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COCYCLIC MAPS AND COEVALUATION SUBGROUPS

BY

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ABSTRACT. For any space X, $DG(X, A)$ is an abelian subgroup of $[X, A]$ when A is an H-group. $DG(X, X)$ is a ring for any H-group X.

1. **Introduction**. The concepts of cocylic maps and coevaluation subgroups were first introduced and studied by Varadarajan [12]. The main object of this paper is an attempt to dualize the results obtained in [10] and [11]. While we are successful, to a certain extent, in obtaining some dual results, there still remain many open questions (see the remark at the end of Section 4 and those after Definitions 5.7 and 5.9) in the dual case. The approach adopted here is quite different from that given in [5]. Existence of cocylcic maps is shown in Section 3. In Section 4 we provide a characterization of a co-H-space in terms of cocyclicity of maps. Among other things, we prove that if f is cocyclic, then Σf is cocentral. In Section 5, we settle a problem of [12] by showing that $DG(X, A)$ is an abelian subgroup of $[X, A]$ when A is an H-group. A result of [6] is also generalized. Section 6 contains our main result, namely, $DG(X, X)$ is a ring for any H-group X. In the course of achieving this, it is also proved that $DG(X, A)$ is a covariant functor of A from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms.

Unless otherwise stated, we shall work in the category of spaces with base points and having the homotopy type of locally finite CW-complexes. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by $*$. For simplicity, we use the same symbol for a map and its homotopy class.

The diagonal map $\triangle: X \to X \times X$ is given by $\triangle(x) = (x, x)$ for each $x \in X$, the folding map $\nabla: X \vee X \to X$ by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$, and the switching map $T: X \times Y \to Y \times X$ by $T(x, y) = (y, x)$ for each $x \in X$, $y \in Y$. Frequently i and j will be reserved for the inclusion maps.

2. **Definition and existence of cocyclic maps**.

DEFINITION 2.1 [12]. *A map $f: X \to A$ is said to be cocyclic if we can find a map ϕ: $X \to X \vee A$ such that the following diagram is homotopy commutative: that is,*

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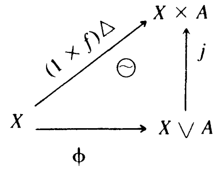
$j\phi \simeq (1 \times f)\Delta$. *We call such a map* ϕ *a coassociated map of* f. *The set of all homotopy classes of cocyclic maps from* X *to* A *is denoted by* $DG(X, A)$.



The following are two elementary examples of cocyclic maps.

EXAMPLE 1. *Every constant map* $*: X \to A$ *is cocylic. In fact, we may take the inclusion map* $i_1: X \to X \vee A$ *to be a coassociated map of* $*$.

EXAMPLE 2. *If* X *is a co-H-space, then every map* $f: X \to A$ *is cocyclic. To see this, let* $\psi: X \to X \vee X$ *be the co-H-structure on* X. *Then we may take* $(1 \vee f)\psi$ *to be a coassociated map of* f.

The next example to be presented is not quite so trivial and reference of [7] is required.

EXAMPLE 3. *Consider the cofibration* $A \vee B \xrightarrow{j} A \times B \to A \wedge B \equiv A \times B/A \vee B$, *where* A *and* B *are spaces of the homotopy type of CW complexes. For simplicity, let* $X = A \vee B$, $Y = A \times B$ *and* $Z = A \wedge B$. *Then the coboundary map* ∂ *in the following Puppe sequence of the given cofibration is cocyclic:*

$$X \xrightarrow{j} Y \to Z \xrightarrow{\partial} \Sigma X \to \Sigma Y \to \dots .$$

In fact, there exists, according to [7, p. 171], a cooperation $\phi: Z \to Z \vee \Sigma X$ with $p_1\phi \simeq 1_Z$ where $p_1: Z \vee \Sigma X \to Z$ is a projection, and $\partial^\# = s^\#$ [7, p. 176] where $s = p_2\phi$ and $p_2: Z \vee \Sigma X \to \Sigma X$ is a projection. Thus $\partial^\#[1_{\Sigma X}] = s^\#[1_{\Sigma X}]$, i.e. $1_{\Sigma X} \circ \partial \simeq 1_{\Sigma X} \circ s$ or $\partial \simeq s$. Claim: s is cocyclic. Indeed, we have

$$\begin{aligned}(1 \times s)\Delta &= (1 \times p_2\phi)\Delta \simeq (p_1\phi \times p_2\phi)\Delta \\ &= (p_1 \times p_2)(\phi \times \phi)\Delta = (p_1 \times p_2)\Delta\phi = \phi = j\phi\end{aligned}$$

where j is the obvious inclusion. Thus s and hence ∂ is cocyclic.

