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LOCAL SPACES WITH THREE CELLS AS H-SPACES

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1. Introduction. The question of which finite CW-complexes are H-spaces has been studied for many years. Since a finite CW-complex is an H-space if and only if its localization at each prime p is an H-space [**21**], an examination of finite local cell complexes as H-spaces yields results concerning CW-complexes. On the other hand, if it is known that a particular CW-complex is not an H-space, one would like to know for which primes p its localization at p fails to be an H-space. The main result of this paper gives a condition equivalent to a three cell local CW-complex's being an H-space for a prime $p > 3$.

An H-space of rank one has the homotopy type of an odd-dimensional sphere S^r. An odd-dimensional sphere S^r is an H-space if and only if $r = 1, 3$ or 7. Its localization $S^r{}_p$ at a prime p fails to be an H-space only for the prime $p = 2$ [**1**].

The 2-torsion free rank two H-spaces have been classified up to homotopy. The only types (q, n) which occur are those such that $\{q, n\} \subset \{1, 3, 7\}$ or $(q, n) = (1, 2)$ or $(3, 5)$. There are exactly sixteen homotopy types of torsion-free 1-connected H-spaces. Again the results depend on the prime 2 behaving differently from the other primes [**2**], [**9**], [**5**], [**14**].

A 1-connected torsion-free CW-complex X which is an H-space of rank two and type (q, n) has the same homotopy type as the total space of an S^q-fibration over the sphere S^n[**16**]. Such a total space is homotopically equivalent to a CW-complex $S^q \cup e^n \cup e^{n+q}$ [**4**]. Localization at a prime p yields another fibration $S^q{}_p \to X_p \to S^n{}_p$ [**19**]. These are the fibrations which will be studied here. Always we assume that q, n and p are odd and that $n > q > 2$.

The main purpose of this paper is to carry through the results of I. M. James and J. H. C. Whitehead [**12**] for local spherical fibrations over spheres without assuming the existence of a cross-section. James and Whitehead considered fiber bundles $S^q \to B \to S^n$ and showed that B is a cell-complex of the form $S^q \cup_\alpha e^n \cup e^{n+q}$. For bundles $S^q \to B_i \to S^n$ with cross-section (i.e. with $\alpha = 0$), there are elements $\lambda(B_i)$ in $\pi_{n+q-1}(S^q)$ such that $\lambda(B_1) = \pm\lambda(B_2)$ if and only if (B_1, S^q) and (B_2, S^q) have the same homotopy type. Also, for a bundle with cross-section, $\lambda(B) = 0$ if and only if B and $S^q \times S^n$ have the same homotopy type. Furthermore, B is an H-space if and only if $\lambda(B) = 0$ and the spheres S^q and S^n are H-spaces.

In Section 2, it will be shown that the total space of a local spherical fibra-

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tion $S^q{}_p \to E \to S^n{}_p$ is homotopically equivalent to a local cell complex $S^q{}_p \cup_\alpha e^n{}_p \cup e^{n+q}{}_p$.

Section 3 is devoted to fibrations with fixed α and to the construction of an element $\lambda_\alpha(E)$ in $\pi_{n+q-1}(S^q{}_p \cup_\alpha e^n{}_p)$ for each of these fibrations. If a cross-section exists, then the injection $i: S^q{}_p \to S^q{}_p \cup_\alpha e^n{}_p$ induces a monomorphism in homotopy, and the element $i_*^{-1}\lambda_\alpha(E)$ in $\pi_{n+q-1}(S^q{}_p)$ is uniquely defined; this element corresponds to James and Whitehead's $\lambda(B)$. Certain subsets of $\operatorname{Im} i_*$ in $\pi_{n+q-1}(S^q{}_q \cup e^n{}_p)$ will be defined in such a way that $\lambda_\alpha(E_1)$ and $\lambda_\alpha(E_2)$ are in the same subset if and only if $(E_1, S^q{}_p)$ and $(E_2, S^q{}_p)$ are homotopically equivalent. Each subset for fixed α corresponds to James and Whitehead's set $\{\pm\lambda(B)\}$ for fixed $\alpha = 0$.

In Section 4, again α in $\pi_{n-1}(S^q{}_p)$ is a fixed element. The main result is:

THEOREM 4.4. *Suppose that q and n are odd integers and that p is an odd prime. Let $S^q{}_p \to E \to S^n{}_p$ be a fibration such that E has first attaching map α. If $p > 3$, then $\lambda_\alpha(E) = 0$ if and only if E is an H-space. For $p = 3$, if E is an H-space, then $\lambda_\alpha(E) = 0$.*

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2. The total space as a local *CW*-complex. In this section it will be shown that the total space of a fibration $S^q{}_p \to E \to S^n{}_p$ is homotopically equivalent to a local CW-complex. First, some notation and definitions are needed.

The local sphere $S^r{}_p$ can be considered as the suspension of $S^{r-1}{}_p$ for $r > 2$ because the localization of the suspension of a simply-connected space X has the homotopy type of the suspension of X localized, i.e., $(\sum X)_p \simeq \sum(X_p)$ [**19**]. Let

$$S^r{}_p = \{[x, t] \mid x \in S^{r-1}{}_p,\ -1 \leqq t \leqq 1;\ [x_1, 1] = [x_2, 1] \text{ and } [x_1, -1] = [x_2, -1] \text{ for all } x_1, x_2 \in S^{r-1}{}_p\},$$

and let the base point a_r of $S^r{}_p$ be $[x, 1]$, where $x \in S^{r-1}{}_p$. The local r-cell $e^r{}_p$ is defined to be the cone on $S^{r-1}{}_p$ with vertex $b_r = [x, 0]$, where $x \in S^{r-1}{}_p$. As defined by Sullivan [**19**], a local CW-complex is a space constructed inductively from a point or local sphere $S^m{}_p$ by attaching local cells $e^r{}_p$ by maps of local spheres $S^{r-1}{}_p$ into the cells of lower dimension.

Define a map $u_r: e^r{}_p \to S^r{}_p$ by $u_r([x, t]) = [x, 2t - 1]$ for $x \in S^{r-1}{}_p$ and $0 \leqq t \leqq 1$. Then, on the boundary $S^{r-1}{}_p$, we have that $u_r([x, 1]) = a_r$.

