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RESEARCH ARTICLE

A model of the Axiom of Determinacy in which every set of reals is universally Baire

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Abstract

The consistency of the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + 'every set of reals is universally Baire' is proved relative to ZFC + 'there is a cardinal that is a limit of Woodin cardinals and of strong cardinals'. The proof is based on the derived model construction, which was used by Woodin to show that the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + 'every set of reals is Suslin' is consistent relative to ZFC + 'there is a cardinal λ that is a limit of Woodin cardinals and of $<\lambda$-strong cardinals'. The Σ^2_1 reflection property of our model is proved using genericity iterations as in Neeman [18] and Steel [22].

Contents

1. Introduction

Universal Baireness has come to be seen in modern set theory as a sort of master regularity property for sets of real numbers, implying for instance both Lebesgue measurability and the property of Baire. Moreover, assuming the existence of suitable large cardinals, it is a property shared by every set of reals with a sufficiently simple definition. For instance, if there exist proper class many Woodin cardinals, then every set of reals in the inner model $L(\mathbb{R})$ (the smallest transitive model of ZF containing the reals and the ordinals) is universally Baire in the full universe V (see, for instance, Theorems 3.3.9 and 3.3.13 of [10]). In this case, $L(\mathbb{R})$ also satisfies the Axiom of Determinacy (AD). However, if AD holds in $L(\mathbb{R})$, then not every set of reals in $L(\mathbb{R})$ satisfies the definition of universal Baireness in $L(\mathbb{R})$, since in this case, $L(\mathbb{R})$ does not satisfy the statement that every set of reals is Suslin (as defined in Definition 2.1; see, for instance, Theorems 6.24 and 6.28 of [11]). In this paper, we produce a model of $\mathsf{ZF} + \mathsf{AD}$ in which every set of reals is universally Baire. The large cardinal hypothesis used in our argument has since been shown to be optimal by Sandra Müller [16].

We let ω denote the set of nonnegative integers, with the discrete topology, and we let ω^ω denote the set of ω-sequences from ω, with the product topology. Following set-theoretic convention, we will also denote ω^ω by $\mathbb{R}$ and refer to its elements as *reals*, despite the fact that it is homeomorphic to the space of irrational numbers and not to the real line.

The definition of universal Baireness that we will use in this paper is the set-theoretic definition involving trees, which we postpone to Section 2. For now, we remark that every universally Baire set of reals $A \subseteq \omega^\omega$ has the following property:

(*) For every topological space X with a regular open base and every continuous function $f\colon X \to \omega^\omega$, the preimage $f^{-1}[A]$ has the Baire property in X.

Note that while every regular topological space has a regular open basis, some Hausdorff spaces do not. The property $(*)$ was the original definition of universal Baireness given by Feng, Magidor and Woodin in [2], where it was shown to be equivalent in ZFC to the definition we use in this paper [2, Corollary 2.1(3)]. We do not know if the equivalence of the two definitions follows from ZF, but Lemma 2.4 shows that the definition we use implies $(*)$, so that all sets of reals in the model we will build are universally Baire according to both definitions.

The following is an immediate consequence of our main theorem (for which see Section 5).

Theorem 1.1. *If the theory* ZFC + *'there is a cardinal that is a limit of Woodin cardinals and a limit of strong cardinals' is consistent, then so is the theory* $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ + *'every set of reals is universally Baire'.*

The axiom $\mathsf{AD}_{\mathbb{R}}$ asserts the determinacy of all two-player games of length ω on the real numbers. The relationships between AD, $\mathsf{AD}_{\mathbb{R}}$ and AD^+ (a technical strengthening of AD whose definition we give in Section 5) are discussed in Section 5; see, for instance, [11] for more background.