3. **Some basic results related to cocyclic maps**. We shall now record some basic results related to cocyclic maps. The following useful lemma is due to Varadarajan:

LEMMA 3.1 [12]. *If* $f: X \to A$ *is a cocyclic map and* $\theta: A \to B$ *is an arbitrary map, then the map* $\theta f: X \to B$ is cocyclic.

An immediate consequence of this lemma is the next proposition which provides a characterization of a co-H-space in terms of cocyclicity of maps.

PROPOSITION 3.2. *Let* X *be a space. Then the following are equivalent:*

a) X *is a co H-space.*
b) 1_X *is cocyclic.*
c) $DG(X, A) = [X, A]$ *for any space* A.

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PROOF. The proof is exactly dual to that of Proposition 3.3 of [10].

Let $f: X \to A$ be a cocyclic map. Observe that if $h: Y \to X$ is any map which has a left homotopy inverse, then $fh: Y \to A$ is also cocyclic.

LEMMA 3.3. *Let A be an H-space and $f: X \to A$ a map. Then f is cocyclic iff $e'f: X \to \Omega\Sigma A$ is cocyclic where $e': A \to \Omega\Sigma A$ is the usual map.*

The following lemma which shows that cocyclicity of maps is closed under the wedge product is clear.

LEMMA 3.4. *If the maps $f: X \to A$ and $g: Y \to B$ are cocyclic, then so is $f \vee g: X \vee Y \to A \vee B$.*

We shall now introduce a concept for a map which is closely related to cocyclicity.

DEFINITION 3.5. *Let (G, ϕ, ν) be a H-cogroup and A any space. We say that a map $f: G \to A$ is cocentral if $(1 \vee f)c \simeq *$ where $c: G \to G \vee G$ is the basic cocommutator map (that is, $c \equiv \nabla(1 \vee 1 \vee \nu \vee \nu)\ (\phi \vee \phi)\phi$).*

The following useful lemmas are immediate consequences of the definition.

LEMMA 3.6. *If $f: G \to A$ is a cocentral map and $\theta: A \to B$ is an arbitrary map, then the map $\theta f: G \to B$ is cocentral.*

LEMMA 3.7. *Let $i_1: G \to G \vee A$ and $i_2: A \to G \vee A$ be inclusions. Then $f \in [G, A]$ is cocentral iff $(i_1, i_2 f) = 0 \in [G, G \vee A]$.*

PROOF. Let $j_1, j_2: G \to G \vee G$ be the obvious inclusions. Then

$$\begin{aligned}(1 \vee f)c &= (1 \vee f)\ \{\phi + (\nu \vee \nu)\phi\} \\ &= (1 \vee f)\phi + (\nu \vee f\nu)\phi \\ &= (1 \vee f)\ (j_1 + j_2) + (\nu \vee f\nu)\ (j_1 + j_2) \\ &= (i_1 + i_2 f) + (-i_1 - i_2 f) = (i_1, i_2 f).\end{aligned}$$

Hence the assertion follows.

Next we establish an important lemma (to be applied in Section 4).

LEMMA 3.8. *Any cocentral map $f: G \to A$ lies in the center of $[G, A]$.*

PROOF. We first note that $(1 \vee f)c \simeq *$ *iff* $(1 \vee f)\phi = T(f \vee 1)\phi$ where T is the switching map. Let $g \in [G, A]$. Then

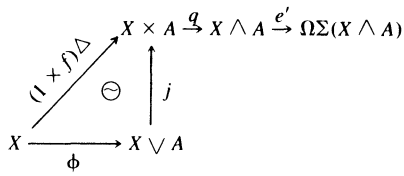
$$\begin{aligned}\nabla(f \vee g)\phi &= \nabla(1 \vee g)\ (f \vee 1)\phi \\ &= \nabla(g \vee 1)T\ (f \vee 1)\phi \\ &= \nabla(g \vee 1)\ (1 \vee f)\phi = \nabla(g \vee f)\phi.\end{aligned}$$

Thus $f + g = g + f$ for all $g \in [G, A]$ and the assertion follows.

