A CHARACTERIZATION OF THE ALGEBRA OF FUNCTIONS VANISHING AT INFINITY

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1. In this paper, X will always denote a locally compact Hausdorff space, $C_0(X)$ the algebra of all complex-valued continuous functions vanishing at infinity on X and $B(X)$ the algebra of all bounded continuous complex-valued functions defined on X. If X is compact, $C_0(X)$ is identical to $B(X)$ and all the results of this paper are obvious. Therefore, we will assume at the outset that X is not compact. If A represents an algebra of functions, $A_{\mathbf{R}}$ will denote the algebra of all real-valued functions in A.

A continuous function f defined on X is said to vanish at infinity if for every positive number ϵ the set $\{x: |f(x)| \geqq \epsilon\}$ is a compact subset of X. We are interested in the following type of question: if it is only known that for each f in an algebra A, $\{x: f(x) = 1\}$ is a compact subset of X, must f belong to $C_0(X)$ for each f in A? If we know that A is a closed subalgebra of $B_{\mathbf{R}}(X)$, the answer is affirmative. If, however, A is a closed subalgebra of $B(X)$, we exhibit an example showing that A need not be contained in $C_0(X)$. We will say that an algebra A satisfies property (P) if $x: f(x) = 1$ is compact for all f in A. We show that there can be many algebras satisfying property (P) which are distinct from $C_0(X)$ and even maximal with respect to satisfying property (P). However, $C_0(X)$ is characterized as the unique closed subalgebra of $B(X)$ maximal with respect to satisfying property (P).

2. Let $C_{\mathbf{R}}(X)$ denote the algebra of real-valued functions in $C_0(X)$.

THEOREM 1. *Let A be a closed subalgebra of $B_{\mathbf{R}}(X)$ satisfying property* (P). *Then A is a subalgebra of $C_{\mathbf{R}}(X)$.*

Proof. Let f be any function in A and suppose that $\epsilon > 0$ is arbitrary. Let K be the closed set $\{x \in X: |f(x)| \geqq \epsilon\}$. We proceed to show that K is compact by selecting an interval $[-b, b]$ containing $f(X)$ and a continuous real-valued function F defined on $[-b, b]$. We can choose F such that $F(0) = 0$ and F is identically one on $[-b, -\epsilon] \cup [\epsilon, b]$. In view of the Weierstrass approximation theorem, there exists a sequence of polynomials $\{P_n\}$ with real coefficients such that $P_n(0) = 0$ for $n = 1, 2, \ldots$ and $\{P_n\}$ converges uniformly to F on $[-b, b]$. It follows that $P_n \circ f$ belong to A for $n = 1, 2, \ldots$ and $\{P_n \circ f\}$ converges uniformly on X to a function $h = F \circ f$ in A. Since $|f(x)| \geqq \epsilon$ on K, F is one on $f(K)$ and h is one on K. By assumption, $\{x: h(x) = 1\}$ is compact and K being a closed subset must be compact.

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COROLLARY. *If A is a closed self-adjoint subalgebra of $B(X)$ satisfying property* (P), *then A is contained in $C_0(X)$.*

Proof. If $f \in A$, then $f\bar{f} = |f|^2 \in A$. Let

$$K = \{x\colon |f(x)| \geqq \sqrt{\epsilon}\} = \{x\colon |f(x)|^2 \geqq \epsilon\}.$$

K is seen to be compact from the argument in Theorem 1.

Remark. To see that the closure of A in Theorem 1 is necessary, consider $X = [0, 1)$ and A the algebra of all polynomial functions on X which are zero at $x = 0$. The following is an example of a closed subalgebra A of $B(X)$ satisfying property (P) yet not contained in $C_0(X)$.

Example. **C** will denote the field of complex numbers. Let

$$X = \{z \in \mathbf{C}\colon |z| = 2\} \cup \{z \in \mathbf{C}\colon 0 \leqq \operatorname{Re} z < 1, \operatorname{Im} z = 0\}.$$

X is a locally compact Hausdorff space if it is given the induced topology of the plane. Let $A_0 = \{f \in B(X)\colon f(0) = 0$ and f is the restriction of a function continuous on $|z| \leqq 2$ and holomorphic in $|z| < 2.\}$

Clearly, A_0 is a closed subalgebra of $B(X)$. A_0 satisfies property (P); otherwise, $\{x\colon f(x) = 1\}$ is not compact for some f in A_0. Then, there would necessarily exist a sequence $\{x_n\}$ in X such that $\{x_n\} \to 1$ and $f(x_n) = 1$ for $n = 1, 2, \ldots$. By the Identity Theorem of one complex variable, f would have to be identically one on X which cannot be since $f(0) = 0$. Thus, A_0 satisfies property (P). Since the function g defined by $g(z) = z$ for all z in X does not vanish at infinity, A_0 is not contained in $C_0(X)$.