The following is a special case of the local form of Proposition 1 in [**17**] with a modification of the proof [**11**].

PROPOSITION 2.1. *Let $F \to E \to S^n{}_p$ be a fibration, and suppose that the fiber F is a local CW-complex. Then the total space E has the homotopy type of a local CW-complex $K = F \cup (e^n{}_p \times F)$.*

Proof. Let χ be the map u_n: $e^n{}_p \to S^n{}_p$, and let π: $E \to S^n{}_p$ be the fiber map. Consider the induced fibration π_χ: $E_\chi \to e^n{}_p$. Since the cone $e^n{}_p$ is contractible, the induced fiber space E_χ is fiber homotopy equivalent to the product $e^q{}_p \times F$. Let ϕ: $e^n{}_p \times F \to E_\chi$ and ψ: $E_\chi \to e^n{}_p \times F$ be fiber homotopy inverses of each other such that the map

$$\phi|b_n \times F\colon b_n \times F \to \pi_\chi{}^{-1}(b_n)$$

is homotopic to the identity mapping of the fiber F. Also, let ζ: $E_\chi \to E$ and ρ: $e^n{}_p \times F \to e^n{}_p$ be the natural projections. Let $\bar{F} = \pi^{-1}(a_n)$. Then, for $x \in S^{n-1}{}_p$ and $y \in F$, we have that $\zeta\phi(x, y) \in \bar{F}$. Let $\nu = \zeta\phi|S^{n-1}{}_p \times F$, and use the map ν to construct the complex $K = \bar{F} \cup_\nu (e^n{}_p \times F)$. The following lemma completes the proof.

LEMMA 2.2. *The spaces E and K are homotopically equivalent.*

Proof. Let θ: $K \to E$ be the map induced by $\zeta\phi$. A map β: $E \to K$ will be defined such that θ and β are homotopy inverses of each other.

Let h_t: $E_\chi \to E_\chi$ be a homotopy such that $h_1 = 1$ and $h_0 = \phi\psi$. Using the definition of $e^n{}_p$ as the cone on $S^{n-1}{}_p$, define a map s: $e^n{}_p \to e^n{}_p$ by:

$$\begin{aligned} s([x, t]) &= [x, 2t] \quad \text{if } 0 \leqq t \leqq 1/2, \quad x \in S^{n-1}{}_p; \\ &= [x, 1] \quad \text{if } 1/2 \leqq t \leqq 1, \quad x \in S^{n-1}{}_p. \end{aligned}$$

Then the map s is homotopic to the identity on $e^n{}_p$ under a homotopy G which keeps each point of $S^{n-1}{}_p$ fixed; assume that $G|e^n{}_p \times 0 = 1$ and $G|e^n{}_p \times 1 = s$. Since $e^n{}_p$ is a metric space, the fibration π_χ: $E_\chi \to e^n{}_p$ is regular [**10**]. This means that any homotopy into $e^n{}_p$ that keeps certain points stable can be lifted to a homotopy which keeps the same points stable. Then, since $G(\pi_\chi \times 1)(x, t) = G(\pi_\chi \times 1)(x, t')$ for $0 \leqq t, t' \leqq 1$ and $\pi_\chi(x) \in S^{n-1}{}_p$, there is a homotopy H: $E_\chi \times 1 \to E$ such that $\pi_\chi H = G(\pi_\chi \times 1)$, $H|E_\chi \times 0$ is the identity on E_χ, and $H(x, t) = H(x, t')$ for $0 \leqq t, t' \leqq 1$ and $x \in E_\chi$ such that $\pi_\chi(x) \in S^{n-1}{}_p$.

Define a map v: $E_\chi \to E_\chi$ by $v = H|E_\chi \times 1$. Then, by the properties of the map H listed above, we have that $\pi_\chi v = s\pi_\chi$, the map v is homotopic to the identity on E_χ, and $v(x) = x$ for all $x \in E_\chi$ such that $\pi_\chi(x) \in S^{n-1}{}_p$.

Let e be a point of $E - \bar{F}$. Then $\pi(e) \in S^n{}_p - a_n$, and there is exactly one point $x \in e^n{}_p$ such that $\chi(x) = \pi(e)$. Then the set $\zeta^{-1}(e)$ consists of the one point $(x, e) \in E_\chi$. Let j: $\bar{F} \to K$ be the inclusion, and let η: $e^n{}_p \times F \to K$ be the map induced by ν. Define a map β: $E \to K$ extending the identity on $\bar{F}$ by: if $e \in E - \bar{F}$ such that $\pi_\chi\zeta^{-1}(e) = [y, t]$,

$$\begin{aligned} \beta(e) &= \eta\psi v\zeta^{-1}(e) \qquad \text{if } 0 \leqq t \leqq 1/2 \\ &= j\zeta h_{2t-1} v\zeta^{-1}(e) \quad \text{if } 1/2 \leqq t \leqq 1. \end{aligned}$$

Then β is a continuous map because the two definitions for $t = 1/2$ agree, and $\zeta h_{2t-1} v\zeta^{-1}(e)$ lies in $\bar{F}$ for $t \geqq 1/2$ and equals e for $t = 1$.

The maps $\beta\theta$ and $\theta\beta$ are homotopic to the appropriate identities. This completes the proof of the lemma.

COROLLARY 2 3 *Let $S^q{}_p \to E \to S^n{}_p$ be a fibration. Then the total space E is homotopically equivalent to a local CW-complex K with decomposition $S^q{}_p \cup e^n{}_p \cup e^{n+q}{}_p$.*

Proof. The total space E is homotopically equivalent to a complex $K = S^q{}_p \cup_\nu (e^n{}_p \times S^q{}_p)$ by Proposition 2.1. Let h: $e^n{}_p \times S^q{}_p \to K$ be the map determined by ν; let $k = h(1 \times u_q)$: $e^n{}_p \times e^q{}_p \to K$. (This notation, which will be used throughout the rest of this paper, is that used by James and Whitehead [**12**] in discussing the cellular decomposition of the total space of a bundle.)