LEMMA 3.9 [3]. *Let* $f: X \to A$ *be a map. Then* Σf *is cocentral iff* $e'q(1 \times f)\Delta \simeq *$ *where* $e': X \wedge A \to \Omega\Sigma(X \wedge A)$ *is the usual map and* $q: X \times A \to X \wedge A$ *is the quotient map.*

COROLLARY 3.10. *If* $f: X \to A$ *is a cocyclic map, then the map* Σf *is cocentral.*

PROOF. This follows from the above lemma and the existence of a map $\phi: X \to X \vee A$ such that $j\phi \simeq (1 \times f)\Delta$. In fact, we have $e'q(1 \times f)\Delta \simeq e'qj\phi \simeq *$ from the following homotopy commutative diagram



REMARK. While we were able to investigate the relationship between cyclicity of maps and maps of finite order [10], we do not know whether those results can be dualized. The main obstacle for obtaining the dual results is our lack of knowledge on the existence of the dual of $G(A, X) = \omega_{\#}([A, X^X])$ (see [10]) and those of the results of [4] on homology.

4. **Basic properties and related results of** $DG(X, A)$. In this section we shall derive some basic properties of $DG(X, A)$ and $DC(X, A)$.

PROPOSITION 4.1. *Let* $\Omega A \xrightarrow{f} Y \to C$ *be a cofibration. If* $DG(C, A) = 0$, *then there exists a map* $\rho: Y \to \Omega A$ *such that* $\rho f \simeq 1_{\Omega A}$.

PROOF. Sinc $DG(C, A) = 0$, we have $\partial^{\#}([\Sigma\Omega A, A]) = 0$, so that $(\Sigma f)^{\#}$ is onto. Thus for $e: \Sigma\Omega A \to A$, we can find a map $g: \Sigma Y \to A$ such that $(\Sigma f)^{\#}[g] = [e]$, that is, $g(\Sigma f) \simeq e$. Taking adjoints, we get $\tau(g)f \simeq 1_{\Omega A}$.

In [12], Varadarajan raised the question as to whether $DG(X, A)$ is a subgroup of $[X, A]$ when A is an H-group. We provide an answer in the affirmative. In fact, we have the following theorem.

THEOREM 4.2. *If* A *is an* H*-group, then* $DG(X, A)$ *is a subgroup (in fact, it is abelian, see Theorem 6.2) of* $[X, A]$ *for any space* X.

PROOF. Let m and μ be the H-structure and the inverse on A respectively. Then the inverse of f in the group $[X, A]$ is the homotopy class of $\mu f: X \to A$. According to Lemma 3.1, $\mu f \in DG(X, A)$ if $f \in DG(X, A)$. Hence $DG(X, A)$ is closed under inversion. To see that it is closed under the operation + in $[X, A]$, let $f, g \in DG(X, A)$. Then we can find maps ϕ, ψ: $X \to X \vee A$ such that $j\phi \simeq (1 \times f)\Delta$ and $j\psi \simeq (1 \times g)\Delta$. Let $i: (X \vee A) \vee A \to X \vee (A \times A)$ and $i': (X \times A) \vee A \to X \times (A \times A)$ be the obvious inclusions. Then we have the following homotopy commutative diagram:

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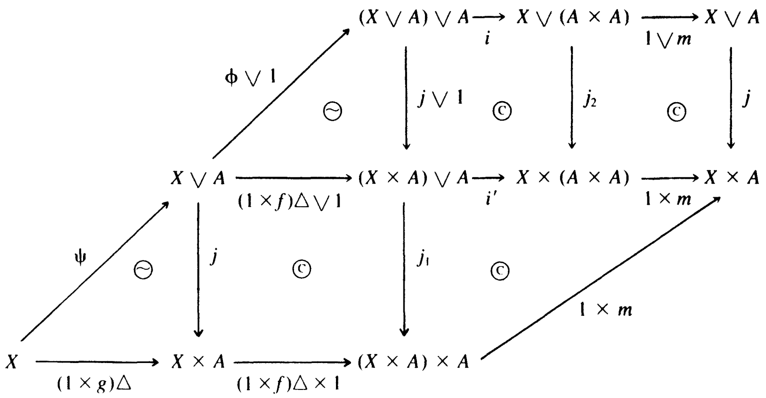

Here all vertical arrows are inclusions. Let $\lambda \equiv (1 \vee m)i(\phi \vee 1)\psi$. Then we have

$$\begin{aligned} j\lambda &\simeq (1 \times m)\{(1 \times f)\triangle \times 1\}\{(1 \times g)\triangle\} \\ &= (1 \times m)\{(1 \times f)\triangle \times g\}\triangle = \{1 \times m(f \times g)\triangle\}\triangle \\ &= \{1 \times (f + g)\}\triangle . \end{aligned}$$

Thus $f + g \in DG(X, A)$, so that $DG(X, A)$ is closed under $+$. Hence $DG(X, A)$ is a subgroup of $[X, A]$.

