AN ARITHMETIC SUM WITH AN APPLICATION TO QUASI k-FREE INTEGERS

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1. Introduction

Let n be a positive integer and let T be any nonempty set of positive integers. By $(n,T)=1$ we mean n is relatively prime to each element of T. Hence n can be written as $n = n_1 n_2$ where n_1 is the largest divisor of n such that $(n_1, T) = 1$. Let P be a property associated with positive integers. We shall say that a positive integer n is a P-number if it satisfies the property P. If in the above representation of n, the integer n_1 is a P-number, then we shall say that n is a quasi P-number relative to T, or, simply, a quasi P-number. In particular, for k a positive integer > 1, n is quasi k-free (for given set T) if n_1 is k-free.

Property P may be the property of belonging to a set A. In such a case we use the following notations:

$Q_{A;T}$ = *the set of all quasi belonging to A numbers*;

$Q_{A;T}(x)$ = *the number of positive integers* $\leqq x$ *belonging to* $Q_{A;T}$, *where* x *is real and* $\geqq 1$;

$Q_{A;T}(x; h)$ = *the number of positive integers* $\leqq x$ *belonging to* $Q_{A;T}$ *and relatively prime to a fixed positive integer* h;

$q_{A;T}(n)$ = *the characteristic function of* $Q_{A;T}$, *that is* $q_{A;T}(n) = 1$ *if* n *is in* $Q_{A;T}$ *and* $= 0$ *otherwise.*

In Section 2 we establish a sum for $Q_{A;T}(x)$ where the set A is multiplicative. Some known and some new results follow as corollaries. In Section 3 we give an estimate for a certain class of quasi k-free numbers.

2. The arithmetic sum

LEMMA 2.1. *Let T be any nonempty set of positive integers and let A be any nonempty set of positive integers whose characteristic function $a(n)$ is*

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multiplicative. Then

$$Q_{A;T}(x) = \sum_{\substack{n \leqq x \\ (n,T)=1}} a^*(n) \left[\frac{x}{n}\right]$$

where $[x]$ *is the bracket function and* $a^*(n) = \sum_{d|n} a(d)\mu(n/d)$ *or, equivalently,* $a(n) = \sum_{d|n} a^*(d)$. *More generally, for h a positive integer,*

$$Q_{A;T}(x; h) = \sum_{(a,h)=(d,T)=1} a^*(d)\phi\left(\frac{x}{d}, h\right)$$

where $\phi(x,h)$ *is the number of positive integers* $\leqq x$ *and relatively prime to h.*

PROOF. Let $n = n_1 n_2$ where n_1 is the largest divisor of n such that $(n_1, T) = 1$. Then

$$q_{A;T}(n) = q_{A;T}(n_1 n_2) = q_{A;T}(n_1) = a(n_1) = \sum_{d|n_1} a^*(d)$$

so that

$$q_{A;T}(n) = \sum_{\substack{d|n \\ (d,T)=1}} a^*(d).$$

Now let $\varepsilon(n, T)$ be defined by

$$\varepsilon(n, T) = \begin{cases} 1, & (n,T) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$q_{A;T}(n) = \sum_{d|n} a^*(d)\varepsilon(d,T).$$

Hence

$$Q_{A;T}(x) = \sum_{n \leqq x} q_{A;T}(n) = \sum_{\substack{n \leqq x \\ d|n}} a^*(d)\varepsilon(d, T)$$

$$= \sum_{\substack{n \leqq x \\ (n.T)=1}} a^*(n) \left[\frac{x}{n}\right].$$

The proof of the more general result is similar and will be omitted.

In the rest of this paper we take $T = \{m\}$, the set consisting of the single element m, where m is a square-free integer. For convenience we write $Q_{k;m}$ in the place of $Q_{k;\{m\}}$.

COROLLARY 2.2. *For K the set of all k-free integers,*

$$Q_{K;m}(x) = \sum_{\substack{n \leqq x \\ (n.m)=1}} \mu_k(n) \left[\frac{x}{n}\right]$$

where $\mu_k(n)$ *is the multiplicative function given for powers of an arbitrary prime p by*

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$$\mu_k(p^a) = \begin{cases} 1, & a=0 \\ -1, & a=k \\ 0, & \text{otherwise.} \end{cases}$$

The corollary follows from the well-known result:

$$q_K(n) = \sum_{d|n} \mu_k(d) = \sum_{d^k|n} \mu(d).$$

COROLLARY 2.3. *More generally,*

$$Q_{K;m}(x;h) = \sum_{\substack{n \leqq x \\ (n,mh)=1}} \mu_k(n)\phi\left(\frac{x}{n}, h\right).$$

COROLLARY 2.4. *Let U be the set $\{1\}$. For $A = U$, and $m = p$, a prime,*

$$Q_{U;p}(x) = \sum_{\substack{n \leqq x \\ (n,p)=1}} \mu(n)\left[\frac{x}{n}\right],$$

a result attributed to Newman by Gupta [I, p. 445].

