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A SERIES FOR $\zeta(s)$

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We obtain a series representation for $\zeta(s)$ valid in $\operatorname{Re} s > -k(<0)$. This representation is obtained by a sequence of regrouping of the series $1-2^{-s}+3^{-s}-4^{-s}+\cdots(=(1-2^{1-s})\zeta(s)$, $\operatorname{Re} s>0)$. We can obtain asymptotic relations like

$$\frac{3^{3^2}5^{5^2}7^{7^2}\ldots(2N+1)^{(2N+1)^2}}{2^{2^2}4^{4^2}6^{6^2}\ldots(2N)^{(2N)^2}} \sim \exp\{3/2-7\zeta(3)/(4\pi^2)\}(2N)^{2N^2+3N+1}e^{3N/2} \tag{1}$$

as an application of our series representation for $\zeta(s)$. We first have the following:

Theorem. *Let k be a positive integer, $\alpha>0$ and $s=\sigma+it$. Then the Hurwitz zeta function*

$$\zeta(s,\alpha)=\alpha^{-s}+(1+\alpha)^{-s}+(2+\alpha)^{-s}+\ldots,\sigma>1$$

satisfies the relation

$$\begin{aligned}
2^{k-s}\left\{\zeta(s,\alpha/2)-\zeta\left(s,\frac{1+\alpha}{2}\right)\right\} &= \alpha^{-s}-\left\{(1+\alpha)^{-s}-\binom{k}{1}\alpha^{-s}\right\} \\
&+\left\{(2+\alpha)^{-s}-\binom{k}{1}(1+\alpha)^{-s}+\binom{k}{2}\alpha^{-s}\right\}-\cdots+ \\
&+(-1)^{k-1}\left\{(k-1+\alpha)^{-s}-\binom{k}{1}(k-2+\alpha)^{-s}+\binom{k}{2}(k-3+\alpha)^{-s}-\cdots+(-1)^{k-1}\binom{k}{k-1}\alpha^{-s}\right\} \\
&+\sum_{n=0}^{\infty}(-1)^{n+k}\left\{(n+k+\alpha)^{-s}-\binom{k}{1}(n+k-1+\alpha)^{-s}+\binom{k}{2}(n+k-2+\alpha)^{-s}-\cdots+\right. \\
&\left.+(-1)^{k}(n+\alpha)^{-s}\right\},
\end{aligned} \tag{2}$$

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valid in $\sigma > -k$. *This gives, in particular, the analytic continuation of* $\zeta(s,\alpha/2)-\zeta(s,(1+\alpha)/2)$. *Also* (2) *implies, for* $\sigma > -k$,

$$2^{-s}\left\{\zeta(s,\alpha/2)-\zeta\left(s,\frac{1+\alpha}{2}\right)\right\}=\alpha^{-s}-(1+\alpha)^{-s}+(2+\alpha)^{-s}-\cdots+(2N+\alpha)^{-s}-f_k(2N+\alpha)+o(1), \tag{3}$$

as $N\to\infty$ *where*

$$f_k(x)=\langle k,0\rangle(x+1)^{-s}-\langle k,1\rangle(x+2)^{-s}+\langle k,2\rangle(x+3)^{-s}-\cdots+(-1)^{k-1}\langle k,k-1\rangle(x+k)^{-s} \tag{4}$$

with

$$\langle k,r\rangle=2^{-k}\left\{2^k-\binom{k}{0}-\binom{k}{1}-\binom{k}{2}-\cdots-\binom{k}{r}\right\}. \tag{5}$$

Observe that all terms in (3), save $f_k(2N+\alpha)$, are independent of k. The case $\alpha=1$ of the above theorem, in view of $\zeta(s,\tfrac{1}{2})=(2^s-1)\zeta(s)$, is the following

