INTERSECTION THEOREMS FOR SYSTEMS OF SETS (III)

Dedicated to the memory of Hanna Neumann

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1. Introduction

A system or family $(A_\gamma: \gamma \in N)$ of sets A_γ, indexed by the elements of a set N, is called an (a, b)-*system* if $|N| = a$ and $|A_\gamma| = b$ for $\gamma \in N$. Expressions such as "$(a, < b)$-system" are self-explanatory. The system $(A_\gamma: \gamma \in N)$ is called a Δ-*system* [1] if $A_\mu \cap A_\gamma = A_\rho \cap A_\sigma$ whenever $\mu, \gamma, \rho, \sigma \in N$; $\mu \neq \gamma$; $\rho \neq \sigma$. If we want to indicate the cardinality $|N|$ of the index set N then we speak of a $\Delta(|N|)$-system. In [1] conditions on cardinals a, b, c were obtained which imply that every (a, b)-system contains a $\Delta(c)$-subsystem. In [2], for every choice of cardinals b, c such that

$$b \geqq 2; \; c \geqq 3; \; b + c \geqq \aleph_0$$

the least cardinal $a = f_\Delta(b, c)$ was determined which has the property that

every $(a, < b)$-system contains a $\Delta(c)$-subsystem.

Let b^+ be the least cardinal greater than b. It is easy to see that the following two statements are equivalent:

every $(a, < b^+)$-system contains a $\Delta(c)$-subsystem,

every (a, b)-system contains a $\Delta(c)$-subsystem.

In the present note we prove a best possible theorem (Theorem 1) on the size of the largest Δ-subsystem that can be found in every (m^+, m)-system $(A_\gamma: \gamma \in N)$ which satisfies $|A_\mu \cap A_\gamma| < n$ for $\mu, \gamma \in N$; $\mu \neq \gamma$. Here $m \geqq \aleph_0$, and n is a given cardinal, $n < m$. In proving this theorem the authors have received valuable help from A. Hajnal.

We now introduce a condition on systems of sets which is less exacting than that of being a Δ-system. The system $(A_\gamma: \gamma \in N)$ is called a *weak* Δ-*system* (*wk*

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Δ-*system*) if

$$|A_\mu \cap A_\gamma| = |A_\rho \cap A_\sigma|$$

whenever $\mu, \gamma, \rho, \sigma \in N$; $\mu \neq \gamma$; $\rho \neq \sigma$.

To avoid misunderstandings we shall henceforth replace the term "Δ-system" by "*strong* Δ-*system* (*st* Δ-*system*). Clearly, every st Δ-system is also a wk Δ-system, and the system $(\{1,2\}, \{1,3\}, \{2,3\})$ is weak but not strong. In Theorem 2 we give an implication in the opposite direction. For cardinals a, b, c, let the relation

$$(1) \qquad (a, b) \to \mathrm{wk}\,\Delta(c)$$

mean that every (a, b)-system contains a wk$\Delta(c)$-subsystem, and similarly for the relation

$$(2) \qquad (a, b) \to \mathrm{st}\,\Delta(c).$$

The negation of a relation involving an arrow $\to$ is obtained by writing $\not\to$ instead of $\to$. The symbol wkΔ by itself denotes the class of all wkΔ-systems, and similarly in other cases, such as st$\Delta(c)$.

In Section 5 we prove a number of results on Δ-systems. In Section 7 we give a complete discussion of the relation (1) for $a, b \geqq \aleph_0$. In this discussion, as well as in some of our theorems, we shall assume the generalised continuum hypothesis (GCH).

2. Terminology and notation

Roman capitals denote sets, and $A \subset B$ denotes inclusion in the wide sense. For every system $(A_\gamma : \gamma \in N)$ and $M \subset N$, we put $A_M = \bigcup (\gamma \in M) A_\gamma$. The system $(A_\gamma : \gamma \in N)$ is called an (a, b)-*system* if $|N| = a$ and $|A_\gamma| = b$ for all $\gamma \in N$. The class of all (a, b)-systems is denoted by $\Omega(a, b)$. For every set A and every cardinal r we put

$$[A]^r = \{X \subset A : |X| = r\}.$$

For cardinals a, c, d, r the *partition relation*

$$a \to (c)^r_d$$

means that whenever A and D are sets; $|A| = a$; $|D| = d$; $[A]^r = \cup (\lambda \in D) I_\lambda$ then there is a set $A' \in [A]^c$ and an element λ of D such that $[A']^r \subset I_\lambda$. For every cardinal m we put $m^+ = \min\{n : n > m\}$. If m has the form p^+ then we put $m^- = p$, and in all other cases $m^- = m$. By $\omega(m)$ we denote the least ordinal λ such that $|\lambda| = m$. For every ordinal α, put $\underline{\alpha} = \{\lambda : \lambda < \alpha\}$, and for every cardinal m put $\underline{m} = \underline{\omega(m)}$. If $m \geqq \aleph_0$, then the symbol cf(m) denotes the least cardinal c such that $m = \Sigma(\gamma \in \underline{c}) m_\gamma$ for some cardinals $m_\gamma < m$. The function cf is the cofinality function. Instead of $(\mathrm{cf}(m))^+$ we write $\mathrm{cf}^+(m)$, and similarly

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in other cases. For objects x, y the symbol $\{x, y\}_{\neq}$ denotes the set $\{x, y\}$ and at the same time expresses the condition that $x \neq y$. If d is a cardinal then the symbol $(A_\gamma : \gamma \in N)_d$ denotes the system $(A_\gamma : \gamma \in N)$ and expresses the condition that $|A_\mu \cap A_\gamma| = d$ for $\{\mu, \gamma\}_{\neq} \subset N$. Symbols like $(A_\gamma : \gamma \in N)_{<d}$ have the obvious meaning.

We use the obliterator ^ ; its effect consists in deleting from a well-ordered sequence the element above which it is placed. Other uses of ^ will be self-explanatory. If $x = (x_0, \cdots, \hat{x}_k)$ and $y = (y_0, \cdots, \hat{y}_k)$ are sequences of the same length k, and $x \neq y$, then there is an ordinal $i < k$, denoted by $x \circ y$, such that $x_j = y_j$ $(j < i)$; $x_i \neq y_i$. We shall occasionally use that

$$\{j < k : (x_0, \cdots, \hat{x}_j) = (y_0, \cdots, \hat{y}_j)\} = \underline{x \circ y + 1},$$

$$\{j < k : (x_0, \cdots, x_j) = (y_0, \cdots, y_j)\} = \underline{x \circ y}.$$

If $(S, \prec)$ is an ordered set and n is an ordinal; $x_0, \cdots, \hat{x}_n \in S$, then the symbol $\{x_0, \cdots, \hat{x}_n\}_{<}$ denotes the set $\{x_0, \cdots, \hat{x}_n\}$ and expresses the condition that $x_\mu \prec x_\gamma$ for $\mu < \gamma < n$. A set $A \subset S$ is said to be cofinal in $(S, \prec)$ if $\bigcup (x \in A) \{y \in S : y \preccurlyeq x\} = S$. It is well known that if $a \geqq \aleph_0$ and $\mathrm{tp}(S, \prec) = \omega(a)$, then $\mathrm{cf}(a)$ is the minimum of the cardinals of the sets A which are cofinal in $(S, \prec)$. Finally, a symbol such as $((A_\gamma)_{\gamma \in N}, B)$ denotes the family $(D_\lambda : \lambda \in L)$, where $L = N \cup \{\rho\}$; $\rho \notin N$; $D_\lambda = A_\lambda$ for $\lambda \in N$, and $D_\rho = B$.

3.

THEOREM 1. *Let m, n be cardinals; $m \geqq \aleph_0$; $n < m$. Let $\mathscr{F} = (A_\gamma : \gamma \in N)_{<n} \in \Omega(m^+, m)$.*

(i) *If $m^n = m$ then the system $\mathscr{F}$ has a st $\Delta(m^+)$-subsystem;*

(ii) *If $m^n > m$ and* GCH *holds, then $\mathscr{F}$ has a st $\Delta(p)$-subsystem for every $p < m$;*

(iii) *the proposition* (ii) *becomes false if the hypothesis $p < m$ is replaced by $p \leqq m$.*

REMARKS. (a) A. Hajnal made valuable contributions towards proving Theorem 1.

(b) It is well known that, for every $m \geqq \aleph_0$, the relation $m^n = m$ holds if and only if $1 \leqq n < \mathrm{cf}(m)$ (assuming GCH).

4. Discretization sequences

Let $\mathscr{F} = (A_\gamma : \gamma \in N)$ be a given system. A *discretization sequence* (*d-sequence*) of $\mathscr{F}$ is any sequence $(N_0, \cdots, \hat{N}_k)$ such that $k = \omega(|N|^+)$ and, for each $\lambda < k$, the set N_λ is maximal with the properties

$$N_\lambda \subset N - N_{\underline{\lambda}};\ (A_\gamma - A_{N_{\underline{\lambda}}} : \gamma \in N_\lambda)_0.$$

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Thus N_0 is maximal such that $N_0 \subset N$; $(A_\gamma : \gamma \in N_0)_0$. Next,

$$N_1 \text{ is maximal such that } N_1 \subset N - N_0;\ (A_\gamma - A_{N_0} : \gamma \in N_1)_0;$$

$$N_2 \text{ is maximal such that } N_2 \subset N - (N_0 \cup N_1);\ (A_\gamma - A_{N_0} \cup {}_{N_1} : \gamma \in N_2)_0,$$

and so on. Let us put $A_{N\underline{\lambda}} = S_\lambda$ for every ordinal $\lambda < k$, and $A_{N_{\underline{p}}} = S_p$ for every cardinal $p < |k|$.

LEMMA 1. *Let* $(N_0, \cdots, \hat{N}_k)$ *be a d-sequence of* $(A_\gamma : \gamma \in N)$.