Section 2 of the paper reviews universally Baire sets, and Section 3 presents some material on semiscales. The models we consider in this paper are introduced in Section 4. The main theorem of the paper is stated in Section 5, which also contains a review of symmetric extensions and homogeneously Suslin sets. The proof of the main theorem is given in Section 5, relative to three facts proved in later sections, including the Σ^2_1-reflection property of our model. Sections 6 (on genericity iterations) and 7 (on absoluteness) develop results needed for the proof of Σ^2_1 reflection, which is given in Section 8. Finally, Section 9 proves a theorem of Woodin (used in the proof of the main theorem), whose proof has not previously appeared in print, and uses the machinery from this proof to fill in the last remaining detail of the proof of the main theorem.

Remark 1.2. The existence of a model of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ in which all sets of reals are universally Baire was independently proved by Hugh Woodin from the stronger large cardinal hypothesis asserting the existence of proper class many Woodin limits of Woodin cardinals. From this hypothesis, he produced an inner model satisfying $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$ + 'ω_1 is supercompact', which in turn implies that every set of reals is universally Baire, and therefore that $\mathsf{AD}_{\mathbb{R}}$ holds. (In the choiceless context, supercompactness is defined in terms of normal fine measures.) Woodin's model is an enlargement of the Chang model $L(\mathrm{Ord}^\omega)$ obtained by adding a predicate for the club filter on $\mathcal{P}_{\omega_1}(\lambda^\omega)$ for every ordinal λ. Woodin showed that these predicates restrict to ultrafilters on the model. The theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$ + 'ω_1 is supercompact' implies that the pointclass of Suslin sets of reals is closed under complementation, by a theorem of Martin and Woodin [14]. Together with AD^+, this implies that every set of reals is Suslin. Using the supercompactness of ω_1 again to take ultrapowers of trees, one can then show that every set of reals is universally Baire.

Remark 1.3. A model of determinacy in which all sets of reals are universally Baire can be shown to exist from the theory $\mathsf{ZF} + \mathsf{AD}^+$ plus the assumption that there is a limit ordinal α such that θ_α (the α-th

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member of the Solovay sequence; see Section 6.3 of [11]) is less than Θ. Assume V is a model of this theory. To construct a model of AD^+ in which all sets of reals are universally Baire, let Δ be the set of reals of Wadge rank less than θ_α. Then $M =_{\text{def}} V_{\theta_{\alpha+1}} \cap \mathrm{HOD}_\Delta$ is a model of $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ in which all sets of reals are universally Baire. This follows from the fact that in V, every set in Δ is κ-homogenously Suslin for every $\kappa < \theta_\alpha$. We do not know if the proof of this fact has appeared in print, but it can be proven by combining the following facts: Theorem 7.5 and Lemma 7.7 of [23], and the results of [7]. See also [14]. We then use Lemma 7.7 of [23] to conclude that M is a model of $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ in which all sets of reals are universally Baire. See also Lemma 7.6 of [23].

We also remark that if θ_α is a regular cardinal of HOD, then M defined above will satisfy the theory $\mathsf{AD}_\mathbb{R}$ + 'Θ is a regular cardinal' (see [1]). By the results of [19], in the minimal model of the Largest Suslin Axiom, there is α such that $\theta_\alpha < \Theta$ and θ_α is regular in HOD. Thus, a model of the theory $\mathsf{AD}_\mathbb{R}$ + 'Θ is a regular cardinal''+''All sets of reals are universally Baire' can be constructed inside the minimal model of the Largest Suslin Axiom.

2. Trees, Suslin sets and universally Baire sets

In this section, we work in ZF, without the Axiom of Choice. A *tree* on a class X is a set of finite sequences $T \subseteq X^{<\omega}$ that is closed under initial segments. For such a tree, we let $[T]$ denote the set of all branches (infinite chains) of T, so that $[T] \subseteq X^\omega$. Note that a set $A \subseteq X^\omega$ is closed in the ω-fold product of the discrete topology on X if and only if $A = \mathrm{p}[T]$ for some tree T on X. For a tree on a product $X \times Y$, we will identify sequences of pairs with pairs of sequences (always of the same length), so for such a tree, we may write $[T] \subseteq X^\omega \times Y^\omega$.

