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# A STRENGTHENED TOPOLOGICAL CARDINAL INEQUALITY

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A new cardinal inequality, $|K(X)| \leq 2^{L^*(X) \cdot psw(X)}$ , is proved in this paper. It strengthens the result of D.K. Burke and R. Hodel that $|K(X)| \leq 2^{e(X) \cdot psw(X)}$ .

A bound on the number of compact sets in a topological space is given by D.K. Burke and R. Hodel [1]: for every $T_1$-space $X$ , we have

$$|K(X)| \leq 2^{e(X) \cdot psw(X)} .$$

Here, $|K(X)| = |\{C : C$ is a compact subset of $X\}|$ ;

$e(X) = \sup\{|D| : D$ is a closed discrete subspace of $X\} + \omega$ ; and $psw(X) = \min\{k :$ there exists some separating open cover $\mathcal{U}$ of $X$ with $\mathrm{ord}(x,\mathcal{U}) \leq k$ for all $x \in X\}$. (The cover $\mathcal{U}$ of $X$ is separating if $\cap\{U \ \mathcal{U} : x \ U\} = \{x\}$ for all $x \in X$ , and $\mathrm{ord}(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$ . ) For this and related results, see the survey paper Hodel [4]. We generalize this result in this paper.

First, we give a definition as follows:

DEFINITION. For every topological space $X$ , the <u>*Lindelöf number of $X$</u> , denoted by $L^*(X)$ , is defined by:

$$L^*(X) = \min\{k : \text{for every open cover } \mathcal{U} \text{ of } X \text{, there exists } A \subseteq X \text{ with } |A| \leq k \text{ such that } \bigcup_{x \in A} st(x,\mathcal{U}) = X\} .$$

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LEMMA 1. (Burke's Lemma, [3, Theorem 1.1])

*If* $\{A_\alpha : \alpha \in \Lambda\}$ *is an indexed collection of sets in which every member has cardinality* $\le \lambda$ *, where* $|\Lambda| > 2^\lambda$ *, and each* $A_\alpha$ *is a disjoint union of two subsets* $A'_\alpha$ *,* $A''_\alpha$ *, then there is a set* $\Lambda' \subseteq \Lambda$ *such that* $|\Lambda'| > 2^\lambda$ *and* $A'_\alpha \cap A''_\beta = \emptyset$ *when* $\alpha$ *,* $\beta \in \Lambda'$ *.*

The main theorem in this paper is as follows:

THEOREM. *For every* $T_1$*-space* $X$ *, we have*

$$|K(X)| \le 2^{L^*(X)\cdot psw(X)} .$$

Proof. The first step is to show that for $x \in T_1$ we have $|X| \le 2^{L^*(X)\cdot psw(X)}$ , using Burke's Lemma.

Let $L^*(X)\cdot psw(X) = \lambda$ , then $psw(X) \le \lambda$ and $L^*(X) \le \lambda$ . Thus there is an open cover $\mathcal{U}$ of $X$ such that $\cap\{U\in\mathcal{U} : x\in U\} = \{x\}$ and $\mathrm{ord}(x,\mathcal{U}) \le \lambda$ for all $x \in X$ . We first construct the sets $A_y = A'_y \cup A''_y$ such that $A'_y \cap A''_y = \emptyset$ and $|A_y| \le \lambda$ for all $y \in X$ as subsets of $X$ . In fact, $\{U\in\mathcal{U} : y\in U\}$ can be indexed and denoted by $\{U_\alpha\}_{\alpha<\lambda}$ . Let $\mathcal{V} = \{U\in\mathcal{U} | y\notin U\}$ and $\mathcal{U}_\alpha = \mathcal{V} \cup \{U_\alpha\}$ for $\alpha < \lambda$ . Since $L^*(X) \le \lambda$ , there exists some $B_\alpha \subset X$ such that $|B_\alpha| \le \lambda$ and $\bigcup_{x\in B_\alpha} st(x,\mathcal{U}_\alpha) = X$ . Since $st(x,\mathcal{U}_\alpha) \subset st(x,\mathcal{V}) \cup U_\alpha$ when $x \ne y$ , but $st(y,\mathcal{U}_\alpha) = U_\alpha$ , then we have $\bigcup_{x\in B_\alpha} st(x,\mathcal{V}) \cup U_\alpha = X$ , and therefore $\bigcup_{x\in B_\alpha} st(x,\mathcal{V}) \supset X - U_\alpha$ . Let $B(y) = \bigcup_{\alpha<\lambda} B_\alpha$ , then $|B(y)| \le \lambda\cdot\lambda = \lambda$ . Then $\bigcup_{x\in B(y)} st(x,\mathcal{V}) = \bigcup_{\alpha<\lambda}\bigcup_{x\in B_\alpha} st(x,\mathcal{V}) \supset \bigcup_{\alpha<\lambda}(X-U_\alpha) = X - \bigcap_{\alpha<\lambda} U_\alpha = X - \{y\}$ . Since $y \notin \bigcup_{x\in B(y)} st(x,\mathcal{V})$, we have $\bigcup_{x\in B(y)} st(x,\mathcal{V}) = X - \{y\}$ . Since $|\bigcup_{x\in B(y)} \{U\in\mathcal{V} | x\in U\}| \le \lambda\cdot\lambda = \lambda$ and