(3) *There is* $k_0 < k$ *such that* $\{\lambda < k : N_\lambda \neq \emptyset\} = \underline{k}_0$;

(4) *if* $\lambda < k$; $\{\mu, \gamma\}_{\neq} \subset N_\lambda$, *then* $A_\mu \cap A_\gamma \subset S_\lambda$;

(5) *if* $\lambda < k$; $\mu \in N - N_{\underline{\lambda+1}}$, *then* $A_{N_\lambda} \cap A_\mu \not\subset S_\lambda$;

(6) *if* $\lambda < k$; $\mu \in N - N_{\underline{\lambda}}$, *then* $|S_\lambda \cap A_\mu| \geqq |\lambda|$.

PROOF OF (3). Let $\lambda < \mu < k$; $N_\lambda = \emptyset$. Then, by definition of N_μ, we have $N_\mu = \emptyset$. Also, $|k| > |N|$.

PROOF OF (4). $A_\mu \cap A_\gamma - S_\lambda = (A_\mu - S_\lambda) \cap (A_\gamma - S_\lambda) = \emptyset$ by definition of N_λ.

PROOF OF (5). The relation $(A_\gamma - S_\lambda : \gamma \in N_\lambda \cup \{\mu\})_0$ is false by the maximality of N_λ. Hence there is $\gamma \in N_\lambda$ such that $(A_\mu - S_\lambda) \cap (A_\gamma - S_\lambda) \neq \emptyset$. Then $A_\mu \cap$ $\cap A_\gamma \not\subset S_\lambda$; $A_\mu \cap A_{N\lambda} \supset A_\mu \cap A_\gamma \not\subset S_\lambda$.

PROOF OF (6). Let $\kappa < \lambda$. Then $\mu \in N - N_{\underline{\lambda}} \subset N - N_{\underline{\kappa+1}}$ and, by (5), there is $x_\kappa \in A_{N_\kappa} \cap A_\mu - S_\kappa$. If $\kappa' < \kappa$ then $x_\kappa \in A_N - A_{N_{\kappa'}} \subset A_N - \{x_{\kappa'}\}$. Hence $|S_\lambda \cap A_\mu| \geqq |\{x_0, \cdots, \hat{x}_\lambda\}_{\neq}| = |\lambda|$. This proves Lemma 1.

PROOF OF THEOREM 1.

Proof of (i). Let $(N_0, \cdots, \hat{N}_k)$ be a d-sequence of $\mathscr{F}$. Then $k = \omega(m^{++})$.

CASE 1. There is $\kappa \in \underline{n}$ with $|N_\kappa| = m^+$. Then there is $\kappa_0 = \min\{\kappa \in n : |N_\kappa| = m^+\}$. Then $|S_{\kappa_0}| \leqq nmm = m$. Put $P = \{\gamma \in N_{\kappa_0} : |A_\gamma \cap S_{\kappa_0}| \geqq n\}$; $Q = N_{\kappa_0} - P$.

CASE 1a. $|P| = m^+$. Then, for $\gamma \in P$, there is $B_\gamma \in [A_\gamma \cap S_{\kappa_0}]^n$. Then $|\{B_\gamma : \gamma \in P\}| \leqq |[S_{\kappa_0}]^n| \leqq m^n = m < |P|$, and there is $\{\mu, \gamma\}_{\neq} \subset P$ such that $B_\mu = B_\gamma$. Then $|A_\mu \cap A_\gamma| \geqq |B_\mu \cap B_\gamma| = |B_\mu| = n > |A_\mu \cap A_\gamma|$ which is a contradiction.

CASE 1b. $|P| \leqq m$. Then $|Q| = m^+$; $|A_\gamma \cap S_{\kappa_0}| < n\ (\gamma \in Q)$. Since $|[S_{\kappa_0}]^{<n}| \leqq \Sigma(t < n)m^t \leqq nm^n = m$, there is $D \in [S_{\kappa_0}]^{<n}$ and $Q' \in [Q]^{m+}$ such that $A_\gamma \cap S_{\kappa_0} = D$ for all $\gamma \in Q'$. Then, by Lemma 1(4), $A_\mu \cap A_\gamma = D$ for $\{\mu, \gamma\}_{\neq} \subset Q'$ and so

$$(A_\gamma : \gamma \in Q') \in \operatorname{st} \Delta(m^+).$$

CASE 2. $|N_\kappa| \leqq m$ $(\kappa \in n)$. Then $|N_{\underline{n}}| \leqq nm = m$; $|N - N_{\underline{n}}| = m^+$. By Lemma 1(6), $|A_\gamma \cap S_n| \geqq n$ $(\gamma \in N - N_{\underline{n}})$. Choose $B_\gamma \in [A_\gamma \cap S_n]^n$ for $\gamma \in N - N_{\underline{n}}$. Then

$$|\{B_\gamma : \gamma \in N - N_{\underline{n}}\}| \leqq |[S_n]^n| \leqq (mm)^n = m < |N - N_{\underline{n}}|,$$

and there is $\{\mu, \gamma\}_{\neq} \subset N - N_{\underline{n}}$ such that $B_\mu = B_\gamma$. Then

$$|A_\mu \cap A_\gamma| \geqq |B_\mu \cap B_\gamma| = |B_\mu| = n > |A_\mu \cap A_\gamma|$$

which is a contradiction. This proves (i).

Before proving (ii) we establish a lemma.

LEMMA 2. *Let*

$$n < m \geqq \aleph_0;\; m^n > m;\; |S| = m;\; |N| = m^+;$$
$$X_\gamma \in [S]^m \;(\gamma \in N).$$

Assume GCH. Then there is $\{\mu, \gamma\}_{\neq} \subset N$ *such that* $|X_\mu \cap X_\gamma| > n$.

PROOF OF LEMMA 2. $n \geqq \mathrm{cf}(m)$. There is a respresentation $S = T_0 \cup \cdots \cup \hat{T}_t$ such that $t = \omega(\mathrm{cf}(m))$; $|T_\lambda| = m_\lambda < m$ $(\lambda < t)$. Let $\gamma \in N$. Then there is $\lambda_\gamma < t$ such that $|X_\gamma \cap T_{\lambda_\gamma}| > n$. For otherwise we obtain the contradiction

$$m = |X_\gamma| \leqq \Sigma(\lambda < t)|X_\gamma \cap T_\lambda| \leqq |t| n < m.$$

Now there is $M \in [N]^{m^+}$ and λ' such that $\lambda_\gamma = \lambda'$ $(\gamma \in M)$. Then

$$|X_\gamma \cap T_{\lambda'}| > n \;(\gamma \in M).$$

Since $|[T_{\lambda'}]^{>n}| \leqq 2^{m_{\lambda'}} < m^+$, there is $\{\mu, \gamma\}_{\neq} \subset M$ with $X_\mu \cap T_{\lambda'} = X_\gamma \cap T_{\lambda'}$. Then $|X_\mu \cap X_\gamma| \geqq |X_\mu \cap X_\gamma \cap T_{\lambda'}| = |X_\mu \cap T_{\lambda'}| > n$.

PROOF OF THEOREM 1 (ii). Let $(N_0, \cdots, \hat{N}_k)$ be a d-sequence of $(A_\gamma : \gamma \in N)$. Then $k = \omega(m^{++})$. Let S_λ and S_p have their previous meaning.

CASE 1. $|N_{\underline{m}}| \leqq m$. Then $|N - N_{\underline{m}}| = m^+$; $|S_m| \leqq m$. By Lemma 1(6), $|S_m \cap A_\gamma| \geqq m$ $(\gamma \in N - N_{\underline{m}})$. By Lemma 2, there is $\{\mu, \gamma\}_{\neq} \subset N - N_{\underline{m}}$ such that

$$|A_\mu \cap A_\gamma| \geqq |(S_m \cap A_\mu) \cap (S_m \cap A_\gamma)| > n > |A_\mu \cap A_\gamma|$$

which is false.

CASE 2. $|N_{\underline{m}}| = m^+$. Then there is $\lambda_0 = \min\{\lambda \in \underline{m} : |N_\lambda| = m^+\}$. Then

$$|A_\gamma \cap S_{\lambda_0}| \leqq |S_{\lambda_0}| \leqq m \;(\gamma \in N).$$

CASE 2*a*. There is $M \in [N_{\lambda_0}]^{m^+}$ such that $|A_\gamma \cap S_{\lambda_0}| = m$ $(\gamma \in M)$. Then, by Lemma 2, there is $\{\mu, \gamma\}_{\neq} \subset M$ such that

$$\left|(A_\mu \cap S_{\lambda_0}) \cap (A_\gamma \cap S_{\lambda_0})\right| > n > \left|A_\mu \cap A_\gamma\right|.$$

This is a contradiction.

CASE 2*b*. There is $M \in [N_{\lambda_0}]^{m^+}$ such that $|A_\gamma \cap S_{\lambda_0}| < m \ (\gamma \in M)$.

Then there is $M' \in [M]^{m^+}$ such that the cardinal $|A_\gamma \cap S_{\lambda_0}|$ is constant for $\gamma \in M'$, say $|A_\gamma \cap S_{\lambda_0}| = q \ (\gamma \in M')$. There are sets X_γ, B_γ such that $((X_\gamma)_{\gamma \in M'}, A_N)_0$ and $|B_\gamma| = p + q = p_0$, say $(\gamma \in M')$, where $B_\gamma = (A_\gamma \cap S_{\lambda_0}) \cup X_\gamma \ (\gamma \in M')$. Then $(B_\gamma\colon \gamma \in M') \in \Omega(\geqq p_0^{++}, p_0)$, and by [1], Theorem I, there is $M'' \subset M'$ such that $(B_\gamma\colon \gamma \in M'') \in \mathrm{st}\ \Delta(p_0^{++})$. Then $(A_\gamma \cap S_{\lambda_0}\colon \gamma \in M'') \in \mathrm{st}\ \Delta(p_0^{++})$ and, by Lemma 1, $(A_\gamma\colon \gamma \in M'') \in \mathrm{st}\ \Delta(p_0^{++})$. This proves Theorem 1 (ii).