Corollary 1. *For* $\sigma > -k$, *we have the representation*

$$\begin{aligned}2^k(1-2^{1-s})\zeta(s)=1&-\left\{2^{-s}-\binom{k}{1}\right\}+\left\{3^{-s}-\binom{k}{1}2^{-s}+\binom{k}{2}\right\}-\cdots+\\&+(-1)^{k-1}\left\{k^{-s}-\binom{k}{1}(k-1)^{-s}+\binom{k}{2}(k-2)^{-s}-\cdots+(-1)^{k-1}\binom{k}{k-1}\right\}\\&+\sum_{n=1}^{\infty}(-1)^{n+k-1}\left\{(n+k)^{-s}-\binom{k}{1}(n+k-1)^{-s}+\binom{k}{2}(n+k-2)^{-s}-\cdots+(-1)^k n^{-s}\right\}.\end{aligned} \tag{6}$$

This gives the analytic continuation of $(s-1)\zeta(s)$. *Further, for* $\sigma > -k$,

$$(1-2^{1-s})\zeta(s)=1-2^{-s}+3^{-s}-\cdots+(2N+1)^{-s}-f_k(2N+1)+o(1), \tag{7}$$

as $N\to\infty$, *where* $f_k(x)$ *is given by* (4).

We have the following special cases of (6) and (7) as examples.

$$2(1-2^{1-s})\zeta(s)=1-(2^{-s}-1)+(3^{-s}-2^{-s})-(4^{-s}-3^{-s})+\dots,\quad \text{for } \sigma>-1, \tag{6.1}$$

$$\begin{aligned}16(1-2^{1-s})\zeta(s)=1&-\{2^{-s}-4\}+\{3^{-s}-4.2^{-s}+6\}-\{4^{-s}-4.3^{-s}+6.2^{-s}-4\}\\&+\{5^{-s}-4.4^{-s}+6.3^{-s}-4.2^{-s}+1\}\\&-\{6^{-s}-4.5^{-s}+6.4^{-s}-4.3^{-s}+2^{-s}\}+\dots,\quad \text{for } \sigma>-4,\end{aligned} \tag{6.2}$$

$$(1-2^{1-s})\zeta(s)=\lim_{N\to\infty}(1-2^{-s}+3^{-s}-\cdots+(2N+1)^{-s}-\tfrac{1}{2}(2N+2)^{-s}),\quad \text{for } \sigma>-1, \tag{7.1}$$

$$(1-2^{1-s})\zeta(s)=\lim_{N\to\infty}(1-2^{-s}+3^{-s}-\cdots+(2N+1)^{-s}-\tfrac{1}{16}\{15(2N+2)^{-s}$$
$$-11(2N+3)^{-s}+5(2N+4)^{-s}-(2N+5)^{-s}\}),\quad \text{for } \sigma>-4, \tag{7.2}$$

and so on.

Denote by h' the differential coefficient of a function h with respect to s. From the proof of the theorem it will be clear that term by term differentiation of the right side of (2) is justified in $\sigma>-k$. The differentiation of (2) yields a relation which would be the same as obtained by differentiating (3) with the $o(1)$ retained as such. By taking exponentials on both sides we obtain an asymptotic relation. Let us state this result only for the case $k=4$ as the expression for general k is a bit complicated.

Corollary 2. *For $\sigma>-4$ and $\alpha>0$ we have, as $N\to\infty$,*

$$\frac{(1+\alpha)^{(1+\alpha)^{-s}}(3+\alpha)^{(3+\alpha)^{-s}}\dots(2N-1+\alpha)^{(2N-1+\alpha)^{-s}}}{\alpha^{\alpha^{-s}}(2+\alpha)^{(2+\alpha)^{-s}}\dots(2N+\alpha)^{(2N+\alpha)^{-s}}}$$
$$\sim C(s,\alpha)\left(\frac{(2N+2+\alpha)^{11(2N+2+\alpha)^{-s}}(2N+4+\alpha)^{(2N+4+\alpha)^{-s}}}{(2N+1+\alpha)^{15(2N+1+\alpha)^{-s}}(2N+3+\alpha)^{5(2N+3+\alpha)^{-s}}}\right)^{1/16}, \tag{8}$$

with

$$C(s,\alpha)=\exp\left\{2^{-s}\left(\left\{\zeta'(s,\alpha/2)-\zeta'\left(s,\frac{1+\alpha}{2}\right)\right\}-\log 2\left\{\zeta(s,\alpha/2)-\zeta\left(s,\frac{1+\alpha}{2}\right)\right\}\right)\right\}. \tag{9}$$