The trees we consider will usually be trees on the class $\omega \times \mathrm{Ord}$ where Ord is the class of ordinals.[1] By the *projection* of such a tree T, we mean the projection of $[T]$ onto its first coordinate, which is a set of reals:

$$\mathrm{p}[T] = \{x \in \omega^\omega : \exists f \in \mathrm{Ord}^\omega\ (x, f) \in [T]\}.$$

An important fact that we will often use without further comment is that membership in the projection of a tree is absolute: if M is a transitive model of ZF containing a real x and a tree T on $\omega \times \mathrm{Ord}$, then $x \in \mathrm{p}[T]$ if and only if $M \models x \in \mathrm{p}[T]$. This follows from the absoluteness of wellfoundedness of the tree $T_x = \{t \in \mathrm{Ord}^{<\omega} : (x{\restriction}|t|, t) \in T\}$. The same fact applies with V and a generic extension of V in place of M and V, respectively.

Definition 2.1. A set of reals A is *Suslin* if $A = \mathrm{p}[T]$ for some T on $\omega \times \mathrm{Ord}$. Given a set X, A is X-Suslin if it is the projection of a tree on $\omega \times X$.

Recall that a set of reals A is $\mathbf{\Sigma}^1_1$ (analytic) if and only if $A = \mathrm{p}[T]$ for some T on $\omega \times \omega$, so the Suslin sets generalize the analytic sets.

For a set $A \subseteq \omega^\omega \times \omega^\omega$, we can define '$A$ is Suslin' and 'A is κ-Suslin' in terms of trees on $\omega \times \omega \times \mathrm{Ord}$ and $\omega \times \omega \times \kappa$, respectively, and similarly for $A \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ and so on. In this paper, we will typically state and prove results for sets of reals and leave the straightforward higher-dimensional generalizations to the reader.

Suslin sets of reals are important objects of study under the Axiom of Determinacy. Under the Axiom of Choice, however, every set of reals is Suslin (in a useless way). More generally, every wellordered set of reals $A = \{x_\alpha : \alpha < \kappa\}$ is Suslin as witnessed by the tree

$$\{(x_\alpha{\restriction}n, \bar{\alpha}{\restriction}n) : \alpha < \kappa \text{ and } n \in \omega\}$$

on $\omega \times \kappa$, where $\bar{\alpha}$ denotes the infinite sequence with constant value α. Universal Baireness is a strengthening of Suslinness that is nontrivial in the presence of the Axiom of Choice.

[1]A set T is a tree on $\omega \times \mathrm{Ord}$ if and only if it is a tree on $\omega \times \kappa$ for some cardinal κ. In this paper, we will usually not be concerned with what κ is, so we will speak of trees on $\omega \times \mathrm{Ord}$ for simplicity.

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Instead of the original definition of universal Baireness, which we called $(*)$ above, we will use the following definition.

Definition 2.2. Let $\mathbb{P}$ be a poset.

- A pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$ is $\mathbb{P}$-*absolutely complementing* if $1_{\mathbb{P}} \Vdash \mathrm{p}[T] = \omega^\omega \setminus \mathrm{p}[\tilde{T}]$.
- A set of reals A is $\mathbb{P}$-*Baire* if $A = \mathrm{p}[T]$ for some $\mathbb{P}$-absolutely complementing pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$.

A set of reals A is *universally Baire* if it is $\mathbb{P}$-Baire for every poset $\mathbb{P}$. We write uB for the pointclass $\{A \subseteq \omega^\omega : A \text{ is universally Baire}\}$.