PROOF OF THEOREM 1 (iii). It suffices to find a system

$$(A_\gamma\colon \gamma \in N)_{<\mathrm{cf}(m)} \in \Omega(m^+, m)$$

which has no st $\Delta(m)$-subsystem. Put $k = \omega(\mathrm{cf}(m))$. There are cardinals m_λ such that $m_0, \cdots, \hat{m}_k < m = m_0 + \cdots + \hat{m}_k$. Put

$$N = \{\gamma = (\gamma_0, \cdots, \hat{\gamma}_k)\colon \gamma_\lambda \in \underline{m}_\lambda (\lambda < k)\},$$

$$B_\gamma = \{(\gamma_0, \cdots, \hat{\gamma}_\lambda)\colon \lambda < k\} \ (\gamma = (\gamma_0, \cdots, \hat{\gamma}_k) \in N).$$

Then $(B_\gamma\colon \gamma \in N) \in \Omega(\Pi\, m_\lambda, |k|)$. We have $\Pi\, m_\lambda = m^+$; $|k| = \mathrm{cf}(m) < m$. Let $|X_\gamma| = m \ (\gamma \in N)$ and $((X_\gamma)_{\gamma \in N}, B_N)_0$, and put $A_\gamma = B_\gamma \cup X_\gamma \ (\gamma \in N)$. Then $(A_\gamma\colon \gamma \in N) \in \Omega(m^+, m)$. Let $\{\mu, \gamma\}_{\neq} \subset N$. Then there is $\lambda_0 = \mu \circ \gamma$, and we have

$$|A_\mu \cap A_\gamma| = |(B_\mu \cup X_\mu) \cap (B_\gamma \cup X_\gamma)| = |B_\mu \cap B_\gamma| = |\lambda_0| < |k| = \mathrm{cf}(m).$$

Now let $M \subset N$ and $(A_\gamma\colon \gamma \in M) \in \mathrm{st}\ \Delta$. Then $(B_\gamma\colon \gamma \in M) \in \mathrm{st}\ \Delta$. But then there is $\lambda_1 < k$ such that $\mu \circ \gamma = \lambda_1$ and $B_\mu \cap B_\gamma = \{(\rho_0, \cdots, \hat{\rho}_\lambda)\colon \lambda \leqq \lambda_1\}$ for all $\{\mu, \gamma\}_{\neq} \subset M$. Here $\rho_\lambda \in \underline{m}_\lambda (\lambda < \lambda_1)$, and $\rho_0, \cdots, \hat{\rho}_{\lambda_1}$ are independent of μ, γ. Therefore

$$|M| = |\{\gamma_{\lambda_1}\colon (\gamma_0, \cdots, \hat{\gamma}_k) \in M\}| \leqq m_{\lambda_1} < m,$$

and the proof of Theorem 1 is completed.

5. Some special Theorems

THEOREM 2. *Let* $(A_\gamma\colon \gamma \in N) \in \mathrm{wk}\ \Delta$. *Assume that*

(i) $|A_\gamma| \leqq n < \aleph_0$ *for* $\gamma \in N$,

(ii) $|A_\mu \cap A_\gamma| = k$ *for* $\{\mu, \gamma\}_{\neq} \subset N$,

(iii) $|N| > 1 + n\binom{n}{k}$.

Then $(A_\gamma\colon \gamma \in N) \in \mathrm{st}\ \Delta$.

PROOF. Let $\gamma_0 \in N$. By (i) and (ii),

$$|\{A_\gamma \cap A_{\gamma_0}\colon \gamma \in N - \{\gamma_0\}\}| \leqq \binom{n}{k}.$$

Hence, by (iii), there are sets M, D with $M \in [N - \{\gamma_0\}]^{n+1}$ and $D \in [A_{\gamma_0}]^k$ such that $A_\mu \cap A_{\gamma_0} = D$ for $\mu \in M$.

CASE 1. There is $\gamma_1 \in N - \{\gamma_0\}$ with $D \not\subset A_{\gamma_1}$. Then, for every $\mu \in M$, we have $A_\mu \cap A_{\gamma_1} \neq D$, and there is $x_\mu \in A_\mu \cap A_{\gamma_1} - D$. Then

$$\left|\{x_\mu : \mu \in M\}\right| \leqq \left|A_{\gamma_1}\right| \leqq n < |M|,$$

and there is $\{\rho, \sigma\}_{\neq} \subset M$ with $x_\rho = x_\sigma$. Then $x_\rho \in A_\rho \cap A_\sigma = D$ which is a contradiction.

CASE 2. $D \subset A_\gamma$ for all $\gamma \in N - \{\gamma_0\}$. Then $A_\mu \cap A_\gamma = D$ for $\{\mu, \gamma\}_{\neq} \subset N$ and the theorem follows.

Definitions: $(A_\gamma : \gamma \in N)$ is called a system without repetition if $A_\mu \neq A_\gamma$ for $\{\mu, \gamma\}_{\neq} \subset N$. For $n < \aleph_0$, denote by $g(n)$ the largest integer such that there exists a $(g(n),n)$-system without repetition which has no wk $\Delta(3)$-subsystem. Let $h(n)$ be defined similarly but with repetitions allowed.

It is easy to see that $g(1) = 1$; $g(2) = 5$; $g(3) \geqq 10$. D. Hanson proved that $g(3) = 10$.

THEOREM 3. *For all n with* $0 < n < \aleph_0$,

$$\text{(i) } h(n) = 2g(n), \qquad \text{(ii) } g(n+1) \geqq 2g(n).$$

COROLLARY. $g(n) \geqq 5.2^{n-2}$ *for* $n \geqq 2$.

PROOF OF (i). If $(A_1, A_2, \cdots, A_x)$ is a $(g(n), n)$-system without repetition which has no wk $\Delta(3)$-subsystem, then $(A_1, \cdots, A_x, A_1, \cdots, A_x)$ is a $(2g(n), n)$-system, with repetition, and again without wk $\Delta(3)$-subsystem. Hence $h(n) \geqq 2g(n)$. If, for some n, we have $h(n) > 2g(n)$ then there is a $(> 2g(n), n)$-system without wk $\Delta(3)$-subsystem. Such a system contains at least $g(n) + 1$ distinct members, and these form a system whose existence contradicts the definition of $g(n)$. Hence (i).

PROOF OF (ii). There is a $(g(n), n)$-system $(A_\gamma : \gamma \in N)$ without repetition and without wk $\Delta(3)$-subsystem. Let $x_{\gamma\lambda}$ be any $2g(n)$ distinct objects, for $\gamma \in N$ and $\lambda \in \underline{2}$ which do not belong to A_N. Then it is easily verified that

$$(A_\gamma \cup \{x_{\gamma\lambda}\} : \gamma \in N;\ \lambda \in \underline{2})$$

is a $(2g(n), n+1)$-system without repetition and without wk $\Delta(3)$-subsystem. This proves (ii).

THEOREM 4. *Let* $a > 0$ *and* $1 \leqq n \leqq \aleph_0$. *Then there is an* (a^n, n)*-system* $(A_x : x \in X)_{<n}$ *which has no* wk $\Delta(a^+)$*-subsystem.*

PROOF. Put $X = \{x = (x_0, \cdots, \hat{x}_n) : x_0, \cdots, \hat{x}_n \in \underline{a}\}$;

$$A_x = \{(x_0, \cdots, x_\gamma) : \gamma \in \underline{n}\} \quad (x \in X).$$

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Then $(A_x: x \in X)_{<n} \in \Omega(a^n, n)$. If $\{x, y\}_{\neq} \subset X$ then

$$\left|A_x \cap A_y\right| = \left|\{(x_0, \cdots, x_\gamma): \gamma < x \circ y\}\right| = x \circ y < n.$$

Let $X' \subset X$ and $(A_x: x \in X') \in \text{wk}\,\Delta$. Then there is $m < n$ such that $x \circ y = m$ for $\{x, y\}_{\neq} \subset X'$, and hence $\left|X'\right| = \left|\{x_m: x \in X'\}\right| \leqq a$. The theorem follows.

THEOREM 5. *Let α be a non-zero ordinal, and put $d_\alpha = 2^{|\alpha|}$. Then there is a $(d_\alpha, \aleph_\alpha)$-system $(A_\gamma: \gamma \in N)_{<\aleph_\alpha}$ without* wk $\Delta(3)$*-subsystem. In particular, we have $(d_\alpha, \aleph_\alpha) \nrightarrow$ wk $\Delta(3)$. If (i) $2^{|\beta|} \leqq \aleph_\alpha$ for $\beta < \alpha$, (ii) $\aleph_\alpha = \left|\alpha\right|$, then we can stipulate that, in addition, $\left|A_N\right| = \aleph_\alpha$.*

REMARK. The condition (i) is a weak version of the generalized continuum hypothesis, and the condition (ii) is equivalent to $\omega_\alpha = \alpha$ and is known to hold for some α.