REMARK 1. In view of the preceding theorem, we can naturally call $DG(X, A)$ the coevaluation subgroup of X with respect to A when it is a subgroup.

REMARK 2. If $A = K(\pi, n)$, an Eilenberg–MacLane complex of type (π, n), then $DG(X, A)$ reduces to $G^n(X; \pi)$ [6] which is called the n^{th} coevaluation subgroup of X (with respect to π).

Our next result is straightforward.

PROPOSITION 4.3. *Let X and Y be spaces having the homotopy type of CW-complexes and A any space. Then the sets* $DG(X \vee Y, A)$ *and* $DG(X, A) \times DG(Y, A)$ *are isomorphic where* $\times$ *denotes the cartesian product.*

We are now in a position to establish the following theorem.

THEOREM 4.4. *Let X and Y be spaces having the homotopy type of CW-complexes and A an H-group. Then* $DG(X \vee Y, A) \cong DG(X, A) \oplus DG(Y, A)$ *as groups, where* $\cong$ *and* $\oplus$ *denote isomorphism and the direct product respectively.*

PROOF. In view of Theorem 4.2 and the preceding proposition, it suffices to show that the function Φ defined in the proof of the latter is a homomorphism of groups. To do this, let $f, g \in DG(X \vee Y, A)$ and m the given H-structure on A. Then we have

$$\begin{aligned} \Phi(f + g) &= \Phi\{m(f \times g)\triangle\} \\ &= (m(f \times g)\triangle i_1, m(f \times g)\triangle i_2) \end{aligned}$$

$$= (m(fi_1 \times gi_1)\triangle,\ m(fi_2 \times gi_2)\triangle)$$
$$= (fi_1 + gi_1, fi_2 + gi_2)$$
$$= (fi_1, fi_2) + (gi_1, gi_2) = \Phi(f) + \Phi(g).$$

Hence Φ is a homomorphism of groups.

When $A = K(\pi, n)$, the above theorem reduces to the following result of Haslam.

COROLLARY 4.5 [6].

$$G^n(X \vee Y; \pi) = G^n(X; \pi) \oplus G^n(Y; \pi)$$

for all integers $n \geqslant 0$ and abelian groups π.

EXAMPLE 1. *Let T be the torus. Then*

$$DG(S^2 \vee S^1, T) \cong DG(S^2, T) \oplus DG(S^1, T)$$
$$= 0 \oplus (Z \oplus Z) = Z \oplus Z.$$

EXAMPLE 2. *Let X be the figure-eight space. Then*

$$DG(X, S^1) \cong DG(S^1, S^1) \oplus DG(S^1, S^1) = Z \oplus Z$$

and

$$DG(X, \Omega S^1) = \pi_1(\Omega S^1) \oplus \pi_1(\Omega S^1) = 0.$$

Let $\{X_\alpha\}$ be a collection of spaces having the homotopy type of CW-complexes. Let $\vee X_\alpha$ be the subspace of the product space ΠX_α defined as follows:

$$\vee X_\alpha \equiv \{\langle x_\alpha \rangle |\ \text{all coordinates } x_\alpha, \text{ except possibly one, are base points}\}.$$

Then the preceding theorem (resp. proposition) can be extended to the following proposition.

PROPOSITION 4.6.

$$DG(\vee X_\alpha, A) \cong \oplus DG(X_\alpha, A)$$

as groups (resp. as sets), where $\oplus$ denotes the direct product (resp. the cartesian product).

We shall now introduce a subset $DC(X, A)$ of $[X, A]$ which is the dual of $C(A, X)$ defined in [11]. If A is an H-space, then the function Σ: $[X, A] \to [\Sigma X, \Sigma A]$, given by $f \mapsto \Sigma f$, is injective. Let $[\Sigma X, \Sigma A]_{C\Sigma}$ denote the subset of $[\Sigma X, \Sigma A]$ consisting of those homotopy classes of maps Σf which are cocentral.

DEFINITION 4.7. *Let A be an H-space. We define*

$$DC(X, A) \equiv \Sigma^{-1}[\Sigma X, \Sigma A]_{C\Sigma}.$$

REMARK. Clearly, $DG(X, A) \subset DC(X, A)$ if A is an H-space. The appearance of the factor e' in Lemma 3.9 changes some dual results. In particular, it is not known to us whether $DG(\Omega X, A) = DC(\Omega X, A)$. However, the distributive laws and Proposition 4.6 of [11] dualize without any change.

