By Theorem 1.4 we prove Corollary 1.5.

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