PROOF. Let the letter λ denote elements of 2, and the letters β, γ, δ elements of $\underline{\alpha}$. Let $\left|X(\lambda_0, \cdots, \lambda_\beta)\right| = \aleph_{\beta+1}$ for all $\beta, \lambda_0, \cdots, \lambda_\beta$, and

$$(X(\lambda_0, \cdots, \lambda_\beta): \beta \in \underline{\alpha};\ \lambda_0, \cdots, \lambda_\beta \in \underline{2})_0.$$

Put $N = \{(\lambda_0, \cdots, \hat{\lambda}_\alpha): \lambda_0, \cdots, \hat{\lambda}_\alpha \in \underline{2}\}$ and $A(\lambda_0, \cdots, \hat{\lambda}_\alpha) = \bigcup (\beta < \alpha)\, X(\lambda_0, \cdots, \lambda_\beta)$ for $(\lambda_0, \cdots, \hat{\lambda}_\alpha) \in N$. Then $\left|N\right| = 2^{|\alpha|}$; $\left|A(\lambda_0, \cdots, \hat{\lambda}_\alpha)\right| = \sum (\beta < \alpha) \aleph_{\beta+1} = \aleph_\alpha$. Now suppose that $\{(\lambda_0, \cdots, \hat{\lambda}_\alpha), (\lambda'_0, \cdots, \hat{\lambda}'_\alpha), (\lambda''_0, \cdots, \hat{\lambda}''_\alpha)\}_{\neq} \subset N$. Put $\rho = \lambda \circ \lambda'$. Then $\left|A(\lambda) \cap A(\lambda')\right| = \sum (\gamma < \rho) \aleph_{\gamma+1} \leqq \aleph_\rho < \aleph_\alpha$. Put $\sigma = \lambda \circ \lambda''$; $\tau = \lambda' \circ \lambda''$. Change the notation, if necessary, so that $\rho \leqq \sigma \leqq \tau$. Then

$$\rho < \tau;\ \left| A(\lambda) \cap A(\lambda') \right| \leqq \aleph_\rho < \aleph_{\rho+1} \leqq \aleph_\tau = \sum (\gamma < \tau) \aleph_{\gamma+1} = \left|A(\lambda') \cap A(\lambda'')\right|.$$

Hence the $(2^{|\alpha|}, \aleph_\alpha)$-system $(A(\lambda): \lambda \in N)_{<\aleph_\alpha}$ has no wk $\Delta(3)$-subsystem. Now suppose that (i) and (ii) hold. Then

$$\left| \bigcup (\lambda \in N) A(\lambda)\right| = \left| \bigcup (\beta < \alpha; \lambda_0, \cdots, \lambda_\beta \in \underline{2})\, X(\lambda_0, \cdots, \lambda_\beta)\right|$$
$$= \sum (\beta < \alpha) 2^{|\beta+1|} \aleph_{\beta+1} = \aleph_\alpha;\ \left|N\right| = 2^{|\alpha|} = 2^{\aleph_\alpha}.$$

Hence, on changing the notation slightly, we obtain a $(2^{\aleph_\alpha}, \aleph_\alpha)$-system $(A_\mu: \mu \in M)$ without wk $\Delta(3)$-subsystem, and now $\left|A_M\right| = \aleph_\alpha$.

THEOREM 6. *Let $a = \aleph_\omega$. Then (i) assuming GCH, there is an $(a^+, \aleph_0)$-system $(A_\lambda: \lambda \in L)_{<\aleph_0}$ with $\left|A_L\right| \leqq a$; (ii) no $(a^+, \aleph_0)$-system $(B_\lambda: \lambda \in L)_{<\aleph_{..}}$ with $\left|B_L\right| \leqq a$ has a* wk $\Delta(a^+)$*-subsystem; (iii) if GCH holds then*

$$(\aleph_{\omega+1}, \aleph_0) \nrightarrow \text{wk}\,\Delta(\aleph_{\omega+1}).$$

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REMARKS. The result (i) is due to A. Tarski. For the convenience of the reader we give a proof. In Section 7, Case 1 b2a1, we prove $(\aleph_{\omega+1}, \aleph_0) \nrightarrow wk\Delta(\aleph_\omega)$, a relation which is stronger than (iii).

PROOF OF (i). Let L be the set of all sequences $\lambda = (l_0, \cdots, \hat{l}_\omega)$ such that $l_\gamma \in \underline{\omega}_\gamma$ for $\gamma < \omega$. Put $A_\lambda = \{(l_0, \cdots, \hat{l}_\mu) \colon \mu < \omega\}$ for $\lambda \in L$. Then $(A_\lambda \colon \lambda \in L) \in \Omega(a^+, \aleph_0)$;

$$|A_L| = |\{(l_0, \cdots, \hat{l}_\mu) \colon \mu < \omega;\ l_\gamma \in \underline{\omega}_\gamma \text{ for } \gamma < \mu\}| = \Sigma\,(\mu < \omega) \prod\,(\gamma < \mu) \aleph_\gamma = a.$$

If $\{\lambda, \lambda'\}_{\neq} \subset L$ then there is $\gamma_0 = \lambda \circ \lambda'$, and we have $|A_\lambda \cap A_{\lambda'}| = \gamma_0 + 1 < \aleph_0$.

PROOF OF (ii). Let the $(a^+, \aleph_0)$-system $(B_\lambda \colon \lambda \in L)_{<\aleph_0}$ satisfy $|B_L| \leqq a$. Let $(B_\lambda \colon \lambda \in L') \in \text{wk}\ \Delta$ for some $L' \in [L]^{a^+}$. Choose $\{\lambda', \lambda''\}_{\neq} \subset L'$. Then $|B_{\lambda'} \cap B_{\lambda''}| = p < \aleph_0$. Choose $D_\lambda \in [B_\lambda]^{p+1}$ for $\lambda \in L'$. Then $|\{D_\lambda \colon \lambda \in L'\}| \leqq |B_L| < |L'|$ and therefore there is $\{\rho, \sigma\}_{\neq} \subset L'$ such that $D_\rho = D_\sigma$. Then

$$p = |B_\rho \cap B_\sigma| \geqq |D_\rho| = p + 1$$

which is the required contradiction.

6. Some Lemmas

It is convenient to use the function $\psi(a) = |\{x \colon x \leqq a\}|$, where a ranges over cardinals. Thus, $\psi(\aleph_\alpha) = \aleph_0 + |\alpha|$.

Throughout the rest of this paper we use the following notation for two fixed cardinals:

$$a = \aleph_\alpha; \quad b = \aleph_\beta.$$

Furthermore, GCH is assumed without reference being made to this fact.

LEMMA 3. *Let* $a > \text{cf}(a)$. *Then* $(a, b) \nrightarrow \text{wk}\ \Delta(a)$.

PROOF. If $n = \omega(\text{cf}(a))$ then there are cardinals a_γ with

$$a_0, \cdots, \hat{a}_n < a = a_0 + \cdots + \hat{a}_n.$$

Choose sets B_γ with $|B_\gamma| = b\ (\gamma < n)$ and $(B_0, \cdots, \hat{B}_n)_0$, and put $D_{\gamma\lambda} = B_\gamma$ for $\gamma < n$ and $\lambda \in \underline{a}_\gamma$. Then $(D_{\gamma\lambda} \colon \gamma < n;\ \lambda \in \underline{a}_\gamma) \in \Omega(a, b)$. Let $D_\gamma \subset \underline{a}_\gamma\ (\gamma < n)$;

$$(D_{\gamma\lambda} \colon \gamma < n;\ \lambda \in D_\gamma) \in \text{wk}\ \Delta(c).$$

CASE 1. There is $\gamma_0 < n$ such that $|D_{\gamma_0}| \geqq 2$. Choose $\{\sigma, \tau\}_{\neq} \subset D_{\gamma_0}$. Then $|D_{\gamma_0\sigma} \cap D_{\gamma_0\tau}| = b > 0$. Hence $D_\gamma = \varnothing$ for $\gamma \in \underline{n} - \{\gamma_0\}$, and so

$$c = \Sigma\,(\gamma < n)|D_\gamma| = |D_{\gamma_0}| \leqq a_{\gamma_0} < a.$$

CASE 2. $|D_\gamma| < 2$ for $\gamma < n$. Then $\Sigma\,(\gamma < n)|D_\gamma| \leqq |n| = \text{cf}(a) < a$.

LEMMA 4. *Let* $b < \text{cf}(c)$. *Then* $(c^+, b) \rightarrow \text{st}\ \Delta(c^+)$.

PROOF. In [2], p. 471, the function $s(x, y)$ was defined for all cardinals x, y such that $x \geqq 2$; $y \geqq 3$; $x + y \geqq \aleph_0$, by putting

$$s(x, y) = \sup\{ \Sigma(\gamma \in \underline{x}) y_0 \cdots \hat{y}_\gamma : y_0, \cdots, \hat{y}_{\omega(x)} < y\}.$$

We have

$$s(b^+, c^+) = \Sigma(\gamma \in \underline{b}^+) c^{|\gamma|} \leqq \Sigma(\gamma \in \underline{b}^+) c = b^+ c = c \leqq s(b^+, c^+).$$

Here, the first inequality follows from $|\gamma| \leqq b < \mathrm{cf}(c)$, and the second inequality from $b > 0$. By [2], Theorem IV,

$$f_\Delta(b^+, c^+) = s^+(b^+, c^+),$$

and therefore

$$(s^+(b^+, c^+), \leqq b) \to \mathrm{st}\ \Delta(c^+); (c^+, \leqq b) \to \mathrm{st}\ \Delta(c^+);$$

$$(c^+, b) \to \mathrm{st}\ \Delta(c^+).$$

LEMMA 5. *Let* $a = a^- = \mathrm{cf}(a) > b$. *Then* $(a, b) \to \mathrm{st}\ \Delta(a)$.

PROOF. $s(b^+, a) \leqq \Sigma(\gamma \in \underline{b}^+) a^{|\gamma|} \leqq \Sigma(\gamma \in \underline{b}^+) a = b^+ a = a;$

$$s(b^+, a) \geqq \sup\{a_0 : a_0 < a\} = a.$$

Hence $s(b^+, a) = a$. We now prove $f_\Delta(b^+, a) = s(b^+, a)$. We want to apply [2] Theorem IV (*a*) (iii). To do this we must prove

(i) $\aleph_0 \leqq b^+ < \mathrm{cf}(a) \leqq a^- = a$;

(ii) if $\sup\{a_0^b : a_0 < a\} = d$ then $d = \mathrm{cf}(d) > a_1^b$ for $a_1 < a$.

Now, (i) is true. Also,

$$\sup\{a_0^b : a_0 < a\} \leqq \sup\{a_0^+ b^+ : a_0 < a\} \leqq a$$

$$\leqq \sup\{a_0^b : a_0 < a\}; \sup\{a_0^b : a_0 < a\} = a = \mathrm{cf}(a).$$

Finally, let $a_1 < a$. Then $a_1^b \leqq a_1^+ b^+ < a$. This proves (ii), and we have, by [2], $f_\Delta(b^+, a) = s(b^+, a) = a$; $(a, < b^+) \to \mathrm{st}\ \Delta(a)$; $(a, b) \to \mathrm{st}\ \Delta(a)$.