Then we have that

$$k(S^{n-1}{}_p \times e^q{}_p) \subset \bar{S}^q{}_p,\ k(\text{Int}\, e^n{}_p \times S^{q-1}{}_p) \subset e^n{}_p \times a_q = e^n{}_p, \text{ and}$$
$$k(a_{n-1} \times S^{q-1}{}_p) = \text{a point } e^0.$$

This yields a decomposition of K as the local CW-complex $e^0 \cup e^q{}_p \cup_\alpha e^n{}_p \cup e^{n+q}{}_p$, where

$$e^0 = k(a_{n-1}, a_{q-1}),\ S^q{}_p = e^0 \cup e^q{}_p,\ e^n{}_p = k(e^n{}_p \times a_{q-1}),\ \alpha = \nu|S^{n-1}{}_p \times a_q,$$
$$\text{and } e^{n+q}{}_p = k(e^n{}_p \times e^q{}_p),$$

which is attached by the map $k|(e^n{}_p \times e^q{}_p)^\cdot$.

3. Homotopy type of $(E, S^q{}_p)$. Let $\alpha \in \pi_{n-1}(S^q{}_p)$ be a fixed homotopy class. We will consider only those fibrations $S^q{}_p \to E \to S^n{}_p$ such that E has the homotopy type of a local CW-complex K with first attaching map α. Then K has the form:

$$K = S^q{}_p \cup_\alpha e^n{}_p \cup e^{n+q}{}_p.$$

Let L denote the subcomplex of K defined by: $L = S^q{}_p \cup_\alpha e^n{}_p$. Certain subsets of $i_*\pi_{n+q-1}(S^q{}_p) \subset \pi_{n+q-1}(L)$ will be designated in such a way that each subset corresponds to a homotopy class of pairs $(E, S^q{}_p)$.

The map k of the preceding section determines particular generators i_n of $\pi_n(L, S^q{}_p)$ and i_q of $\pi_q(S^q{}_p)$. Let $i_n = [k|e^n{}_p \times a_{q-1}]$, and let $i_q = [k|a_{n-1} \times e^q{}_p]$.

In order to study the homotopy class of the boundary of the map k, maps f and g of the boundary of $e^n{}_p \times e^q{}_p$ into itself will be defined. Composing the boundary of k with these two maps will lead to expressing the homotopy class of the boundary of k as a sum of two elements. One of these elements determines the homotopy type of the pair $(E, S^q{}_p)$, and the other element is similar to a Whitehead product of i_q and i_n. We first define this product in general.

Suppose that A is an H-space and a subspace of a space X. Let β in $\pi_q(A)$

and γ in $\pi_n(X, A)$ be represented by the maps:

$b: (e^q, S^{q-1}) \to (A, *)$, and

$c: (e^n{}_1, e^{n-1}, e^{n-1}{}_+) \to (X, A, *)$, where

$e_1{}^n = \{[x, t] \in \sum e^{n-1} | \, t \geqq 0\}$,

$e^{n-1} = \{[x, t] \in \sum e^{n-1} | \, t = 0\}$, and

$e^{n-1}{}_+ = \{[x, t] \in \sum S^{n-2} | \, t \geqq 0\}$.

Let Y be the space $(e^n{}_1 \times S^{q-1}) \cup (e^{n-1}{}_+ \times e^q) \cup (e^{n-1} \times e^q)$, which is homotopically equivalent to e^{n+q-1}. Consider the map $(b, c): Y \to X$ defined by:

$$\begin{aligned} (b, c)(u, v) &= c(u) && \text{if } (u, v) \in e^n{}_1 \times S^{q-1}, \\ &= b(v) && \text{if } (u, v) \in e^{n-1}{}_+ \times e^q, \\ &= c(u) \cdot b(v) && \text{if } (u, v) \in e^{n-1} \times e^q, \end{aligned}$$

where the product means multiplication in the H-space A. The first two parts of this definition give a representative of the relative Whitehead product $[\beta, \gamma]$ in $\pi_{n+q-1}(X, A)$, and the last part is the usual map for showing that any Whitehead product (and, in this case, $\partial[\beta, \gamma] = [\beta, \partial\gamma]$) is trivial for an H-space. Since any homotopies b_t and c_t yield a homotopy (b_t, c_t), we can define the product:

Definition 3.1. $[\beta, \gamma]_X$ is the homotopy class of (b, c) in $\pi_{n+q-1}(X)$.

Alternately, the representative of the product $[\beta, \gamma]_X$ could be defined as follows: use the H-structure of A to deform a representative of the relative Whitehead product $[\beta, \gamma]$ to a map which is trivial on the boundary of $e^n{}_1 \times e^q$. The next proposition lists the properties of this product.

PROPOSITION 3.2. *Suppose that A is an H-space and subspace of a space X. Consider homotopy classes β, β_1 and β_2 in $\pi_q(A)$ and γ, γ_1 and γ_2 in $\pi_n(X, A)$. Then:*

1. $j_*([\beta, \gamma]_X) = [\beta, \gamma]$, *where $j: (X, *) \to (X, A)$ is the inclusion.*
2. $[\beta_1 + \beta_2, \gamma]_X = [\beta_1, \gamma]_X + [\beta_2, \gamma]_X$.
3. $[\beta, \gamma_1 + \gamma_2]_X = [\beta, \gamma_1]_X + [\beta, \gamma_2]_X$.
4. *Suppose that B is an H-space and subspace of a space Y and that $f: (X, A) \to (Y, B)$ is a map. Then $f_*([\beta, \gamma]_X) = [f_*\beta, f_*\gamma]_Y$.*

Proof. The first three properties follow immediately from the definition of the product. We now prove the last property.