PROPOSITION 4.8. *Let A, B and X be spaces. If $f \in DC(X, \Omega A)$, then $[f, g]' = 0$ for all $g \in [X, \Omega B]$ where $[\ \ ]'$ is the dual of the generalized Whitehead product* [1].

PROOF. Let $i_1: A \to A \vee B$ and $i_2: B \to A \vee B$ be the usual inclusions. According to [1], we have

$$(\Omega i)\ [f, g]' = (\Omega i_1)f + (\Omega i_2)g - (\Omega i_1)f - (\Omega i_2)g.$$

Taking τ^{-1}, we obtain

$$i\tau^{-1}([f, g]') = i_1\{e_1(\Sigma f)\} + i_2\{e_2(\Sigma g)\} - i_1\{e_1(\Sigma f)\} - i_2\{e_2(\Sigma g)\},$$

where $e_1: \Sigma\Omega A \to A$ and $e_2: \Sigma\Omega B \to B$ are the usual maps. Since Σf is cocentral, so is $i_1\{e_1(\Sigma f)\}$ by Lemma 3.6. According to Lemma 3.8, we have $i\tau^{-1}([f, g]') = 0$. Hence $[f, g]' = 0$ as $i_\#$ is mono and τ^{-1} is an isomorphism.

EXAMPLE. *Let X be a co-H-space, A and B any spaces. Then $[f, g]' = 0$ for all $f \in [X, \Omega A]$* and $g \in [X, \Omega B]$.

DEFINITION 4.9.

$$DW(X, \Omega A) \equiv \{\alpha \in [X, \Omega A] \,|\, [\alpha, \beta]' = 0 \text{ for all } \beta \in [X, \Omega B] \text{ and for all } B\}.$$

$$DP(X, \Omega A) \equiv \{\alpha \in [X, \Omega A] \,|\, [\alpha, \beta]' = 0 \text{ for all } \beta \in [X, \Omega^l A] \text{ and for all } l \geq 1\}.$$

REMARKS. Clearly we have the following inclusions:

$$DG(X, \Omega A) \subset DC(X, \Omega A) \subset DW(X, \Omega A) \subset DP(X, \Omega A) \subset [X, \Omega A].$$

It will be interesting to have examples which show that some of the inclusions are proper. It is also not known whether $DP(X, \Omega A)$ is a subgroup of $[X, \Omega A]$ or $DG(\Omega X, \Omega X) = DC(\Omega X, \Omega X) = DW(\Omega X, \Omega X) = DP(\Omega X, \Omega X)$ for any space X.

5. ***DG(X, X)* as ring**. We first show that $DC(X, A)$ and $DG(X, A)$ are covariant functors of A from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms. The fact that $DC(X, X)$ and $DG(X, X)$ are rings will then follow immediately.

LEMMA 5.1 [8]. *Let A be an H-space with H-structure m, and let X be a space. Let $f, g: X \to A$ be maps. Then $\Sigma(f + g) = J(m)\Sigma\{q(f \times g)\Delta\} + \Sigma f + \Sigma g$ where $J(m): \Sigma(A \wedge A) \to \Sigma A$ is the Hopf construction on m and $q: A \times A \to A \wedge A$ is the quotient map.*

THEOREM 5.2. *Let A be an H-space with a right homotopy inverse μ, and let X be a space. Then $DC(X, A)$ and $[\Sigma X, \Sigma A]_{C\Sigma}$ are subgroups contained in the centers of $[X, A]$ and $[\Sigma X, \Sigma A]$ respectively, and $\Sigma: DC(X, A) \to [\Sigma X, \Sigma A]_{C\Sigma}$ is an isomorphism of abelian groups. In particular, $DG(X, A)$ is contained in the center of $[X, A]$.*

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PROOF. We first show that $\Sigma(f+g) = \Sigma f + \Sigma g$ if f or g is in $DC(X, A)$. To do this, consider the map $J(m)\Sigma\{q(f \times g)\Delta\}$. Taking adjoint, we obtain

$$\tau(J(m))\{q(f \times g)\Delta\} = \Omega(J(m))e'q(f \times g)\Delta \simeq *.$$

Thus $J(m)\Sigma\{q(f \times g)\Delta\} \simeq *$ and hence $\Sigma(f+g) = \Sigma f + \Sigma g$. With the aid of Lemma 3.7, it can be easily verified that $\Sigma f + \Sigma g$ is cocentral for all $f, g \in DC(X, A)$.