LEMMA 6. *Let* $a = \mathrm{cf}(a)$; $f(\mu, \gamma) \in \underline{2}$ for $\mu < \gamma \in \underline{a}^+$. *Then there is an* (a^+, a).-*system* $(F_\gamma : \gamma \in \underline{a}^+)$ *such that, for* $\mu < \gamma \in \underline{a}^+$,

$$\begin{aligned} |F_\mu \cap F_\gamma| &< a \quad \textit{if } f(\mu, \gamma) = 0 \\ &= a \quad \textit{if } f(\mu, \gamma) = 1. \end{aligned}$$

PROOF. 1. We begin by showing that, given any (a, a)-system $(A_\gamma : \gamma \in N)_{<a}$, there is a set T (called a ($< a$)-*transversal* of the system) such that

$$T \in [A_N]^a;\ 1 \leqq |T \cap A_\mu| < a\ (\mu \in N).$$

We may assume $N = \underline{a}$. Then there are elements x_γ, for $\gamma \in \underline{a}$, such that $x_\gamma \in A_\gamma - (A_{\underline{\gamma}} \cup \{x_0, \cdots, \hat{x}_\gamma\})\ (\gamma \in \underline{a})$. We may put $T = \{x_\gamma : \gamma \in \underline{a}\}_{\neq}$. For, let $\mu \in \underline{a}$. If $\xi \in T \cap A_{\underline{\mu}}$, then there is $\gamma \in \underline{a}$ such that $\xi = x_\gamma \in A_\gamma - A_{\underline{\gamma}}$. Also, $\xi \in A_\mu$. Hence $\mu \notin \underline{\gamma}$; $\mu \geqq \gamma$, so that $1 \leqq |T \cap A_\mu| \leqq |\{x_0, \cdots, x_\mu\}| = |\underline{\mu + 1}| < a$.

2. Choose a system $(S_{\alpha\beta} : \alpha \in \underline{a}^+ : \beta \in \underline{a})_0 \in \Omega(a^+, a)$. We now choose sets B_μ, for $\mu \in \underline{a}^+$, by the following procedure. Let $\mu_0 \in \underline{a}^+$, and suppose that $B_0, \cdots, \hat{B}_{\mu_0}$ have already been defined in such a way that

(*) $$\begin{cases} B_\mu \text{ is a } (< a)\text{-transversal of the family} \\ ((S_{\alpha\beta} : \alpha \leqq \mu;\ \beta \in \underline{a}),\ B_0, \cdots, \hat{B}_\mu)_{<a} \text{ for } \mu < \mu_0. \end{cases}$$

We show that

(**) $$((S_{\alpha\beta} : \alpha \leqq \mu_0;\ \beta \in \underline{a}),\ B_0, \cdots, \hat{B}_{\mu_0})_{<a}.$$

Let $\mu < \mu_0$. Then

$$B_{\underline{\mu}} \subset \bigcup (\alpha \leqq \mu;\ \beta \in \underline{a})\ S_{\alpha\beta} \cup B_{\underline{\mu}} = S_{\underline{\mu+1},\underline{a}} \cup B_{\underline{\mu}},\ \text{say}.$$

By induction over μ, we deduce that $B_\mu \subset S_{\underline{\mu+1},\underline{a}}\ (\mu < \mu_0)$.

(i) Let $\alpha \leqq \mu_0$; $\beta \in \underline{a}$; $\gamma < \mu_0$. If $\alpha \leqq \gamma$, then $|S_{\alpha\beta} \cap B_\gamma| < a$ by (*) with $\mu = \gamma$. If $\alpha > \gamma$, then $|S_{\alpha\beta} \cap B_\gamma| \leqq |S_{\alpha\beta} \cap S_{\underline{\gamma+1},\underline{a}}| = 0$ since $\alpha \notin \underline{\gamma + 1}$.

(ii) Let $\rho < \sigma < \mu_0$. Then $|B_\rho \cap B_\sigma| < a$ by (*) with $\mu = \sigma$. This proves (**). Now let B_{μ_0} be a $(< a)$-transversal of the family (**). Put $S_\alpha = \bigcup (\beta \in \underline{a}) S_{\alpha\beta}$ $(\alpha \in \underline{a}^+)$;

$$A_{\alpha\mu} = S_\alpha \cap B_\mu\ (\alpha \leqq \mu \in \underline{a}^+).$$

Then it follows, by induction on μ, that

$$B_\mu \subset \bigcup (\alpha \leqq \mu;\ \beta \in \underline{a})\ S_{\alpha\beta} = \bigcup (\alpha \leqq \mu) S_\alpha;$$

$B_\mu = \bigcup (\alpha \leqq \mu) S_\alpha \cap B_\mu = \bigcup (\alpha \leqq \mu) A_{\alpha\mu}\ (\mu \in \underline{a}^+)$. Since $|S_{\alpha\beta} \cap B_\mu| \geqq 1$ $(\alpha \leqq \mu \in \underline{a}^+;\ \beta \in \underline{a})$, we have $|A_{\alpha\mu}| = a$ $(\alpha \leqq \mu \in \underline{a}^+)$. Put $F_\gamma = S_\gamma \cup \bigcup (\mu < \gamma;\ f(\mu, \gamma) = 1) A_{\mu\gamma}$ $(\gamma \in \underline{a}^+)$. Then $S_\gamma \subset F_\gamma \subset S_{\underline{\gamma+1}}$ $(\gamma \in \underline{a}^+)$;

$$(F_\gamma : \gamma \in \underline{a}^+) \in \Omega(a^+, a).$$

Now let $\mu < \gamma \in \underline{a}^+$. If $f(\mu, \gamma) = 1$, then $A_{\mu\gamma} \subset F_\gamma$; $A_{\mu\gamma} \subset S_\mu \subset F_\mu$; $|F_\mu \cap F_\gamma| \geqq |A_{\mu\gamma}| = a$. Now suppose $f(\mu, \gamma) = 0$. Then $F_\mu \cap F_\gamma = (S_\mu \cup \bigcup (\alpha < \mu;\ f(\alpha, \mu) = 1) A_{\alpha\mu}) \cap (S_\gamma \cup \bigcup (\beta < \gamma;\ f(\beta, \gamma) = 1) A_{\beta\gamma})$. We note that $S_\mu \cap S_\gamma = \varnothing$; if

$f(\beta,\gamma) = 1$ then $\beta \neq \mu$ and hence $S_\mu \cap A_{\beta\gamma} \subset S_\mu \cap S_\beta = \emptyset$. If $\alpha < \mu$, then $A_{\alpha\mu} \cap S_\gamma \subset S_\alpha \cap S_\gamma = \emptyset$; if $\alpha \neq \beta$, then $A_{\alpha\mu} \cap A_{\beta\gamma} \subset S_\alpha \cap S_\beta = \emptyset$. All this shows that $F_\mu \cap F_\gamma \subset \bigcup(\alpha < \mu) A_{\alpha\mu} \cap A_{\alpha\gamma} \subset B_\mu \cap B_\gamma$; $|F_\mu \cap F_\gamma| \leqq |B_\mu \cap B_\gamma| < a$. This proves Lemma 6.

LEMMA 7. *Let* $a = \mathrm{cf}(a)$. *Then* $(a^+, a) \nrightarrow \mathrm{wk}\ \Delta(a^+)$.

PROOF. By [3], $a^+ \nrightarrow (a^+)^2_2$. Hence there is a function f: $[a^+]^2 \nrightarrow \underline{2}$ such that, whenever $M \subset \underline{a}^+$ and f is constant on $[M]^2$, then $|M| < a^+$. By Lemma 6, there are sets F_γ such that $|F_\gamma| = a$ for $\gamma \in \underline{a}^+$ and, for $\mu < \gamma \in \underline{a}^+$, $|F_\mu \cap F_\gamma| < a$ if $f(\mu,\gamma) = 0$; $|F_\mu \cap F_\gamma| = a$ if $f(\mu,\gamma) = 1$. Then the (a^+, a)-system $(F_\gamma : \gamma \in \underline{a}^+)$ has no wk $\Delta(a^+)$-subsystem.

LEMMA 8. *Let* $a \rightarrow (c)^2_{\psi(b)}$. *Then* $(a, b) \rightarrow \mathrm{wk}\ \Delta\ (c)$.

PROOF. Let $(A_\gamma : \gamma \in N) \in \Omega(a,b)$. Then

$$[N]^2 = \bigcup(b_0 \leqq b)\{\{\mu,\gamma\}_{\neq} \subset N : |A_\mu \cap A_\gamma| = b_0\}.$$

By Hypothesis there are M and b_0 such that $M \in [N]^c$; $b_0 \leqq b$; $|A_\mu \cap A_\gamma| = b_0$ for $\{\mu,\gamma\}_{\neq} \subset M$. Then

$$(A_\gamma : \gamma \in M)_{b_0} \in \mathrm{wk}\ \Delta(c).$$

LEMMA 9. *Let* $a > a^-$. *Then* $(a^+, a) \rightarrow \mathrm{wk}\ \Delta(a)$.

PROOF. $\psi(a) = \psi(a^-) \leqq a^- < a$. Hence, clearly, $a \rightarrow (a)^1_{\psi(a)}$ and therefore, by the "stepping-up lemma" of [3], $a^+ \rightarrow (a)^2_{\psi(a)}$. Now Lemma 8 yields $(a^+, a) \rightarrow$ wk $\Delta(a)$.

LEMMA 10. *Let* $(a, b) \nrightarrow \mathrm{wk}\ \Delta(c)$. *Then* $(a', b') \nrightarrow \mathrm{wk}\ \Delta(c')$ *if* $a \geqq a'$; $b \leqq b'$; $c \leqq c'$.

REMARK. This lemma will be applied without reference.

PROOF. There is an (a, b)-system $(A_\gamma : \gamma \in N)$ without wk $\Delta(c)$-subsystem. Choose sets B_γ such that $A_\gamma \subset B_\gamma$ and $|B_\gamma| = b'$ for $\gamma \in N$, and $((B_\gamma - A_\gamma)_{\gamma \in N}, A_N)_0$. Let $N' \in [N]^{a'}$. Then the (a', b')-system $(B_\gamma : \gamma \in N')$ has no wk $\Delta(c')$-subsystem.

LEMMA 11. $(\psi(b), b) \nrightarrow \mathrm{wk}\ \Delta(3)$.