COROLLARY 2.5. *For $A = U = \{1\}$,*

$$Q_{U;m}(x) = \sum_{\substack{n \leqq x \\ (n,m)=1}} \mu(n)\left[\frac{x}{n}\right].$$

This result is due to Gupta [I] (as being the number of divisors of m^s, where $s = \log_2 x$, which do not exceed x).

COROLLARY 2.6. *Let V be the set of all positive integers which are k-th powers, $k \geqq 2$. Then*

$$Q_{V;m}(x) = \sum_{\substack{n \leqq x \\ (n,m)=1}} \lambda_k(n)\left[\frac{x}{n}\right]$$

where $\lambda_k(n)$ is the multiplicative function defined for powers of an arbitrary prime p by

$$\lambda_k(p^a) = \begin{cases} 1, & a \equiv 0 \pmod{k} \\ -1, & a \equiv 1 \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

This result is due to Gupta [2].

COROLLARY 2.7. *Let R be the set of all (k,r)-numbers, that is, the set of all positive integers whose k-free parts are r-free, where $0 < r < k$ (Subbarao and Harris [3]). Then*

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$$Q_{R,m}(x) = \sum_{\substack{n \leqq x \\ (n,m)=1}} \lambda_{k,r}(x) \left[\frac{x}{n}\right],$$

where $\lambda_{k,r}(n)$ is the multiplicative function defined for powers of an arbitrary prime p by

$$\lambda_{k,r}(p^a) = \begin{cases} 1, & a \equiv 0 \pmod{k} \\ -1, & a \equiv r \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

3. Application to quasi k-free numbers

Let $J_k(n)$ be the Jordan totient, $\phi(n)$ be Euler's phi-function, $\zeta(k)$ be the Riemann zeta function and $\sigma_s^*(m)$ be the sum of the s-th powers of the square-free divisors of m.

THEOREM 3.1. *Let $h = h_1 h_2$ where h_1 is the largest divisor of h such that $(h_1, m) = 1$. Then*

$$Q_{K;m}(x; h) = \frac{m^k h^k \phi(h_1)\phi((m,h))}{(m,h)h_1 J_k(mh)} \frac{x}{\zeta(k)} + 0\left(\frac{\phi(mh)}{mh} \sigma_{-s}^*(h) x^{1/k}\right)$$

uniformly with respect to m, h and x for any s, $0 < s < 1/k$.

PROOF. By Corollary 2.3,

$$\begin{aligned} Q_{K;m}(x; h) &= \sum_{\substack{d \leqq x \\ (d.mh)=1}} \mu_k(d)\phi\left(\frac{x}{d}, h\right) \\ &= \sum_{\substack{d \leqq x \\ (d,mh)=1}} \mu_k(d) \left\{\frac{x}{d} \frac{\phi(h)}{h} + 0\left(\frac{x^s}{d^s}\sigma_{-s}^*(h)\right)\right\} \end{aligned}$$

for every s where $0 < s < 1$, by Cohen [4]. Continuing,

$$\begin{aligned} Q_{K;m}(x; h) &= \frac{\phi(h)}{h} x \sum_{\substack{d \leqq x \\ (d,mh)=1}} \frac{\mu_k(d)}{d} + 0\left(\sum_{\substack{d \leqq x \\ (d,mh)=1}} \mu_k(d) \frac{x^s}{d^s} \sigma_{-s}^*(h)\right) \\ &= \frac{\phi(h)}{h} x \sum_{\substack{d=1 \\ (d,mh)=1}}^{\infty} \frac{\mu_k(d)}{d} - \frac{\phi(h)}{h} \sum_{\substack{d > x \\ (d,mh)=1}} \frac{\mu_k(d)}{d} \\ &\qquad + 0\left(\sum_{\substack{d \leqq x \\ (d,mh)=1}} \mu_k(d) \frac{x^s}{d^s} \sigma_{-s}^*(h)\right), \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say. We investigate I_1, I_2, I_3 in turn. First,