Let the maps b and c represent β and γ. Then

$$\begin{aligned} f(b, c)(u, v) &= fc(u) && \text{if } (u, v) \in e^n{}_1 \times S^{q-1}, \\ &= fb(v) && \text{if } (u, v) \in e^{n-1}{}_+ \times e^q, \\ &= f(c(u) \cdot b(v)) && \text{if } (u, v) \in e^{n-1} \times e^q. \end{aligned}$$

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The only difficulty lies in the third line; here we know that

$$f(c(u)\cdot b(v)) = fm(c(u), b(v)) = fm(c \times b)(u, v),$$

where m is the multiplication in A. We want to show that $fm(c \times b)$ is homotopic to $m'(fc \times fb)$, where m' is the multiplication in B. For $[a]$ in $\pi_r(A \times A)$, projections $p_i\colon A \times A \to A$, and diagonal map $\Delta\colon S^r \to S^r \times S^r$, we have that

$$[fma] = f_*[m(p_1a \times p_2a)\Delta] = f_*([p_1a] + [p_2a]) = [fp_1a] + [fp_2a]$$
$$= [m'(fp_1a \times fp_2a)\Delta] = [m'(f \times f)a].$$

Then, letting $a = c \times b$, we find that

$$[fm(c \times b)] = [m'(f \times f)(c \times b)] = [m'(fc \times fb)],$$

and thus $fm(c \times b)$ is homotopic to $m'(fc \times fb)$. Therefore,

$$f_*[\beta, \gamma]_X = [f_*\beta, f_*\gamma]_Y,$$

and the proposition is proved.

Since $e^r{}_p$ is the cone on $S^{r-1}{}_p$ and $S^r{}_p$ is the suspension of $S^{r-1}{}_p$, local spheres and cells are related in ways analogous to those of the usual spheres and cells. For example, the boundary $(e^n{}_p \times e^q{}_p)^\cdot$ of $e^n{}_p \times e^q{}_p$ is

$$(S^{n-1}{}_p \times e^q{}_p) \cup (e^n{}_p \times S^{q-1}{}_p)$$

and $e^n{}_p \times e^q{}_p$ is homeomorphic to $e^{n+q}{}_p$. The following notation will be used (Figure 1):

$$e^r{}_+ = \{[x, t] \in \textstyle\sum S^{r-1}{}_p | t \geqq 0\} \quad \text{and} \quad e^r{}_- = \{[x, t] \in \textstyle\sum S^{r-1}{}_p | t \leqq 0\} \subset S^r{}_p;$$
$$e^r{}_1 = \{[x, t] \in \textstyle\sum e^{r-1}{}_p | t \geqq 0\} \quad \text{and} \quad e^r{}_2 = \{[x, t] \in \textstyle\sum e^{r-1}{}_p | t \leqq 0\} \subset e^r{}_p.$$

Define a map $f\colon (e^n{}_p \times e^q{}_p)^\cdot \to (e^n{}_p \times e^q{}_p)^\cdot$ as follows (Figures 1 and 2): for $[x, t] \in e^n{}_p = \sum e^{n-1}{}_p$, $x \in e^{n-1}{}_p$, $y \in e^q{}_p$,

$$f([x, t], y) = ([x, 2t + 1], y) \quad \text{if } -1 \leqq t \leqq 0;$$
$$= ([x, 1], y) \quad \text{if } 0 \leqq t \leqq 1.$$

The map f is homotopic to the identity on $(e^n{}_p \times e^q{}_p)^\cdot$.

The points of $e^n{}_p$ can be parametrized in the unusual form $([x, r], t)$, where $x \in S^{n-2}{}_p$, $[x, r] \in e^{n-1}{}_p = CS^{n-1}{}_p$, $0 \leqq r \leqq 1$, and $r - 1 \leqq t \leqq 1 - r$. In this representation, boundary points of $e^n{}_p$ have the form $([x, r], \pm(1 - r))$. We refer to lines where $[x, r]$ is fixed and t varies as lines orthogonal to $e^{n-1}{}_p$. Define a map $g\colon (e^n{}_p \times e^q{}_p)^\cdot \to (e^n{}_p \times e^q{}_p)^\cdot$ by (Figures 1 and 2):

$$g([x, r], t, y) = ([x, r], r - 1, y) \quad \text{if } r - 1 \leqq t \leqq 0, y \in S^{q-1}{}_p;$$
$$= ([x, r], 2t - 1 + r, y) \quad \text{if } 0 \leqq t \leqq 1 - r, y \in S^{q-1}{}_p;$$
$$= ([x, r], t, y) \quad \text{if } y \in e^q{}_p, ([x, r], t) \in S^{n-1}{}_p.$$

The map g is homotopic to the identity on $(e^n{}_p \times e^q{}_p)^\cdot$, and $g|S^{n-1}{}_p \times e^q{}_p$ is the