Remark. It is interesting to note that the algebra A_0 has the somewhat stronger property that $\{x\colon f(x) = 1\}$ is a compact subset of X for every f in A_0. If such were not the case there would be some function f_0 in A_0 such that $f_0(x_n) = 1$ for $n = 1, 2, \ldots$, where $\{x_n\} \to 1$. If F is the holomorphic extension of f_0 with $\sum_0^\infty a_n \cdot z^n$ representing the power series expansion of F, then $G(z) = \sum_0^\infty \bar{a}_n \cdot z^n$ is also continuous on $|z| \leqq 2$ and holomorphic in $|z| < 2$. Since $G(x) = \overline{F(x)}$ for each real number x in $[0, 1)$, it follows that $G(x_n) \cdot F(x_n) = 1$ for $n = 1, 2, \ldots$. Letting h be the function in A_0 which is the restriction of GF to X, we conclude that h is identically one on X, which cannot be.

Thus, the property that $\{x\colon |f(x)| = 1\}$ is a compact subset of X for all functions in a closed subalgebra of $B(X)$ is still not sufficient to imply that all the functions in the algebra vanish at infinity.

3. The example in § 2 can be used to demonstrate the existence of algebras other than $C_0(X)$ which are maximal with respect to property (P). A simple application of Zorn's lemma will show that there exists an algebra A_M (not necessarily closed) containing A_0 and maximal with respect to property (P), i.e., if any algebra A satisfies property (P), then A cannot properly contain

A_M. $A_M \neq C_0(X)$ since $g(z) = z$ defines a function g in A_0 not in $C_0(X)$. Note that the conjugate of A_M, namely $\bar{A}_M = \{\bar{f}: f \in A_M\}$, is another algebra distinct from A_M and $C_0(X)$. The fact that $A_M \neq \bar{A}_M$ follows easily from the corollary to Theorem 1.

Before we show that $C_0(X)$ is the unique closed subalgebra of $B(X)$ maximal with respect to property (P), we need the following lemma.

LEMMA. *Let X be a locally compact Hausdorff space and let Y be a closed subset of X. Considering $Y_\infty = Y \cup \{\infty\}$ as a subset of $X_\infty = X \cup \{\infty\}$, let T_1 and T be the one-point compactification topologies of Y_∞ and X_∞, respectively. Then T_1 is the subspace topology on Y_∞ induced from (X_∞, T).*

Proof. Let T_2 denote the subspace topology on Y_∞ induced from (X_∞, T). Clearly, (Y_∞, T_2) is a Hausdorff space. Since (Y_∞, T_1) is a compact space, we need only show that the identity mapping $\Psi: (Y_\infty, T_1) \to (Y_\infty, T_2)$ from Y_∞ onto Y_∞ is continuous in order to conclude that $T_1 = T_2$. In this direction, let U be an element in T_2. If ∞ belongs to U, then $U = V \cap Y_\infty$ for some V such that $X \backslash V$ is compact in X. Thus, $U = Y_\infty \backslash (Y \backslash V)$, where $Y \backslash V$ is compact in Y. In this case, $U = \Psi^{-1}(U)$ is a member of T_1. If $\infty \notin U$, then U is the intersection of an open set in X with Y in which case $U = \Psi^{-1}(U)$ is a member of T_1. Thus, Ψ is continuous, and $T_1 = T_2$.

THEOREM 2. *Let X be a locally compact Hausdorff space and let A be a subalgebra (not necessarily closed) of $B(X)$ such that A contains $C_0(X)$ and satisfies property* (P). *Then $A = C_0(X)$.*

Proof. Suppose, on the contrary, that there exists a function f in A which does not vanish at infinity. For some $\epsilon > 0$, $\{x: |f(x)| \geqq \epsilon\}$ is not compact.