Remark 2.3. If a poset $\mathbb{P}$ regularly embeds into a poset $\mathbb{Q}$, then every $\mathbb{Q}$-Baire set of reals is $\mathbb{P}$-Baire. A classical result of McAloon says that every partial order $\mathbb{P}$ regularly embeds into the partial order $\mathrm{Col}(\omega, \mathbb{P})$, which adds a surjection from ω to $\mathbb{P}$ by finite approximations (see, for instance, the appendix to [10]). To show that a set of reals is universally Baire, then, it suffices to show that it is $\mathrm{Col}(\omega, Z)$-Baire for every set Z. In the context of the Axiom of Choice, the set Z can be taken to be an infinite cardinal. We show in Lemma 5.7 that a similar implication holds in the models we consider in this paper.

It is immediate from the definition that every universally Baire set of reals is Suslin and that the collection of universally Baire sets is closed under complements. The proof of Shoenfield's absoluteness theorem shows that all $\mathbf{\Sigma}^1_1$ (analytic) sets of reals are universally Baire, from which it follows that all $\mathbf{\Pi}^1_1$ (coanalytic) sets are as well. Universal Baireness for more complex sets of reals is tied to large cardinals. For example, every $\mathbf{\Sigma}^1_2$ set of reals is universally Baire if and only if every set has a sharp, as shown by by Feng, Magidor and Woodin [2, Theorem 3.4]. As noted above, if there is a proper class of Woodin cardinals, then every set of reals in $L(\mathbb{R})$ is universally Baire in V.

Lemma 2.4 below shows that universal Baireness implies property $(*)$ in ZF. We do not know whether the converse can be proved in ZF. The proof uses the standard notion of the leftmost branch of a tree. That is, if S is an ill-founded tree on Ord, the *leftmost branch* of S is the sequence $\mathrm{lb}(S) \in \mathrm{Ord}^\omega$ defined recursively by letting $\mathrm{lb}(S)(n)$ be the least ordinal α such that the tree

$$\{s \in S : s \subseteq (\mathrm{lb}(S) \restriction n)^\frown \langle \alpha \rangle \vee (\mathrm{lb}(S) \restriction n)^\frown \langle \alpha \rangle \subseteq s\}$$

is ill-founded. Usually this operation is applied to a tree of the form T_x (as above), where $x \in \omega^\omega$ and T is a tree on $\omega \times \mathrm{Ord}$, to find a witness to the statement $x \in \mathrm{p}[T]$.

Lemma 2.4. *Let $A \subseteq \omega^\omega$ be universally Baire, let X be a topological space with a regular open base, and let $f : X \to \omega^\omega$ be a continuous function. Then the preimage $f^{-1}[A]$ has the Baire property in X.*

Proof. The proof is a minor modification of the proof given in [2], which uses Choice for some steps. Let $\mathrm{RO}(X)$ denote the collection of all regular open subsets of X, which is a complete boolean algebra with negation given by the complement of the closure, and suprema given by the interior of the closure of the union. Considering $\mathrm{RO}(X)$ as a poset (under inclusion, with the empty set excluded), let S and T be trees witnessing that A is $\mathrm{RO}(X)$-Baire, with $\mathrm{p}[S] = A$. For each pair $n, m \in \omega$, let $X_{n,m}$ be the set of $x \in X$ with $f(x)(n) = m$. Then each $X_{n,m}$ is in $\mathrm{RO}(X)$. Let $\dot{g}$ be the $\mathrm{RO}(X)$-name for an element of ω^ω consisting of the pairs $(X_{n,m}, (n, m))$ for each $n, m \in \omega$, so that each $X_{n,m}$ forces that $\dot{g}(n) = m$.

Densely many conditions in $\mathrm{RO}(X)$ decide whether the realization of $\dot{g}$ is in the projection of S or T. Let U be the union of the conditions forcing $\dot{g}$ into $\mathrm{p}[S]$, and let V be the corresponding set for T. Since intersections of regular open sets are regular open, U and V are disjoint. Since X has a regular open base, $U \cup V$ is dense. It suffices to see that $f^{-1}[A] \triangle U$ is meager in X.