$\mathrm{ord}(y,\mathcal{U}) \le \lambda$ , we can define the set $A_y = A'_y \cup A''_y$ , where $A'_y = \bigcup_{x\in B_y} \{U\in\mathcal{U} : x\in U\}$ and $A''_y = \{U\in\mathcal{U} : y\in U\}$ . It is clear that $A'_y \cap A''_y = \emptyset$ and $|A_y| \le |A'| + |A''| \le \lambda + \lambda = \lambda$ .

Now, we have defined the sets $A_y$ for all $y \in X$. We can obtain $|X| \leq 2^{\lambda}$, immediately. Otherwise, $|X| > 2^{\lambda}$. Let $A = \{A_x\}_{x \in X}$. Then, by Burke's Lemma, there is a subset $X' \subset X$ such that $|X'| > 2^{\lambda}$ and $A'_x \cap A''_y = \emptyset$ for any $x, y \in X'$. But this is impossible, because $y \in X - \{x\} = \bigcup_{x' \in B(x)} st(x', V)$ where $V' = \{U \in \mathcal{U} | X \not\subset U\}$, whenever $x \neq y$. However, there exists some $U \in A'_x$ such that $y \in U$ and, of course, $U \in A''_y$ so $A'_x \cap A''_y = \emptyset$, a contradiction. This contradiction shows that we must have $|X| \leq 2^{\lambda} = 2^{L^*(X) \cdot psw(X)}$.

A standard argument now establishes that $|K(X)| \leq 2^{L^*(X) \cdot psw(X)} = 2^{\lambda}$, for example see Hodel [4, proof of Theorem 9.3].

REMARK. The definition of $L^*(X)$ was first introduced by Dai MuMing [5], and an independent proof of the result $|X| \leq 2^{L^*(X) \cdot psw(X)}$ is given in [5]. But our argument is simpler than the original argument.

COROLLARY 1. (D.K. Burke and R. Hodel [1, Theorem 4.4]) *For every $T_1$-space $X$, we have*

$$|K(X)| \leq 2^{e(X) \cdot psw(X)} .$$

Proof. It is sufficient to show $e(X) \geq L^*(X)$. In fact, for every open cover $\mathcal{U}$ of $X$, consider the maximal subset $A \subseteq X$ satisfying the following property (*).

(*): for all $x, y \in A$ if $x \neq y$ then $x \notin st(y, \mathcal{U})$.

Clearly, $X = \bigcup_{y \in A} st(y, \mathcal{U})$. Otherwise, there exists an $x_o \in X - \bigcup_{y \in A} st(y, \mathcal{U})$. But then $st(x_o, \mathcal{U}) \cap A = \emptyset$ and $A \cup \{x_o\}$ satisfies the property (*), contradicting the fact that $A$ is maximal. By definition, $L^*(X) \leq |A|$. To show that $A$ is a closed discrete subspace of $X$, note $A$ is discrete since $st(a, \mathcal{U})$ is open and $\{a\} = st(a, \mathcal{U}) \cap A$ for all $a \in A$ and $A$ is closed since for all $x \in X - A$, there exists an $a_o \in A$ such that $x \in st(a_o, \mathcal{U})$ and $X$ is a $T_1$-space, so $st(a_o, \mathcal{U}) - \{a_o\}$ is open and it is disjoint from $A$. By definition, we have $e(X) \geq |A| \geq L^*(X)$.

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EXAMPLE 1. The Niemytzki plane $X$ is separable, so $L^*(X) = \omega_o$, but it contains a closed discrete subspace of cardinality $c$, and therefore $e(X) \geq c > \omega_o = L^*(X)$.

EXAMPLE 2. Let $Y = N^c$, where $N$ is the discrete countable space. By the Hewitt-Marczewski-Pondiczery theorem, $d(Y) = \omega_o$, and hence $L^*(Y) = \omega_o$. Because if $\bar{A} = X$, then $\bigcup_{x\in A} st(x,U) = X$ for every cover $U$ of $X$. Engelking [2] has proved that the space contains a closed discrete subspace cardinality of $c$, and so $e(Y) \geq c$. Thus $e(Y) > L^*(Y)$, also.

These examples show that the theorem in this paper is a significant extension for Burke and Hodel's result [1].

COROLLARY 2. *For every $T_1$-space $X$, we have*

$$|K(X)| \leq 2^{d(X)\cdot psw(X)} .$$

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