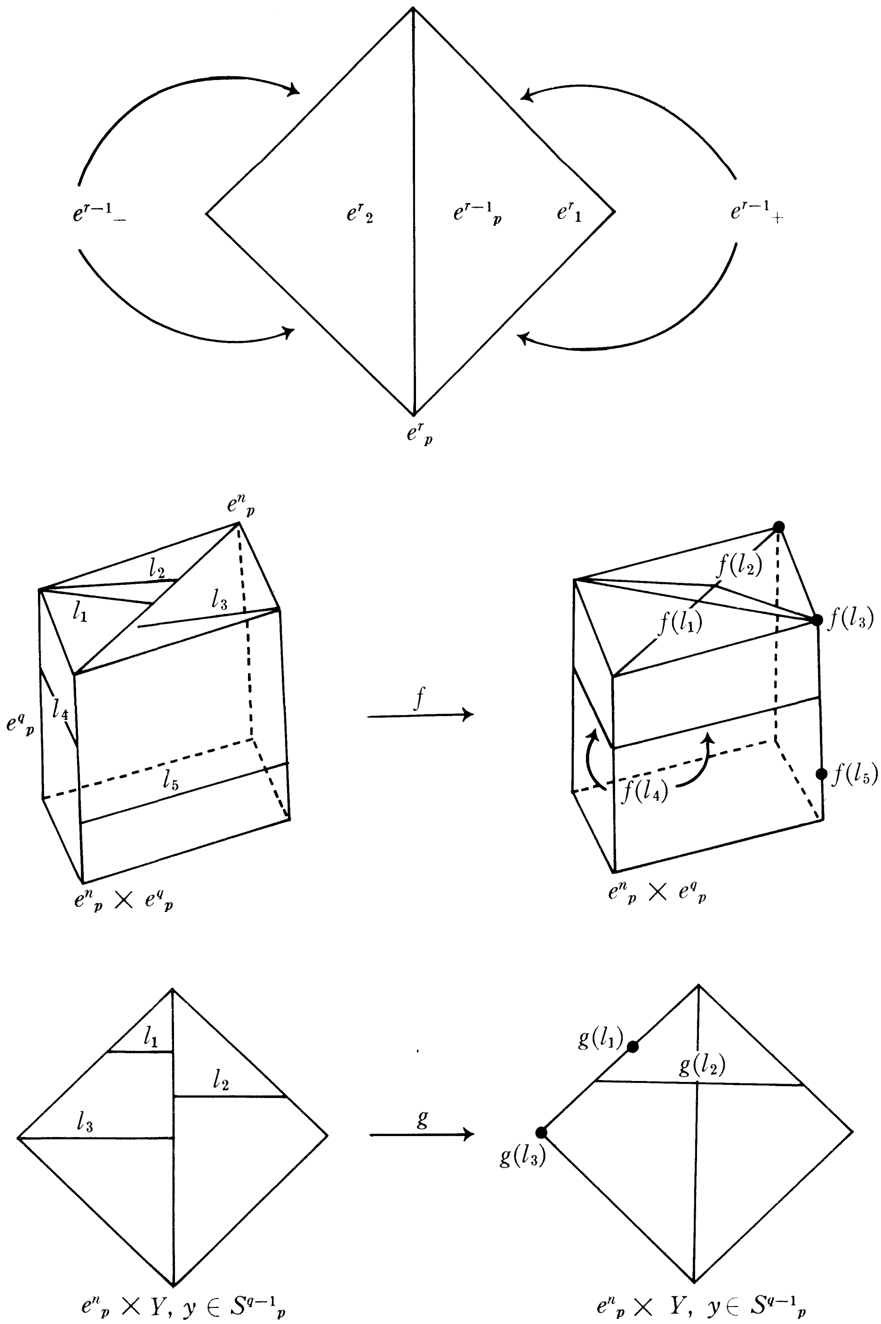

FIGURE 1. Subsets of $e^r{}_p$; the maps f and g.

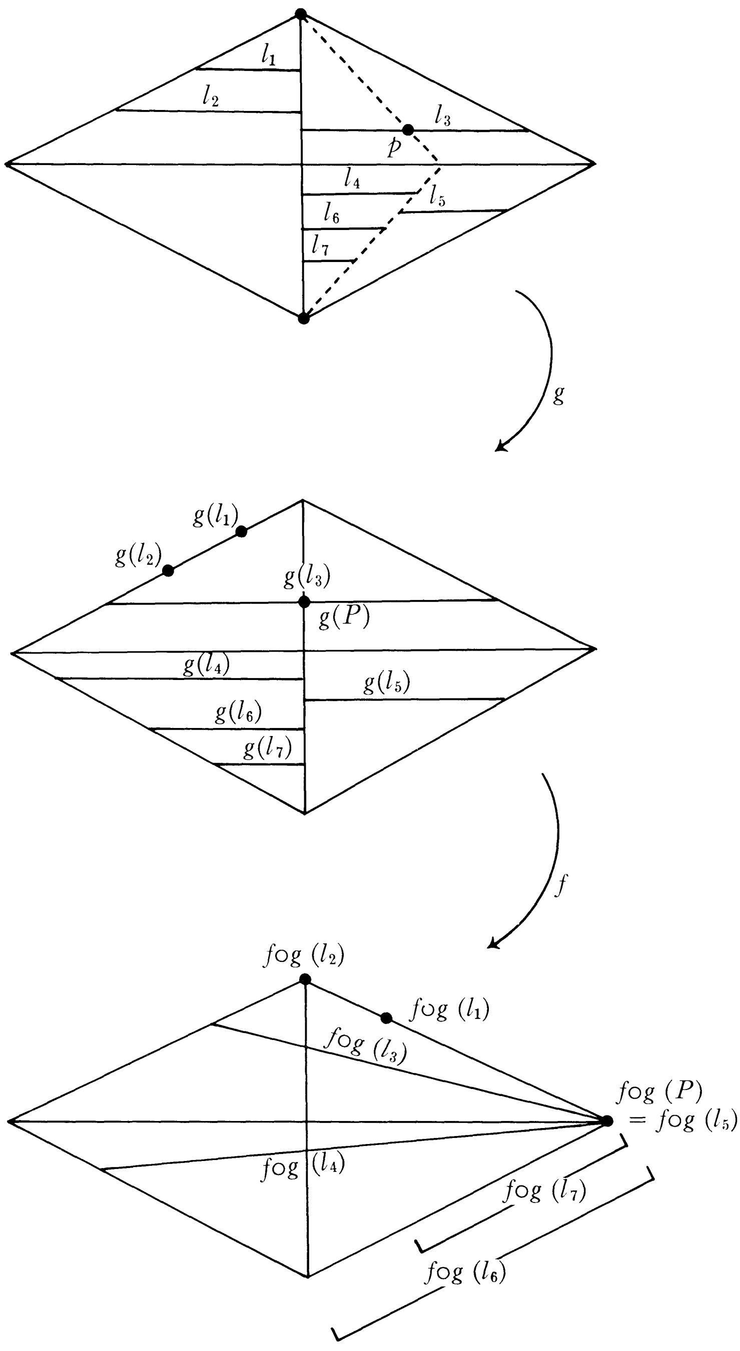

FIGURE 2. The map fg on $e^n{}_p \times y$, $y \in S^{q-1}{}_p$.

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identity. If the points (x_1, y) and (x_2, y) of $e^n{}_2 \times S^{q-1}{}_p$ lie on a line orthogonal to $e^{n-1}{}_p \times y$, then $g(x_1, y) = g(x_2, y)$.