We obtain from (9) that

$$C(s,1)=\exp\{(1-2^{1-s})\zeta'(s)+2^{1-s}\log 2\zeta(s)\}. \tag{10}$$

Also it follows from the functional equation

$$\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)=\xi(1-s)$$

that, for $n=1,2,3,\dots$,

$$\zeta(2n+1)=-4\zeta(2n)\zeta'(-2n)/B_{2n} \tag{10.5}$$

where B_{2n} are Bernoulli numbers. Thus we get from (10)

$$C(-2n,1)=\exp((2^{2n+1}-1)\zeta(2n+1)B_{2n}/\{4\zeta(2n)\}) \tag{11}$$

in view of the fact that $\zeta(-2n)=0$. Now (1) follows from (8) with the value of $C(-2n,1)$ from (11).

We observe that the term

$$(n+k)^{-s}-\binom{k}{1}(n+k-1)^{-s}+\binom{k}{2}(n+k-2)^{-s}-\cdots+(-1)^k n^{-s}$$

is an integer for $s=0, -1, -2,\ldots,$ where n is any integer. Since the infinite series in (6) is convergent for $s=0, -1, -2,\ldots,1-k$, we arrive at the identity

$$x^r-\binom{k}{1}(x-1)^r+\binom{k}{2}(x-2)^r-\cdots+(-1)^k\binom{k}{k}(x-k)^r\equiv 0 \tag{12}$$

in x, for $r=0,1,2,\ldots,k-1$. Hence we deduce from the Theorem the following:

Corollary 3. *Let k be a positive integer. Then, for $r=0,1,2,\ldots,k-1$ we have, with $\langle k,r\rangle$ defined in* (5), *that*

$$2^r\left\{\zeta(-r,\alpha/2)-\zeta\left(-r,\frac{1+\alpha}{2}\right)\right\}=\langle k,0\rangle\alpha^r-\langle k,1\rangle(1+\alpha)^r+\langle k,2\rangle(2+\alpha)^r-\cdots$$
$$+(-1)^{k-1}\langle k,k-1\rangle(k-1+\alpha)^r.$$

For $\alpha=1$ this becomes

$$(1-2^{r+1})\zeta(-r)=\langle k,0\rangle-\langle k,1\rangle 2^r+\langle k,2\rangle 3^r-\cdots+(-1)^{k-1}\langle k,k-1\rangle k^r. \tag{12.5}$$

Further in view of $n^r+\binom{r}{1}n^{r-1}+\cdots+1=(n+1)^r$ and (12) we obtain for any positive integer r,

$$\zeta(0)+\binom{r}{1}(2^2-1)\zeta(-1)+\binom{r}{2}(2^3-1)\zeta(-2)+\cdots+$$
$$+\binom{r}{r-1}(2^r-1)\zeta(1-r)+2(2^{r+1}-1)\zeta(-r)+1=0.$$

We come to a proof of the Theorem. We make use of the following:

Lemma. *Define, for real r,*

$$v_r(s,k)=r^{-s}-\binom{k}{1}(r+1)^{-s}+\binom{k}{2}(r+2)^{-s}-\cdots+(-1)^k(r+k)^{-s}.$$

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Then

$$v_r(s,k)=s(s+1)\dots(s+k-1)\int_0^1 du_1\int_0^1 du_2\dots\int_0^1 du_k(r+u_1+u_2+\cdots+u_k)^{-s-k}. \tag{13}$$