Let D be the collection of all compact subsets of X directed by inclusion, i.e., $K_1 \geqq K_2 \Leftrightarrow K_1 \supset K_2$. For each K in D let x_K be a point not in K such that $|f(x_K)| \geqq \epsilon$. If we let Z denote some compact subset of $\mathbf{C}$ containing $f(X)$, we see that $\{x_K, f(x_K)\}_{K \in D}$ is a net in $X_\infty \times Z$. Since $X_\infty \times Z$ is compact, there exists a subnet $\{x_\alpha, f(x_\alpha)\}$ converging to some point (p, q) in $X_\infty \times Z$. Since every neighbourhood of infinity eventually contains all x_α it follows that $p = \infty$. Clearly, $|q| \geqq \epsilon$. Since A satisfies property (P), there exists an open set U containing $\{x: f(x) = q\}$ such that U has compact closure. For each positive integer n let x_n be one of the x_α's outside U such that $|f(x_\alpha) - q| < 1/n$. Clearly, $\{x_n\}$ can have no adherent point in $X \backslash U$ for if P_0 were such a point, $f(P_0)$ would equal q. Thus, $Y = \bigcup_{n=1}^{\infty} \{x_n\}$ is a closed subset of X which is not compact. The function g defined by

$$g(\infty) = 0, \qquad g(x_n) = f(x_n) - q$$

is a continuous function on Y_∞. Since Y_∞ is a compact subset of X_∞ in the induced topology (see Lemma), it follows by the Tietze extension theorem that g can be extended to a function G continuous on X_∞. Since $G(\infty) = 0$, it follows that G belongs to $C_0(X)$, and hence it belongs to A. Letting

$h = (f - G)/g$, we see that h is in A and $h(x_n) = 1$ for $n = 1, 2, \ldots$. We conclude that Y must be compact since it is a closed subset of $\{x: h(x) = 1\}$. This contradicts our original observation that Y is not compact; thus, we must have that A is contained in $C_0(X)$.

THEOREM 3. *$C_0(X)$ is the unique closed subalgebra of $B(X)$ maximal with respect to property* (P).

Proof. Let A be a closed subalgebra of $B(X)$ maximal with respect to property (P). Let us first suppose that $A \cap C_{\mathbf{R}}(X)$ fails to separate the points of X. Let p and q be two distinct points in X such that $f(p) = f(q)$ for all f in $A \cap C_{\mathbf{R}}(X)$. If K is a compact set whose interior contains p and q, there exists a function g in $C_{\mathbf{R}}(X)$ such that $g(p) = 1$, $g(q) = 0$, and g vanishes outside the interior of K. Note that g does not belong to A. Let A_1 be the smallest algebra containing g and A. Clearly, $A_1|Y = A|Y$, where Y denotes the complement in X of the interior of K. If f is any function in A_1, let $\mathbf{C} = \{x \in X: f(x) = 1\}$. Choose a function h in A such that $h(x) = f(x)$ for all x in Y. Then $\mathbf{C} = \{x \in \operatorname{Int} K: f(x) = 1\} \cup \{x \in Y: h(x) = 1\}$ is a closed subset of the compact set $K \cup \{x \in X: h(x) = 1\}$. We conclude that $\mathbf{C}$ is compact and A_1 satisfies property (P). This cannot be since A is maximal with respect to property (P) and A_1 properly contains A. We must therefore look at the only remaining case, namely, that $A \cap C_{\mathbf{R}}(X)$ separates the points of X. By the Stone-Weierstrass theorem, A contains $C_0(X)$ and by Theorem 2, we can finally conclude that $A = C_0(X)$.

Added in proof. In the conclusion of Theorem 3, we tacitly assume that $A \cap C_{\mathbf{R}}(X)$ separates strongly. If it did not separate strongly, let p be a point in X such that $f(p) = 0$ for all f in $A \cap C_{\mathbf{R}}(X)$ and let K_0 be a compact set containing p as an interior point. Let

$$A_0 = \{f + g: f \in A,\ g \in B(X) \text{ and } g \text{ vanishes outside } K_0\};$$

then A_0 is an algebra satisfying property (P) which contains $C_0(X)$. From Theorem 2, it follows that $A_0 = C_0(X)$, and hence $A = C_0(X)$.

REFERENCES

1. J. L. Kelley, *General topology* (Van Nostrand, New York, 1955).
2. R. E. Mullins, *Some results on algebras of functions*, Ph.D. Thesis, Northwestern University, Evanston, Illinois, 1965.

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