PROOF. Put $N = \underline{\omega} \cup \{\omega_0, \cdots, \hat{\omega}_\beta\}$;

$$A_\gamma = \underline{\gamma} \cup \{\xi : \omega_\beta \gamma \leqq \xi < \omega_\beta(\gamma + 1)\}\ (\gamma \in N).$$

Then the $(\psi(b), b)$-system $(A_\gamma : \gamma \in N)$ has no wk$\Delta(3)$-subsystem. For if $\{\mu, \gamma, \lambda\}_< \subset N$ then

$$|A_\mu \cap A_\gamma| = |\mu| < |\gamma| = |A_\gamma \cap A_\lambda|.$$

LEMMA 12. *Let* $b = b^-$. *Then* $(b^+, b) \nrightarrow \text{wk}\,\Delta(b)$.

PROOF. Put $N = \{\gamma = (\gamma_0, \cdots, \hat{\gamma}_{\omega_\beta}) : \gamma_0, \cdots, \hat{\gamma}_{\omega_\beta} \in \underline{2}\}$;

$$A_\gamma = \{(\gamma_0, \cdots, \gamma_\lambda) : \lambda \in \underline{b}\} \ (\gamma \in N).$$

Then $(A_\gamma : \gamma \in N) \in \Omega(b^+, b)$. Assume that there is $M \in [N]^b$ such that $(A_\gamma : \gamma \in M)_p$ for some p. Let $\{\mu, \gamma\}_{\neq} \subset M$. Then $p = |A_\mu \cap A_\gamma| = |\mu \circ \gamma| < b$; $\mu \circ \gamma \in p^+$. Put $\sigma = \omega(p^+)$. Then $|\{(\gamma_0, \cdots, \hat{\gamma}_\sigma) : (\gamma_0, \cdots, \hat{\gamma}_{\omega_\beta}) \in M$ for some $\gamma_\sigma, \cdots, \hat{\gamma}_{\omega_\beta}\}| \leqq 2^{|\sigma|} = p^{++} < b = |M|$, and there is $\{\mu, \gamma\}_{\neq} \subset M$ such that $(\mu_0, \cdots, \hat{\mu}_\sigma) = (\gamma_0, \cdots, \hat{\gamma}_\sigma)$. On the other hand, if $\lambda = \mu \circ \gamma$ then $\lambda < \sigma$; $\mu_\lambda \neq \gamma_\lambda$, which is a contradiction.

LEMMA 13. *Let* $b = \psi(b)$. *Then* $(b^+, b) \nrightarrow \text{wk}\,\Delta(3)$.

PROOF. CASE 1. $\beta = 0$. The conclusion follows from the case $a = 2$; $n = \aleph_0$ of Theorem 4.

CASE 2. $\beta > 0$. For $\lambda < \beta$ and $\gamma_0, \cdots, \hat{\gamma}_\lambda \in \underline{2}$, choose a set $X(\gamma_0, \cdots, \hat{\gamma}_\lambda)$ with $|X(\gamma_0, \cdots, \hat{\gamma}_\lambda)| = \aleph_{\lambda+1}$, such that $(X(\gamma_0, \cdots, \hat{\gamma}_\lambda) : \lambda < \beta;\ \gamma_0, \cdots, \hat{\gamma}_\lambda \in \underline{2})_0$. Put $A_\gamma = \bigcup (\lambda < \beta)\, X(\gamma_0, \cdots, \hat{\gamma}_\lambda)$ for $\gamma = (\gamma_0, \cdots, \hat{\gamma}_\beta)$; $\gamma_0, \cdots, \hat{\gamma}_\beta \in \underline{2}$. Then $|A_\gamma| = \Sigma(\lambda < \beta)\ \aleph_{\lambda+1} = \aleph_\beta = b$. We have $|\{\gamma_0, \cdots, \hat{\gamma}_\beta) : \gamma_0, \cdots, \hat{\gamma}_\beta \in \underline{2}\}| = 2^{|\beta|} = |\beta|^+ = b^+$. Let $(\mu, \gamma, \rho)_{\neq}$ and $(A_\mu, A_\gamma, A_\rho) \in \text{wk}\,\Delta(3)$. Put $\mu \circ \gamma = \tau$.

We note that $\{\lambda : (\mu_0, \cdots, \hat{\mu}_\lambda) = (\gamma_0, \cdots, \hat{\gamma}_\lambda)\} = \underline{\tau + 1}$. Hence $|A_\mu \cap A_\gamma| = |\bigcup (\lambda < \tau + 1) X(\gamma_0, \cdots, \hat{\gamma}_\lambda)| = \Sigma(\lambda < \tau + 1) \aleph_{\lambda+1} = \aleph_{\tau+1} = \aleph_{\mu\circ\gamma+1}$. Therefore $\tau = \mu \circ \gamma = \mu \circ \rho = \gamma \circ \rho$, and $(\mu_\tau, \gamma_\tau, \rho_\tau)_{\neq}$ which is impossible. This proves Lemma 13.

LEMMA 14. *Let* $\text{cf}(d) = \aleph_0$. *Then* $(d^+, \aleph_0) \nrightarrow \text{wk}\,\Delta(d)$.

PROOF. There are cardinals d_λ such that $d_0, \cdots, \hat{d}_\omega < d = d_0 + \cdots + \hat{d}_\omega$. Put

$$X = \{x = (x_0, \cdots, \hat{x}_\omega) : x_\lambda \in \underline{d}_\lambda \ (\lambda < \omega)\};$$

$A_x = \{(x_0, \cdots, \hat{x}_\lambda) : \lambda < \omega\}$ $(x \in X)$. Then $(A_x : x \in X) \in \Omega(d^+, \aleph_0)$. Let $L \subset X$ and $(A_x : x \in L) \in \text{wk}\,\Delta$. Then there is $\sigma < \omega$ such that $|A_x \cap A_y| = \sigma + 1$; $x \circ y = \sigma$ for $\{x, y\}_{\neq} \subset L$. Then $|L| = |\{x_\sigma : x \in L\}| \leqq d_\sigma < d$ which proves Lemma 14.

LEMMA 15. *Let* $\text{cf}(d) = \aleph_\delta$. *Then* $(d^+, \aleph_{\omega_\delta}) \nrightarrow \text{wk}\,\Delta(d)$.

PROOF. There are cardinals d_λ such that $d_0, \cdots, \hat{d}_{\omega_\delta} < d = d_0 + \cdots + \hat{d}_{\omega_\delta}$. Let

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$X = \{x = (x_0, \cdots, \hat{x}_{\omega_\delta}) \colon x_\gamma \in \underline{d}_\gamma \ (\gamma < \omega_\delta)\}$. For $x \in X$ and $\lambda < \omega_\delta$, let $|B(x_0, \cdots, \hat{x}_\lambda)| = \aleph_{\lambda+1}$, and $(B(x_0, \cdots, \hat{x}_\lambda) \colon \lambda < \omega_\delta;\ x_\gamma \in \underline{d}_\gamma \ (\gamma < \lambda))_0$. Put

$$A_x = \bigcup (\lambda < \omega_\delta) B(x_0, \cdots, \hat{x}_\lambda)$$

for $x \in X$. Then $|X| = d_0 \cdots \hat{d}_{\omega_\delta} = d^+$; $|A_x| = \Sigma(\lambda < \omega_\delta)\aleph_{\lambda+1} = \aleph_{\omega_\delta}$, so that $(A_x \colon x \in X) \in \Omega(d^+, \aleph_{\omega_\delta})$. Let $L \subset X$ and $(A_x \colon x \in L) \in \text{wk}\,\Delta$. Then there is $\sigma < \omega_\delta$ such that $x \circ y = \sigma$ for $\{x, y\}_{\neq} \subset L$. Hence $|L| = |\{x_\sigma \colon \sigma \in L\}| \leqq d_\sigma < d$, which completes the proof.

Lemma 16. *Let* $0 < d = d^- < \aleph_{\omega_n}$. *Then* $\text{cf}(d) < \aleph_n$.

Proof. We have $d = \aleph_\delta$ for some $\delta < \omega_n$. Since $d = d^-$ we conclude that $d = \Sigma(\pi < \delta)\aleph_\pi$; $\text{cf}(d) \leqq |\delta| < \aleph_n$.

For the last two lemmas we need the following definitions: Consider a system $\mathscr{F} = (A_\gamma \colon \gamma \in N)$. We call $\mathscr{F}$ an $(a, b, \leqq d)$-system if $\mathscr{F} \in \Omega(a, b)$ and $(A_\gamma \colon \gamma \in N)_{\leqq d}$. An $(a, b, < d)$-system is defined similarly. For every set A and every cardinal d we put

$$\mathscr{F}(A, d) = \{\gamma \in N \colon |A \cap A_\gamma| = d\}.$$

Lemma 17. *Let* $\mathscr{F}$ *be an* $(a, b, \leqq d)$*-system;* $a = \text{cf}(a) > b^d$; $|A| = b$; $|\mathscr{F}(A, d)| = a$. *Then* $\mathscr{F}$ *has a* $\text{wk}\,\Delta(a)$*-subsystem.*

Proof. We have $|[A]^d| = b^d < a = \text{cf}(a)$. Hence there is an (a, b)-subsystem $\mathscr{F}' = (A_\gamma \colon \gamma \in N')$ of $\mathscr{F}$ and a set X such that $|X| = d$ and $A \cap A_\gamma = X \ (\gamma \in N')$. Then, for $\{\mu, \gamma\}_{\neq} \subset N'$, we have $d = |X| \leqq |A_\mu \cap A_\gamma| \leqq d$, and $\mathscr{F}'$ is a $\text{wk}\,\Delta(a)$-system.

Lemma 18. *Let* $\mathscr{F} = (A_\gamma \colon \gamma \in N)$ *be an* $(a, b, \leqq d)$*-system, such that*

$$|\mathscr{F}(A_\gamma, d)| < a$$

for every $\gamma \in N$. *Suppose that* $a = \text{cf}(a)$. *Then* $\mathscr{F}$ *has an* $(a, b, < d)$*-subsystem.*

Proof. Assume $N = \underline{a}$. We can construct inductively ordinals γ_ρ for $\rho \in \underline{a}$ such that, for each $\rho \in \underline{a}$, $\gamma_\rho \in (N - \bigcup (\sigma < \rho)\mathscr{F}(A_{\gamma_\sigma}, d)) - \{\gamma_0, \cdots, \hat{\gamma}_\rho\}$. Then $(A_{\gamma_\rho} \colon \rho \in \underline{a})$ is an $(a, b, < d)$-system.