Proof. This is easily proved by induction on k. In fact

$$s(s+1)\dots(s+k)\int_0^1 du_1\int_0^1 du_2\dots\int_0^1 du_{k+1}(r+u_1+u_2+\cdots+u_{k+1})^{s-k-1}$$

$$=s(s+1)\dots(s+k-1)\int_0^1 du_1\int_0^1 du_2\dots\int_0^1 du_k((r+u_1+u_2+\cdots+u_k)^{-s-k}$$

$$-(r+1+u_1+u_2+\cdots+u_k)^{-s-k})$$

$$=v_r(s,k)-v_{r+1}(s,k).$$

Now

$$v_r(s,k)-v_{r+1}(s,k)=v_r(s,k+1), \tag{14}$$

which easily follows from the definition of $v_r(s,k)$, and the proof of the lemma is complete.

It is immediate from the above lemma that

$$|v_r(s,k)|=0(r^{-\sigma-k}), \tag{15}$$

with the 0-constant not depending on r.

For $\sigma>1$,

$$2^{-s}\left\{\zeta(s,\alpha/2)-\zeta\left(s,\frac{1+\alpha}{2}\right)\right\}=\alpha^{-s}-(1+\alpha)^{-s}+(2+\alpha)^{-s}-\cdots=\tfrac{1}{2}(\alpha^{-s}-\{(1+\alpha)^{-s}-\alpha^{-s}\}$$

$$+\{(2+\alpha)^{-s}-(1+\alpha)^{-s}\}-\dots),$$

and a sequence of k such operations bring us to the relation (2). Now we have only to prove that the right side of (2) is convergent for $\sigma>-k$. The infinite series part, $\sum_{n=0}^{\infty}$ there, can be written as $\sum_{n=0}^{\infty}(-1)^n v_{n+\alpha}(s,k)$, following the notation in the above lemma. We see from (15) that, for $\sigma>-k$, $v_r(s,k)\to 0$ as $r\to\infty$. Hence we can make the rearrangement

$$\sum_{n=0}^{\infty}(-1)^n v_{n+\alpha}(s,k)=\sum_{n=0}^{\infty}(v_{2n+\alpha}(s,k)-v_{2n+1+\alpha}(s,k)),$$

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valid in $\sigma > -k$. Now the right side series is convergent in $\sigma > -k$, thanks to (14) and (15) and the proof of the theorem is complete.

Remarks 1. Ramanujan's proof for

$$\pi \sum_{n=0}^{\infty} (-1)^n \{(2n)^{1/2} + (2n+2)^{1/2}\}^{-1} = \sum_{n=0}^{\infty} (2n+1)^{-3/2}$$

reproduced, in essentials, in Corollary 4 to entry 4 in [1] runs as follows:

$$\begin{aligned} \pi \sum_{n=0}^{\infty} (-1)^n \{(2n)^{1/2} + (2n+2)^{1/2}\}^{-1} &= \frac{\pi}{2^{1/2}} \sum_{n=0}^{\infty} (-1)^n \{(n+1)^{1/2} - n^{1/2}\} \\ &= 2^{1/2}\pi \sum_{n=1}^{\infty} (-1)^{n+1} n^{1/2} \qquad (*) \\ &= 2^{1/2}\pi(1-2^{3/2})\zeta(-\tfrac{1}{2}) \\ &= (1-2^{-3/2})\zeta(3/2) \\ &= \sum_{n=0}^{\infty} (2n+1)^{-3/2}. \end{aligned}$$

Thus it turns out that the above proof of Ramanujan is more than sufficient by a step in the sense that if we remove just the step (*) the proof becomes perfectly alright in view of (6.1).

2. Proof of equivalent forms of Corollary 2, (7) and (3) can be given by an application of Euler–Maclaurin summation formula.

3. Relation (12.5) gives an explicit formula for $\zeta(-r)$ and hence for the Bernoulli numbers. Also the method of the paper could be used to obtain the analytic continuation of the Dirichlet series $\sum_{n=1}^{\infty} (-1)^n (f(n))^{-s}$ where $f(n)$ is a polynomial in n.

REFERENCE

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