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$$I_1 = \frac{\phi(h)}{h} x \sum_{\substack{d=1 \\ (d,mh)=1}}^{\infty} \frac{\mu_k(d)}{d} = \frac{\phi(h)}{h} x \prod_{(p,mh)=1} \left(1 - \frac{1}{p^k}\right)$$

$$= \frac{\phi(h)}{h} x \frac{\prod_p \left(1 - \frac{1}{p^k}\right)}{\prod_{p|mh} \left(1 - \frac{1}{p^k}\right)} = \frac{\phi(h)}{h} x \frac{1}{\zeta(k)} \frac{(mh)^k}{J_k(mh)}$$

$$= \frac{x}{\zeta(k)} m^k h^k \frac{1}{J_k(mh)} \frac{\phi(h_1)}{h_1} \frac{\phi(h_2)}{h_2}$$

$$= \frac{x}{\zeta(k)} m^k h^k \frac{1}{J_k(mh)} \frac{\phi((m,h))}{(m,h)} \frac{\phi(h_1)}{h_1}.$$

Next,

$$|I_2| \leqq x \frac{\phi(h)}{h} \sum_{\substack{d>x \\ (d,mh)=1}} \frac{|\mu_k(d)|}{d} \leqq \frac{x\phi(h)}{h} \sum_{\substack{d^k>x \\ (d,mh)=1}} \frac{1}{d^k}$$

$$\leqq x \frac{\phi(h)}{h} 0 \left(\frac{1}{x^{1-1/k}}\right) = 0\left(\frac{\phi(h)}{h} x^{1/k}\right).$$

Finally,

$$I_3 = x^s \sigma^*_{-s}(h)\, 0\left(\sum_{\substack{d \leqq x \\ (d,mh)=1}} \frac{\mu_k(d)}{d^s}\right)$$

$$\leqq x^s \sigma^*_{-s}(h)\, 0\left(\sum_{\substack{d^k \leqq x \\ (d,mh)=1}} \frac{1}{d^{ks}}\right)$$

$$\leqq x^s \sigma^*_{-s}(h)\, 0\left(\sum_{\substack{d \leqq x^{1/k} \\ (d,mh)=1}} \frac{1}{d^{ks}}\right)$$

and so

$$I_3 = x^{1/k} \sigma^*_{-s}(h) \frac{\phi(mh)}{mh}$$

if we can show

$$\sum_{\substack{n \leqq y \\ (n,u)=1}} \frac{1}{n^s} = 0\left(\frac{\phi(u)}{u} y^{1-s}\right), \qquad 0 < s < 1.$$

This result is established as the

LEMMA 3.2. $$\sum_{\substack{1 \leqq n \leqq y \\ (n,u)=1}} \frac{1}{n^s} = 0\left(\frac{\phi(u)}{u} y^{1-s}\right)$$

for u a positive integer and any s in $0 < s < 1$.

PROOF. Let

$$\eta(n,u) = \begin{cases} 1 & \text{if } (n,u)=1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{\substack{1\leqq n\leqq y\\(n,u)=1}} \frac{1}{n^s} = \sum_{1\leqq n\leqq y} \frac{\eta(n,u)}{n^s}.$$

Since by the same lemma of Cohen as used earlier

$$\sum_{n\leqq y} \eta(n,u) = \phi(y,u) = y\frac{\phi(u)}{u} + 0(y^s\sigma^*_{-s}(u)),$$

the sum can be evaluated by a standard method (see, for example, Hardy and Wright [5]) to be

$$\sum_{\substack{1\leqq n\leqq y\\(n,u)=1}} \frac{1}{n^s} = \left[y\frac{\phi(u)}{u} + 0(y^s\sigma^*_{-s}(u))\right]\frac{1}{y^s}$$

$$- \int_1^y \left[\frac{t\phi(u)}{u} + 0(t^s\sigma^*_{-s}(u))\right]\frac{-s}{t^{s+1}}dt$$

$$= 0\left(y^{1-s}\frac{\phi(u)}{u}\right).$$

The theorem follows upon combining the results of evaluating I_1, I_2 and I_3.

COROLLARY 3.3 TO THEOREM 3.1. *Taking* $m=1$ *in theorem* 3.1, *we see that* $h_1 = h$, $h_2 = 1$, *so that we have the following result*: $Q_{K;1}(x;h)$ = *The number of k-free integers* $\leqq x$ *which are relatively prime to h is given by*

$$Q_{K;1}(x;h) = \frac{h^{k-1}\phi(h)}{J_K(h)}\cdot\frac{x}{\zeta(k)} + 0\left(\frac{\phi(h)}{h}\sigma^*_{-s}(h)x^{1/k}\right),$$

uniformly with respect to h and x, for any s such that $0 < s < 1/k$.

This result has been very recently proved under a slightly different notation by Suryanarayana [8] as an improvement in the 0-estimates of the error term obtained in his earlier papers ([6] and [7]).

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