7. Discussion of the *wk* Δ-relation

We consider two fixed infinite cardinals a, b, where

$$a = \aleph_\alpha;\ b = \aleph_\beta,$$

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and we shall determine all cardinals c such that the wk Δ-*relation*

(7) $$(a, b) \to \text{wk}\,\Delta(c)$$

is true. There is a least cardinal $\phi(a, b)$ in $3 \leqq \phi(a, b) \leqq a^+$ such that (7) holds if and only if $c < \phi(a, b)$. We shall determine $\phi(a, b)$. If $\phi(a, b) = 3$ then (7) only holds completely trivially, i.e. for $c \leqq 2$, whereas $\phi(a, b) = a^+$ means that (7) holds for all values of c which are at all admissible, which are the cardinals $c \leqq a$.

Our results show that, for all a, b.

$$\phi(a, b) \in \{3, a^-, a, a^+\}.$$

In our discussion we shall write ϕ instead of $\phi(a, b)$. We remind the reader that throughout this section we assume GCH.

Case 1. $a > b^+$.

Case 1a. $a > a^- > a^{--}$. We prove that $\phi = a^+$. We can write $a = a_0{}^{++}$, and then we have $a_0^{++} = a \geqq b^{++}$; $a_0 \geqq b$. By [2], Theorem I (ii), with a, b in [2] replaced by a_0^+, a_0 respectively, we have $(a_0^{++}, a_0) \to \text{st}\,\Delta(a_0^{++})$. Hence $(a, b) \to \text{st}\,\Delta(a)$.

Case 1b. $a > a^- = a^{--}$.

Case 1b1. $b < \text{cf}(a^-)$. Then $\phi = a^+$. Indeed, by Lemma 4, $(a, b) \to \text{st}\,\Delta(a)$.

Case 1b2. $b \geqq \text{cf}(a^-)$. Let $a_0 < a^-$. Put $a_1 = \max\{a_0, b\}$. Then $(a_1^{++}, a_1) \to \text{st}\,\Delta(a_1^{++})$ by [2]. Hence $(a, b) \to \text{st}\,\Delta(a_0)$ $(a_0 < a^-)$.

Case 1b2a. $\text{cf}(a^-) = \text{cf}^-(a^-)$.

Case 1b2a1. $\text{cf}(a^-) = \aleph_0$. Then $\phi = a^-$. For, by Lemma 14, $(a, \aleph_0) \nrightarrow \text{wk}\,\Delta(a^-)$ and therefore $(a, b) \nrightarrow \text{wk}\,\Delta(a^-)$.

Case 1b2a2. $\text{cf}(a^-) > \aleph_0$. Then $\phi = a^-$. For, we have, by Lemma 15, $(a, \text{cf}(a^-)) \nrightarrow \text{wk}\,\Delta(a^-)$.

To see this, put $\text{cf}(a^-) = \aleph_\delta$. Then δ is a positive limit ordinal; $\aleph_\delta = \text{cf}(\aleph_\delta)$. If $\delta < \omega_\delta$ then $\aleph_\delta = \Sigma(\delta_0 < \delta)\aleph_{\delta_0}$; $\text{cf}(\aleph_\delta) \leqq |\delta| < \aleph_\delta$, which is false. Hence $\delta = \omega_\delta$. By Lemma 15, with $d = a^-$, we have $(a, \aleph_{\omega_\delta}) \nrightarrow \text{wk}\,\Delta(a^-)$, i.e. $(a, \text{cf}(a^-)) \nrightarrow \text{wk}\,\Delta(a^-)$. This implies $(a, b) \nrightarrow \text{wk}\,\Delta(a^-)$.

Case 1b2b. $\text{cf}(a^-) > \text{cf}^-(a^-)$. Then $\text{cf}(a^-)$ has the form $\aleph_{\lambda+1}$.

Case 1b2b1. $\aleph_{\omega_{\lambda+1}} \leqq b$. Then $\phi = a^-$. For, by Lemma 15, $(a, \aleph_{\omega_{\lambda+1}}) \nrightarrow \text{wk}\,\Delta(a^-)$, which implies $(a, b) \nrightarrow \text{wk}\,\Delta(a^-)$.

CASE 1b2b2. $\aleph_{\omega_{\lambda+1}} > b$. We show that $\phi = a^+$. We use the notation $\mathscr{F}(A, d)$ introduced before the statement of Lemma 17. We assume that the (a, b)-system $\mathscr{F}$ has no wk $\Delta(a)$-subsystem, and we have to deduce a contradiction. Since $\mathscr{F}$ is an $(a, b, \leqq b)$-system, it follows that there is a least cardinal d such that $\mathscr{F}$ has an $(a, b, \leqq d)$-subsystem. We have $0 < d \leqq b$. We may assume that $\mathscr{F}$ itself is an $(a, b, \leqq d)$-system. Then $\mathscr{F}$ has no $(a, b, \leqq e)$-subsystem, for every $e < d$. Let $\mathscr{F} = (A_\gamma : \gamma \in N)_{\leqq d}$. Let $\gamma_0 \in N$ and $|\mathscr{F}(A_{\gamma_0}, d)| = a$. Since $b^d \leqq b^b = b^+ < a$, it follows from Lemma 17 that $\mathscr{F}$ has a wk $\Delta(a)$-subsystem, which is a contradiction. Hence $|\mathscr{F}(A_\gamma, d)| < a$ for $\gamma \in N$. Then, by Lemma 18, $\mathscr{F}$ has an $(a, b, < d)$-subsystem. We may assume that $\mathscr{F} = (A_\gamma : \gamma \in N)_{<d}$ is itself an $(a, b, < d)$-system. If $d = e^+$, then $\mathscr{F}$ is an $(a, b, \leqq e)$-system, which contradicts the minimality of d. Hence $0 < d = d^- \leqq b < \aleph_{\omega_{\lambda+1}}$ and, by Lemma 16, $\mathrm{cf}(d) < \aleph_{\lambda+1}$.

We shall now construct a modified d-sequence. There is a maximal set $N_0 \subset N$ such that $(A_\gamma : \gamma \in N_0)_0$. Then $0 < |N_0| < a$. Let $0 < \sigma \in \underline{a}$. Suppose that, for each $\rho < \sigma$, we have already defined a set $N_\rho \in [N]^{<a}$, where $N_\rho \neq \varnothing$, such that, putting $S_\rho = A_{N_\rho}$, we have $|A_\gamma \cap S_\rho| < d$ for $\gamma \in N_\rho$; $A_\mu \cap A_\gamma \subset S_\rho$ for $\{\mu, \gamma\}_{\neq} \subset N_\rho$. Suppose, furthermore, that, for each $\rho < \sigma$, the set N_ρ is maximal such that the above stated conditions hold, i.e.: if $\gamma \in N - N_\rho$, then either $A_\gamma \subset S_\rho$, or there is $\mu \in N_\rho - \{\gamma\}$ with $A_\mu \cap A_\gamma \not\subset S_\rho$. We shall now define N_σ, and in such a way that all these conditions hold for $\rho = \sigma$. Put $S_\sigma = A_{N_\sigma}$. Then $|S_\sigma| \leqq |\sigma| a^- b \underline{\sigma} = a^-$. Well-order S_σ by a relation $\prec$, so that $\mathrm{tp}(S_\sigma, \prec) \leqq \omega(a^-)$. Put $N^* = \{\gamma \in N : |A_\gamma \cap S_\sigma| \geqq d\}$. We now prove $|N^*| < a$. Assume $|N^*| = a$. For each $\gamma \in N^*$, denote by $g(\gamma)$ the initial section of $(A_\gamma \cap S_\sigma, \prec)$ of type $\omega(d)$. If $\{\mu, \gamma\}_{\neq} \subset N^*$ then, by $(A_\gamma : \gamma \in N)_{<d}$, we have $|A_\mu \cap A_\gamma| < d$, and hence $g(\mu) \neq g(\gamma)$. There is an initial section T of $(S_\sigma, \prec)$ such that $|T| < a^-$ and $|\{\gamma \in N^* : g(\gamma) \subset T\}| = a$. For: if $|S_\sigma| < a^-$ then we put $T = S_\sigma$. Now let $|S_\sigma| = a^-$. We have $\mathrm{cf}(d) < \aleph_{\lambda+1} = \mathrm{cf}(a^-)$. For each $\gamma \in N^*$, the set $(g(\gamma), \prec)$ has a cofinal subset of cardinal $\mathrm{cf}(d)$. This subset is not cofinal in $(S_\sigma, \prec)$. Hence $g(\gamma)$ is not cofinal in $(S_\sigma, \prec)$, and there is $x_\gamma \in S_\sigma$ such that $g(\gamma) \subset \{x \in S_\sigma : x \prec x_\gamma\}$. In view of $a = \mathrm{cf}(a)$, there is $x^* \in S_\sigma$ such that $|\{\gamma \in N^* : x_\gamma = x^*\}| = a$. Then we may put $T = \{x \in S_\sigma : x \prec x^*\}$. This completes the definition of T. Now we have $|[T]^d| \leqq 2^{|T|} \leqq a^-$. Hence there is $X \subset T$ such that $|\{\gamma \in N^* : g(\gamma) = X\}| = a$. But then $(A_\gamma : \gamma \in N^*;\ g(\gamma) = X)_{\geqq d}$, which contradicts the relation $(A_\gamma : \gamma \in N)_{<d}$.