Define a map F: $(e^n{}_p \times e^q{}_p)^{\cdot} \to L$ to be the composition kfg. Then F is homotopic to k since g and f are homotopic to the identities. Let σ be the homotopy class of k in the group $\pi_{n+q}(K, L)$. This group is isomorphic to $Z_{(p)}$, the integers localized at p, and σ is a generator. The map F represents $\partial_1\sigma$ in $\pi_{n+q-1}(L, S^q{}_p)$, where ∂_1: $\pi_{n+q}(K, L) \to \pi_{n+q-1}(L, S^q{}_p)$ is the boundary homomorphism. Let G be the restriction of F to $(e^n{}_1 \times S^{q-1}{}_p) \cup (e^{n-1}{}_+ \times e^q{}_p)$. Then the map G represents the relative Whitehead product $[i_q, i_n]$ in $\pi_{n+q-1}(L, S^q{}_p)$.

Let H be the restriction of F to $(e^n{}_2 \times S^{q-1}{}_p) \cup (e^{n-1}{}_- \times e^q{}_p)$. Then the image of H lies in $S^q{}_p$. The restriction of F to the boundary of $e^{n-1}{}_p \times e^q{}_p$ is a map which represents the Whitehead product $[\alpha, i_q]$ in $\pi_{n+q-2}(S^q{}_p)$. Since $S^q{}_p$ is an H-space, the Whitehead product $[\alpha, i_q]$ is trivial. Use the H-structure to deform the map F to a new map which is trivial on $(e^{n-1}{}_p \times e^q{}_p)^{\cdot}$. Now call this new map F, and use the names H and G for the same restrictions of the new F. Then H maps $(e^{n-1}{}_p \times e^q{}_p)^{\cdot}$ to the point e^0, and $[H] \in \pi_{n+q-1}(S^q{}_p)$.

Let ∂: $\pi_{n+q}(K, L) \to \pi_{n+q-1}(L)$ be the boundary homomorphism, and let i: $S^q{}_p \to L$ be the inclusion.

Definition 3.3. $\lambda_\alpha(E) = i_*[H]$ in $\pi_{n+q-1}(L)$.

The next proposition follows immediately from the definitions of the maps G and H as restrictions of the map F (Figure 3).

PROPOSITION 3.4. $\partial\sigma = \lambda_\alpha(E) + [i_q, i_n]_L$ *and* $\partial_1\sigma = [i_q, i_n]$.

Definition 3.5. $\Psi_\alpha(E) = \{\psi \in \pi_{n+q-1}(L) | \psi = c\lambda_\alpha(E)$ for some unit c of $Z_{(p)}\}$.

THEOREM 3.6. *Let q and n be odd integers, and let p be an odd prime. Assume that $S^q{}_p \to E_i \to S^n{}_p$ is a fibration for $i = 1$ and 2 and that the first attaching maps in the local cellular decompositions of the total spaces are the same. Call the common map α. Then $(E_1, S^q{}_p)$ and $(E_2, S^q{}_p)$ have the same homotopy type if and only if $\Psi_\alpha(E_1) = \Psi_\alpha(E_2)$.*

Proof. Suppose first that $\Psi_\alpha(E_1) = \Psi_\alpha(E_2)$. Then, since $\lambda_\alpha(E_1) \in \Psi_\alpha(E_2)$, there exists a unit c of $Z_{(p)}$ such that $\lambda_\alpha(E_1) = c\lambda_\alpha(E_2)$, i.e., $i_*[H_1] = ci_*[H_2]$.

Let $\beta = cj$: $S^q{}_q \to S^q{}_p$, where j is the identity mapping. We will apply the local form of the right distributive law: $(\mu + \eta)\gamma = \mu\gamma + \eta\gamma$ for $\gamma \in \pi_i(S^r)$ and $\mu, \eta \in \pi_r(X)$ such that the Whitehead product $[\mu, \eta] = 0$ [**7**, Lemma 6.5, p. 166]. Since all Whitehead products in the H-space $S^q{}_p$ are trivial, we have that $\beta\gamma = (cj)\gamma = c(j\gamma) = c\gamma$ for $\gamma \in \pi_i(S^q{}_p)$. It follows that $\beta\alpha \simeq c\alpha$. Also, we have that β induces isomorphisms in the homotopy groups, and thus β is a homotopy equivalence. Then the map β can be extended to a homotopy equivalence $\nu: L \to L$ such that $\nu_*(i_{n1}) = ci_{n2}$.

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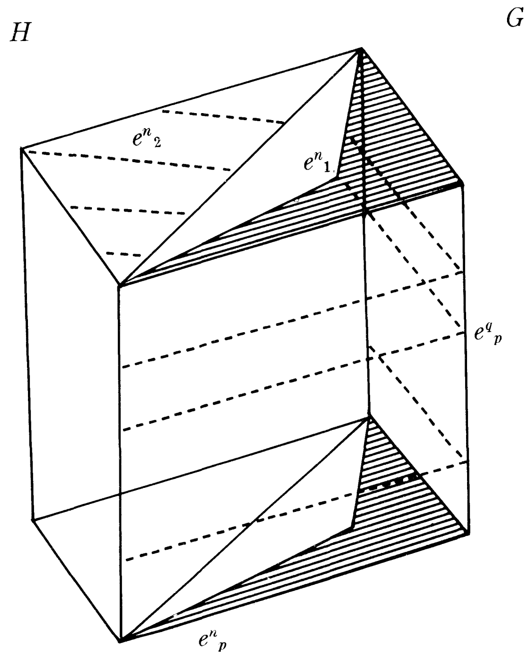

FIGURE 3. The Map F. Points on the same dotted line have the same image under F. Points in the shaded area of $e^n{}_p \times y$, $y \in S^{q-1}{}_p$, are mapped to e^0 by the map F.

Now, letting $\gamma = [H_1]$, we find that $\beta_*[\mathrm{H}_1] = c[H_1]$. Then by Propositions 3.2 and 3.4 we have that:

$$\begin{aligned}\nu_* \partial\sigma_1 &= \nu_* i_*[H_1] + \nu_*[i_{q1}, i_{n1}]_L \\ &= i_*\beta_*[H_1] + [\beta_* i_{q1}, \nu_* i_{n1}]_L \\ &= i_* c[H_1] + [ci_{q2}, ci_{n2}]_L \\ &= c^2 i_*[H_2] + c^2[i_{q2}, i_{n2}]_L \\ &= c^2 \partial\sigma_2.\end{aligned}$$

This means that the second attaching map $k_1|S^{n+q-1}{}_p$ of the total space E_1 is homotopic to $c^2(k_2|S^{n+q-1}{}_p)$. Then the map ν can be extended to a homotopy equivalence $\theta: E_1 \to E_2$. Since $\theta|S^q{}_p = \beta$, this yields a homotopy equivalence $(E_1, S^q{}_p) \to (E_2, S^q{}_p)$.

