# Note on the Problem of the Electrified Disc

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1. Prof. E. T. Copson[1] has discussed the well-known problem of a circular disc kept at a constant potential $V_0$ in an external field of potential $\Phi$ by reducing it to the solution of two integral equations. The solution is however fairly simple if we use oblate spheroidal co-ordinates. This is due to the fact that in this system of co-ordinates the disc can be represented in terms of one co-ordinate only. This method is applied to the above problem and Copson's results are obtained. The solution when $V_0$ is not constant, but any surface function of the disc, is also obtained.

2. The relations between cylindrical polar and oblate spheroidal co-ordinates $\xi$, $\zeta$ are

$$z = a\xi\zeta,\ \rho^2 = a^2(1-\xi^2)(1+\zeta^2), \qquad -1 \leqq \xi \leqq 1, 0 \leqq \zeta < \infty .$$

The circular disc $z = 0$, $\rho \leqq a$ is given by $\zeta = 0$. At the surface of the disc it can easily be seen that

$$\rho^2 = a^2(1-\xi^2), \qquad \frac{\partial V}{\partial z} = \frac{1}{a\xi}\frac{\partial V}{\partial \zeta}.$$

Solutions of Laplace's equation in this system of co-ordinates are oblate spheroidal harmonics and are of the form

$$[A_n^m P_n^m(\xi) + B_n^m Q_n^m(\xi)]\,[A'^m_n P_n^m(i\zeta) + B'^m_n Q_n^m(i\zeta)]\,e^{im\phi},$$

$P_n^m(u)$ and $Q_n^m(u)$ being Legendre associated functions of the first and second kinds respectively.

The potential of any external field which is bounded in the neighbourhood of the disc and at its axis $\xi = 1$ will contain harmonics of the form

$$E_n^m P_n^m(\xi)\, P_n^m(i\zeta) \cos m\phi,$$

$n + m$ being necessarily even, since otherwise the potential would vanish at the surface of the disc.

By similar considerations $V$, the potential of the induced charge, will contain only harmonics of the form

$$I_n^m P_n^m(\xi)\, Q_n^m(i\zeta) \cos m\phi, \qquad n + m \text{ even}.$$

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[1] "On the problem of the electrified disc," *Proc. Edin. Math. Soc.* (2), 8 (1947), 14-19.

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If for simplicity we take $\Phi$ as one harmonic only, then, since $V$ has the value $V_0$ at $\zeta = 0$, its value at any point will be

$$V = V_0\Big(P_0^0(\xi)\, Q_0^0(i\zeta)/Q_0^0(i0)\Big) - E_n^m P_n^m(\xi)\, Q_n^m(i\zeta) \times \Big(P_n^m(i0)/Q_n^m(i0)\Big)\cos m\phi$$

$$= \frac{2}{\pi}\left\{V_0 \cot^{-1}\xi - iE_n^m P_n^m(\xi)\, Q_n^m(i\zeta)\cos m\phi\right\}.$$

The density of the induced charge, $\sigma$, is given by

$$\sigma = -\frac{1}{2\pi}\left[\frac{\partial V}{\partial z}\right]_{\zeta=0} = -\frac{1}{2\pi a\xi}\left[\frac{\partial V}{\partial \zeta}\right]_{\zeta=0}$$

$$= -\frac{1}{\pi^2 a\xi}\left[\frac{-V_0}{1+\zeta^2} + E_n^m P_n^m(\xi)\, Q_n^{m\prime}(i\zeta)\cos m\phi\right]_{\zeta=0}$$

$$= \frac{1}{\pi^2 a\xi}\left\{V_0 - (-1)^{\frac{1}{2}(n-m)}\frac{2.4.6\ldots(n+m)}{1.3.5\ldots(n-m-1)}E_n^m P_n^m(\xi)\cos m\phi\right\}.$$

The first two special cases considered by Copson follow immediately.

(i) If there is no external field, $E_n^m = 0$ and

$$\sigma = \frac{V_0}{\pi^2\sqrt{(a^2-\rho^2)}}.$$

(ii) If $V_0 = 0$, $\Phi = \rho\cos\phi = -ai\,P_1^1(\xi)\,P_1^1(i\zeta)\cos\phi$,

then

$$\sigma = \frac{-ai}{\pi^2 a\xi}P_1^1(\xi)\, Q_1^{1\prime}(i0)\cos\phi$$

$$= \frac{-i}{\pi^2\xi}\sqrt{(1-\xi^2)}\, 2i\cos\phi = \frac{2}{\pi^2}\frac{\rho\cos\phi}{\sqrt{(a^2-\rho^2)}}.$$

Probably this is not the best method for the last application.

It will be seen that the same method is applicable when the potential at the surface of the disc (which will be taken now as non-conducting) is any surface function. Any such function can be expanded in the form

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\xi)\, e^{im\phi}.$$

The potential at any external point $(\xi, \zeta, \phi)$ is then given by

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A_n^m P_n^m(\xi)\Big(Q_n^m(i\zeta)/Q_n^m(i0)\Big)e^{im\phi}.$$

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