We have thus proved $|N^*| < a$. Let $\gamma \in N - N^*$. If $A_\gamma \subset S_\sigma$ then we have $b = |A_\gamma| = |A_\gamma \cap S_\sigma| < d \leqq b$ which is false. Hence $\gamma \in N - N^*$ implies $A_\gamma \not\subset S_\sigma$. Let N_σ be maximal such that $N_\sigma \subset N - N^*$ and $(A_\gamma - S_\sigma : \gamma \in N_\sigma)_0$. Then $N_\sigma \neq \varnothing$ It follows that if $\gamma \in N_\sigma$ then $A_\gamma \not\subset S_\sigma$, and if $\{\mu, \gamma\}_{\neq} \subset N_\sigma$ then $A_\mu \cap A_\gamma \subset S_\sigma$. Also, if $\gamma \in N - N_\sigma$ and $|A_\gamma \cap S_\rho| < d$, then there is $\mu \in N_\sigma$ with $A_\mu \cap A_\gamma \not\subset S_\sigma$. In order to complete the inductive definition of $N_0, N_1, \cdots$ we must now show that $|N_\sigma| < a$. Assume that $|N_\sigma| = a$. Corresponding to every $\gamma \in N_\sigma$, there is $e_\gamma < d$ such that $|A_\gamma \cap S_\sigma| = e_\gamma$. Then there is $e < d$ such that $|\{\gamma \in N_\sigma : e_\gamma = e\}| = a$. For we have $|\{e_\gamma : \gamma \in N_\sigma\}| \leqq d \leqq b < a^-$. Put $N' = \{\gamma \in N_\sigma : |A_\gamma \cap S_\sigma| = e\}$, so that $|N'| = a$. If $\{\mu, \gamma\}_{\neq} \subset N'$, then $|A_\mu \cap A_\gamma| = |A_\mu \cap A_\gamma \cap S_\sigma| \leqq |A_\mu \cap S_\sigma| = e$. Hence $(A_\gamma : \gamma \in N')_{\leqq e} \in \Omega(a,b)$ which contradicts the minimum property of d. This proves $|N_\sigma| < a$, and the inductive definition of N_ρ for $\rho \in \underline{a}$ is accomplished. We have $b^+ < a$, and therefore we can choose $\gamma \in N_{\omega(b_+)}$. For each $\rho \in \underline{b}^+$ there is $\mu_\rho \in N_\rho$ such that $A_{\mu_\rho} \cap A_\gamma \not\subset S_\rho = A_{N_{\underline{\rho}}}$. We can choose $z_\rho \in A_{\mu_\rho} \cap A_\gamma - A_{N_{\underline{\rho}}}$. If $\tau < \rho$ then $z_\tau \in A_{\mu_\tau} \cap A_\gamma \subset A_{\mu_\tau} \subset A_{N_{\underline{\rho}}}$. Hence $z_\rho \neq z_\tau$ for $\tau < \rho \in \underline{b}^+$;

$$|A_\gamma| \geqq |\{z_\rho : \rho \in \underline{b}^+\}_{\neq}| = b^+ > b = |A_\gamma|,$$

which is the required contradiction.

CASE 1c. $a = a^-$.

CASE 1c1. $a = cf(a)$. Then $\phi = a^+$. For, by Lemma 5, $(a,b) \to st\,\Delta(a)$.

CASE 1c2. $a > \mathrm{cf}(a)$. Then $\phi = a$. For, by Lemma 3, $(a,b) \nrightarrow \mathrm{wk}\,\Delta(a)$. Let $a_0 < a$ and put $a_1 = \max\{a_0, b\}$. Then, by [2], $(a_1^{++}, a_1) \to \mathrm{st}\,\Delta(a_1^{++})$. Hence $(a,b) \to \mathrm{st}\,\Delta(a_0)$ $(a_0 < a)$.

CASE 2. $a = b^+$.

CASE 2a. $b = |\beta|$. Then $\phi = 3$. For, by Theorem 5, $(2^{|\beta|}, b) \nrightarrow \mathrm{wk}\,\Delta(3)$. Hence $(a,b) \nrightarrow \mathrm{wk}\,\Delta(3)$.

CASE 2b. $b > |\beta|$.

CASE 2b1. $b > b^-$. Then $\phi = a$. For, by Lemma 7, $(a,b) \nrightarrow \mathrm{wk}\,\Delta(a)$. Also, by Lemma 9, $(a,b) \to \mathrm{wk}\,\Delta(b)$.

CASE 2b2. $b = b^-$. Then $\phi = a^-$. For, by Lemma 12, $(a,b) \nrightarrow \mathrm{wk}\,\Delta(b)$. Now, let $b_0 < b$. Then, by [3], $b \to (b_0)^2_{\psi(b)}$, and Lemma 8 gives $(b,b) \to \mathrm{wk}\,\Delta(b_0)$. Hence $(a,b) \to \mathrm{wk}\,\Delta(b_0)$ $(b_0 < b)$.

CASE 3. $a = b$.

CASE 3a. $b = |\beta|$. Then $\phi = 3$. For, by Lemma 11, $(a,b) \nrightarrow \mathrm{wk}\,\Delta(3)$.

CASE 3b. $b > |\beta|$.

CASE 3b1. $b > b^-$. If $b^- = \mathrm{cf}(b^-)$ then, by Lemma 7, $(b, b^-) \nrightarrow \mathrm{wk}\,\Delta(b)$, and if $b^- > \mathrm{cf}(b^-)$ then, by Lemma 12, $(b, b^-) \nrightarrow \mathrm{wk}\,\Delta(b^-)$. Thus, in either case, $(a, b) \nrightarrow \mathrm{wk}\,\Delta(b)$.

CASE 3b1a. $b^- > b^{--}$. Then $\phi = a$. For we have $\beta = \beta_0 + 1 = \beta_1 + 2$ for some β_0, β_1; $\psi(b) = \aleph_0 + |\beta_1|$; $\aleph_{\beta_1+1} \to (\aleph_{\beta_1+1})^1_{\psi(b)}$ and, by [3], $\aleph_{\beta_1+2} \to (\aleph_{\beta_1+1})^2_{\psi(b)}$. Now Lemma 8 gives $(a,b) \to \mathrm{wk}\,\Delta(b^-)$.

CASE 3b1b. $b^- = b^{--}$. Then, by Lemma 12, $(b, b^-) \nrightarrow \mathrm{wk}\,\Delta(b^-)$ and hence $(a,b) \nrightarrow \mathrm{wk}\,\Delta(b^-)$.

Case 3b1b1. $\psi(b^-) = b^-$. Then $\phi = 3$. For, by Lemma 13, $(b, b^-) \nrightarrow \text{wk}\,\Delta(3)$. Hence $(a, b) \nrightarrow \text{wk}\,\Delta(3)$.

Case 3b1b2. $\psi(b^-) < b^-$. Then $\phi = a^-$. For, let $b_0 < b^-$. Then $b \to (b_0)^2_{\psi(b)}$ and, by Lemma 8,

$$(a, b) \to \text{wk}\,\Delta(b_0)\ (b_0 < b^-).$$

Case 3b2. $b = b^-$. Then $\phi = a$. For, by Lemma 12, $(b^+, b) \nrightarrow \text{wk}\,\Delta(b)$, and hence $(a, b) \nrightarrow \text{wk}\,\Delta(b)$. Let $b_0 < b$. Then $b \to (b_0)^2_{\psi(b)}$ and, by Lemma 8,

$$(a, b) \to \text{wk}\,\Delta(b_0)\ (b_0 < b).$$

Case 4. $a < b$.

Case 4a. $b = |\beta|$. Then $\phi = 3$. For, by Lemma 11, $(\psi(b), b) \nrightarrow \text{wk}\,\Delta(3)$ and hence $(a, b) \nrightarrow \text{wk}\,\Delta(3)$.

Case 4b. $b > |\beta|$.

Case 4b1. $a \leqq 2^{|\beta|}$. Then $\phi = 3$. For, by Theorem 5, $(2^{|\beta|}, b) \nrightarrow \text{wk}\,\Delta(3)$ and therefore $(a, b) \nrightarrow \text{wk}\,\Delta(3)$.

Case 4b2. $a > 2^{\aleph_0 + |\beta|}$. Then $|\beta| < 2^{|\beta|} < a$.

Case 4b2a. $a = a^-$. Then $\phi = a$. For, by Lemma 12, $(a^+, a) \to \text{wk}\,\Delta(a)$, and therefore $(a, b) \nrightarrow \text{wk}\,\Delta(a)$. Let $a_0 < a$. Then $a \to (a_0)^2_{\aleph_0 + |\beta|}$, and Lemma 8 gives $(a, b) \to \text{wk}\,\Delta(a_0)$ $(a_0 < a)$.

Case 4b2b. $a > a^-$.

Case 4b2b1. $a^- > a^{--}$. Then $\phi = a$. For: $|\beta| < 2^{|\beta|} < a$; $a^- \nrightarrow (a^-)^1_{\aleph_0 + |\beta|}$; $a \to (a^-)^2_{\psi(b)}$; $(a,b) \to \text{wk}\Delta(a^-)$. By Lemma 7, $(a, a^-) \nrightarrow \text{wk}\,\Delta(a)$. Since $a^- < a < b$, we deduce $(a, b) \nrightarrow \text{wk}\,\Delta(a)$.

Case 4b2b2. $a^- = a^{--}$. Then $\phi = a^-$. For, Lemma 12 yields $(a, a^-) \nrightarrow \text{wk}\,\Delta(a^-)$, and hence $(a, b) \nrightarrow \text{wk}\,\Delta(a^-)$. Let $a_0 < a^-$. Then $a^- \to (a_0)^1_{\aleph_0 + |\beta|}$; $a \to (a_0)^2_{\psi(b)}$; $(a, b) \to \text{wk}\,\Delta(a_0)$ $(a_0 < a^-)$.

Case 4b3. $2^{|\beta|} < a \leqq 2^{\aleph_0 + |\beta|}$. Then $\phi = 3$. For, we have $\beta < \omega$ and $a \leqq \aleph_1$. By Lemma 13, $(\aleph_1, \aleph_0) \nrightarrow \text{wk}\,\Delta(3)$. Hence $(a, b) \nrightarrow \text{wk}\,\Delta(3)$.

This concludes the dicsussion of the relation $(a, b) \to \text{wk}\,\Delta(c)$ for infinite cardinals a, b.

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