Before considering the converse, we will localize some of James's results [**13**]. If $f: X_p \to Y_p$ is a map, we will use $f': X \to Y$ to denote a map such that $(f')_p = f$. James shows that the homomorphism

$$(i_n')_*: \pi_{n+q-1}(e^n, S^{n-1}) \to \pi_{n+q-1}(S^q \cup_{\alpha'} e^n, S^q)$$

is a monomorphism and that $\pi_{n+q-1}(S^q \cup e^n, S^q) = Z \oplus \mathrm{Im}\,(i_n')_*$, where the Whitehead product $[i_n', i_q']$ is a generator of the infinite cyclic group Z. Then we have that

$$\pi_{n+q-1}(S^q{}_p \cup_\alpha e^n{}_p, S^q{}_p) = Z_{(p)} \oplus (\mathrm{Im}\,(i_n')_*)_p,$$

where $[i_n', i_q']_p = [i_n, i_q]$ is a generator of $Z_{(p)}$. James proves that

$$\partial_1\sigma = m[i_n', i_q'] + i_n'\rho',$$

where

$$i_q' \smile i_n' = m\sigma' \text{ and } \rho' \in \pi_{n+q-1}(e^n, S^{n-1}).$$

Then $\partial_1\sigma = m[i_n, i_q] + i_n\rho$. But we know that $\partial_1\sigma = [i_n, i_q]$ by Proposition 3.4. Then $m = 1$ and $i_n\rho = 0$. Thus $i_q \smile i_n = \sigma$, where i_q and i_n correspond to the homotopy classes of the same name.

Now we assume that $(E_1, S^q{}_p)$ and $(E_2, S^q{}_p)$ have the same homotopy type and let $\theta: (E_1, S^q{}_p) \to (E_2, S^q{}_p)$ be a homotopy equivalence. Then, since the cohomology groups $H^n(L\,S^q{}_p; Z_{(p)})$ and $H^q(S^q{}_p; Z_{(p)})$ are both isomorphic to $Z_{(p)}$,

$$\theta^*(i_{q2}) = bi_{q1} \text{ and } \theta^*(i_{n2}) = ci_{n1}$$

for some units b and c of $Z_{(p)}$. Then

$$\theta^*(i_{q2} \smile i_{n2}) = bc(i_{q1} \smile i_{n1}) \text{ in } H^{n+q}(E_1).$$

For $s = 1$ and 2, let $\sigma_s = (k^*)^{-1}(i_{qs} \smile i_{ns})$, where $k^*: H^{n+q}(E_s, L) \to H^{q+q}(E_s)$ is the isomorphism induced by the inclusion k. Then $\theta^*(\sigma_2) = bc\sigma_1$ in $H^{n+q}(E_1, L)$. Since σ_s in cohomology corresponds to the original σ_s in homotopy, we have that $\theta_*(\sigma_1) = bc\sigma_2$. Also, $\theta_*(i_{q1}) = bi_{q2}$ and $\theta_*(i_{n1}) = ci_{n2}$ in homotopy. Thus,

$$\theta_*(i_*[H_1]) = \theta_*(\partial\sigma_1 - [i_{q1}, i_{n1}]_L) = bc\partial\sigma_2 - bc[i_{q2}, i_{n2}]_L.$$

Then $\theta_* i_*[H_1] = bci_*[H_2]$.

Next we will show that $\theta_*[H_1] = c[H_1]$. Since $\theta_*(i_{q1}) = bi_{q2}$ and each i_{qs} is the homotopy class of the identity on $S^q{}_p$, we have that $\theta|S^q{}_p = bi_{q1}$. Then

$$\theta_*[H_1] = (bi_{q1})[H_1] = b[H_1] \text{ [7], and}$$

$$bi_*[H_1] = i_*b[H_1] = i_*\theta_*[H_1] = \theta_*i_*[H_1] = bci_*[H_2].$$

Since b is a unit, this gives that $i_*[H_1] = ci_*[H_2]$, i.e., $\lambda_\alpha(E_1) = c\lambda_\alpha(E_2)$.

We have shown that $\lambda_\alpha(E_1) \in \Psi_\alpha(E_2)$. Therefore, we have that $\Psi_\alpha(E_1) = \Psi_\alpha(E_2)$.

This completes the proof of the theorem.

4. The total space E as an H-space. Suppose that α is a fixed element of $\pi_{n-1}(S^q{}_p)$ and that $S^q{}_p \to E \to S^n{}_p$ is a fibration such that the total space E has local cellular decomposition $S^q{}_p \cup_\alpha e^n{}_p \cup e^{n+q}{}_p$. The aim of this section is to show that, for p greater than 3, E is an H-space if and only if $\lambda_\alpha(E) = 0$. In the case $\alpha = 0$, Curtis [**4**] shows that E is an H-space if and only if E has the same homotopy type as $S^q{}_p \times S^n{}_p$. For $\alpha \neq 0$, there is a space E_α which plays the role of $S^q{}_p \times S^n{}_p$. The space E_α is defined to be the local CW-complex $S^q{}_p \cup_{\alpha'} (e^n{}_p \times S^q{}_p)$, where $\alpha'(x, y) = \alpha(x) \cdot y$, the product $\cdot$ is multiplication in the H-space $S^q{}_p$, and $x \in S^{n-1}{}_p$, $y \in S^q{}_p$. Stasheff [**18**] proves that E_α is an H-space if $n < (1/2)(p-2)(q+1)$.

PROPOSITION 4.1. *If α is nontrivial, then $\lambda_\alpha(E_\alpha) = 0$. Also $\lambda_0(S^q{}_p \times S^n{}_p) = 0$*

Proof. It suffices to show that the map H (of Section 3) is homotopic to the trivial map for these spaces.