Next, we claim that $-\Sigma f$ is cocentral if $f \in DC(X, A)$. In fact,

$$0 = \Sigma(f + \mu f) = \Sigma f + \Sigma(\mu f)$$

implies that $-\Sigma f = (\Sigma\mu)(\Sigma f)$ which is cocentral by Lemma 3.6. Hence $[\Sigma X, \Sigma A]_{C\Sigma}$ is a subgroup of $[\Sigma X, \Sigma A]$. That $[\Sigma X, \Sigma A]_{C\Sigma}$ lies in the center of $[\Sigma X, \Sigma A]$ follows from Lemma 3.8. The facts that $\Sigma(f+g) = \Sigma f + \Sigma g$ if f or g is in $DC(X, A)$ and that Σ is an injection also imply that $DC(X, A)$ lies in the center of $[X, A]$. Consequently, $DC(X, A)$ and $[\Sigma X, \Sigma A]_{C\Sigma}$ are subgroups contained in the centers of $[X, A]$ and $[\Sigma X, \Sigma A]$ respectively, and

$$\Sigma: DC(X, A) \to [\Sigma X, \Sigma A]_{C\Sigma}$$

is an isomorphism of abelian groups. The proof of the theorem is thus complete.

Let $f: X \to A$ be a map where X is an H-space with H-structure m, and A is a homotopy associative H-space. Then we can find a retraction $\gamma: \Omega\Sigma A \to A$ which is an H-map. Suppose $g_1, g_2: Y \to X$ are maps where Y is any space. Then we can form $f(g_1 + g_2): Y \to A$. Let $J(fm): \Sigma(X \wedge X) \to \Sigma A$ be the Hopf construction on fm. Then we have the following lemma.

LEMMA 5.3 [8]. $f(g_1 + g_2) = \gamma\tau\{J(fm)\}q(g_1 \times g_2)\Delta + fg_1 + fg_2$,

where τ is the adjoint functor and $q: X \times X \to X \wedge X$ is the quotient map.

PROPOSITION 5.4. *If $f: X \to A$ is a map from an H-space into a homotopy associative H-space A, then*

$$f_\#: DC(Y, X) \to DC(Y, A)$$

and

$$f_\#: DG(Y, X) \to DG(Y, A)$$

are homomorphisms for any space Y.

PROOF. It suffices to show that $f(g_1 + g_2) = fg_1 + fg_2$ for all $g_1, g_2 \in DC(Y, X)$. In fact, we have

$$\gamma\tau\{J(fm)\}q(g_1 \times g_2)\Delta = \gamma\Omega(J(fm))e'q(g_1 \times g_2)\Delta,$$

and hence the assertion follows from Lemma 3.9 and the preceding lemma.

Combining Theorem 5.2 with Proposition 5.4, we have the next result.

THEOREM 5.5. *If $f: X \to A$ is a map of H-groups, then*

$$f_{\#}: DC(Y, X) \to DC(Y, A)$$

and

$$f_{\#}: DG(Y, X) \to DG(Y, A)$$

are homomorphisms of abelian groups for any space Y.

In view of the above theorem, we see that for any space Y, $DC(Y, -)$ and $DG(Y, -)$ are covariant functors from the full subcategory of H-groups and maps into the category of abelian groups and homomorphisms.

EXAMPLE. *If $f: K(\pi, n) \to K(\pi, r)$ is a map, then*

$$f_{\#}: G^n(Y; \pi) \to G^r(Y; \pi)$$

is a group homomorphism for any space Y.

We are now in a position to state our main theorem.

THEOREM 5.6. *For any H-group X, $DC(X, X)$ and $DG(X, X)$ are rings.*

PROOF. That $DC(X, X)$ and $DG(X, X)$ are abelian groups follows from Theorem 5.2. Lemmas 3.6 and 3.1 imply, respectively, that $DC(X, X)$ and $DG(X, X)$ are closed under composition of maps which is associative. The left distributive law, that is, $f(g_1 + g_2) = fg_1 + fg_2$, is a consequence of Proposition 5.4 and the right distributive law is trivial.

EXAMPLE 1. *For any space X, $DC(\Omega X, \Omega X)$ and $DG(\Omega X, \Omega X)$ are rings.*

EXAMPLE 2. *For any Eilenberg–MacLane complex $K = K(\pi, n)$, $DC(K, K)$ and $DG(K, K)$ are rings.*

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