The space $S^q{}_p \times S^n{}_p$ can be represented as $S^q{}_p \cup_\gamma (e^n{}_p \times S^q{}_p)$, where, for $(x, y) \in S^{n-1}{}_p \times S^q{}_p$, $\gamma(x, y) = y$. Then, since $\alpha = 0$, we have that $\gamma(x, y) = \alpha(x) \cdot y$. Thus, the map γ corresponds to α' in the definition of E_α, and it will be called α'.

Both spaces $S^q{}_p \times S^n{}_p$ and E_α can be decomposed as local CW-complexes $S^q{}_p \cup e^n{}_p \cup e^{n+q}{}_p$. The first attaching map of E_α is α since, on $S^{n-1}{}_p \times S^q{}_p$,

$$\alpha'(x, a_q) = \alpha(x) \cdot a_q = \alpha(x).$$

The second attaching map is β, where

$$\begin{aligned} \beta(x, y) &= \alpha(x) \cdot u_q(y) && \text{if } (x, y) \in S^{n-1}{}_p \times e^q{}_p; \\ &= x && \text{if } (x, y) \in e^n{}_p \times S^{q-1}{}_p. \end{aligned}$$

The map H, as defined in Section 3, is the composition $\beta f g$ restricted to $(e^n{}_2 \times S^{q-1}{}_p) \cup (e^{n-1}{}_- \times e^q{}_p)$. Then, for $([x, t], y) \in e^{n-1}{}_- \times e^q{}_p$, $x \in (e^{n-1}{}_p)^\cdot$, $-1 \leqq t \leqq 1$, we have that

$$H([x, t], y) = (\beta f g)([x, t], y) = \alpha([x, 2t+1]) \cdot u_q(y).$$

For $(z, y) \in e^n{}_2 \times S^{q-1}{}_p$ such that $g(z, y) = ([x, t], y)$, we have that $H(z, y) = [x, 2t+1]$.

The map H can be extended to a map $J: e^n{}_2 \times e^q{}_p \to S^q{}_p$ by defining

$$J(z, y) = \alpha([x, 2t+1]) \cdot u_q(y),$$

where $z \in e^n{}_2$, $y \in e^q{}_p$ and $g(z, y) = ([x, t], y)$. Since H can be extended to $e^n{}_2 \times e^q{}_p$, H is homotopic to the trivial map, and thus $\lambda_\alpha(E) = i_*[H] = 0$.

PROPOSITION 4.2. *Suppose that p is an odd prime and that q and n are odd. Let $S^q{}_p \to E \to S^n{}_p$ be a fibration such that the total space E has first attaching map α. If E is an H-space, then the spaces E and E_α have the same homotopy type.*

Proof. Let $m: E \times E \to E$ be the multiplication. We can assume that m restricted to $S^q{}_p \times S^q{}_p$ provides an H-structure for the fiber $S^q{}_p$ [**3**], [**6**]. Let

$$\mu = m | S^q{}_p \times S^q{}_p : S^q{}_p \times S^q{}_p \to S^q{}_p.$$

In $E \times E$ define an equivalence relation $\sim$ by: $(u, v) \sim (u', v')$ if and only if $m(u, v) = m(u', v')$ and $u,\ u' \in S^{n-1}{}_p$ and $v,\ v' \in S^q{}_p$. Define the map g: $E_\alpha \to E \times E/\sim$ to be the one induced by the product of inclusions $e^n{}_p \times S^q{}_p \to$

$E \times E$. This map is well-defined because $m|S^q{}_p \times S^q{}_p = \mu$. Now define the map m': $E \times E/\sim \rightarrow E$ to be the one induced by m: $E \times E \rightarrow E$, and let f: $E_\alpha \rightarrow E$ be the composition of m' and g.

We want to show that the map f is a homotopy equivalence. Let i_q, $i_n \in H^*(E)$ and $i_q', i_n' \in H^*(E_\alpha)$ be the generators corresponding to those in homotopy constructed from the map k (in Section 3) for the spaces E and E_α. Then, since $f\,|a_{n-1} \times S^q{}_p$ is the identity onto $S^q{}_p$ and $f\,|e^n{}_p \times a_{q-1}$ is the identity onto $e^n{}_p$, it follows that $f^*(i_q) = i_q'$ and $f^*(i_n) = i_n'$. Thus, we have that

$$f^*(i_q \smile i_n) = f^*i_q \smile f^*i_n = i_q' \smile i_n'.$$

Since these cup products are the generators in dimension $n + q$, f^*: $H^*(E) \rightarrow H^*(E_\alpha)$ is an isomorphism. Then f is a homotopy equivalence. This completes the proof of the theorem.

COROLLARY 4.3. *If $p > 3$, then E_α is an H-space.*

Proof. There exists a fibration $S^q \rightarrow X \rightarrow S^n$ such that the total space X localized at $p > 3$ is an H-space homotopic to a local CW-complex $S^q{}_p \cup_\alpha e^n{}_p \cup e^{n+q}{}_p$ [**6**]. Then, by the preceding proposition, the spaces X_p and E_α have the same homotopy type, and, thus, E_α is an H-space.

THEOREM 4.4. *Suppose that q and n are odd integers and that p is an odd prime. Let $S^q{}_p \rightarrow E \rightarrow S^n{}_p$ be a fibration such that E has first attaching map α. If $p > 3$, then $\lambda_\alpha(E) = 0$ if and only if E is an H-space. For $p = 3$, if E is an H-space, then $\lambda_\alpha(E) = 0$.*

Proof. Suppose that $p \geqq 3$ and that E is an H-space. Then the spaces E and E_α have the same homotopy type (Proposition 4.2), and thus $\lambda_\alpha(E) = \lambda_\alpha(E_\alpha) = 0$ (Theorem 3.6 and Proposition 4.1).

Now let $p > 3$ and suppose that $\lambda_\alpha(E) = 0$. Then E and E_α have the same homotopy type (Theorem 3.6 and Proposition 4.1). Since E_α is an H-space (Corollary 4.3), the space E also is an H-space.

This concludes the discussion of H-spaces with three local cells.

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