For each $\sigma \in \omega^{<\omega}$, let X_σ be $\bigcap_{n<|\sigma|} X_{n,\sigma(n)}$ (i.e., the set of $x \in X$ for which σ is an initial segment of $f(x)$). For each $n \in \omega$, let $\mathcal{D}_n$ be the set of conditions Y of $\mathrm{RO}(X)$ which are contained in either U or V and also in some set of the form X_σ, for some $\sigma \in \omega^n$, and which decide the first n elements of the leftmost branch of whichever of $S_{\dot{g}}$ or $T_{\dot{g}}$ is ill-founded (i.e., $S_{\dot{g}}$ for sets contained in U and $T_{\dot{g}}$ for sets contained in V). Let $\tau_n(Y)$ denote the length-n initial segment of this leftmost branch as decided by Y.

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Note then that if $Y \in \mathcal{D}_n$ is contained in some such X_σ, then the pair $(\sigma, \tau_n(Y))$ is in the corresponding tree.

For each $n \in \omega$, let D_n be the union of all the members of $\mathcal{D}_n$. Then each D_n is dense open. It suffices now to see that if x is in $U \cap \bigcap_{n\in\omega} D_n$, then $f(x) \in A$, and if $x \in V \cap \bigcap_{n\in\omega} D_n$, then $f(x) \notin A$.

To see the former claim, fix $x \in U$ and regular open sets $Y_n \in \mathcal{D}_n (n \in \omega)$ with $x \in \bigcap_{n\in\omega} Y_n$. Each Y_n is contained in some X_{σ_n}, in such a way that $f(x) = \bigcup_{n\in\omega} \sigma_n$. Furthermore, since the conditions Y_n are compatible, the values $\tau_n(Y_n)$ are compatible, and the pair $(f(x), \bigcup_{n\in\omega} \tau_n(Y_n))$ gives a branch through S, showing that $f(x) \in A$. The proof for the latter claim is the same. □

As a trivial consequence of the previous lemma, if A is universally Baire, then A itself has the Baire property. Note that if the Axiom of Choice holds, then there is a set of reals that does not have the Baire property and therefore is not universally Baire.

The following standard lemma on $\mathbb{P}$-absolutely complementing pairs of trees is often useful.

Lemma 2.5. *Let T be a tree on $\omega \times \mathrm{Ord}$, let $\mathbb{P}$ be a poset, and let $(U, \tilde{U})$ be a $\mathbb{P}$-absolutely complementing pair of trees on $\omega \times \mathrm{Ord}$. If $\mathrm{p}[T] \subseteq \mathrm{p}[U]$, then $1_{\mathbb{P}} \Vdash \mathrm{p}[T] \subseteq \mathrm{p}[U]$.*

Proof. If $\mathrm{p}[T] \subseteq \mathrm{p}[U]$, then $\mathrm{p}[T] \cap \mathrm{p}[\tilde{U}] = \emptyset$. This implies that

$$1_{\mathbb{P}} \Vdash \mathrm{p}[T] \cap \mathrm{p}[\tilde{U}] = \emptyset,$$

by the absoluteness of wellfoundedness of the tree

$$\{(s, t, u) : (s, t) \in T \wedge (s, u) \in \tilde{U}\}$$

on $\omega \times \mathrm{Ord} \times \mathrm{Ord}$. Because $(U, \tilde{U})$ is $\mathbb{P}$-absolutely complementing, it follows that $1_{\mathbb{P}} \Vdash \mathrm{p}[T] \subseteq \mathrm{p}[U]$. □

Applying Lemma 2.5 in both directions, we immediately obtain the following result, which shows that although there may not be a canonical way to choose trees witnessing universal Baireness of a given set of reals, any such pair of trees gives canonical information about how to expand the set of reals in generic extensions.

Lemma 2.6. *Let $\mathbb{P}$ be a poset, let A be a $\mathbb{P}$-Baire set of reals, and let $(T, \tilde{T})$ and $(U, \tilde{U})$ be $\mathbb{P}$-absolutely complementing pairs of trees witnessing that A is $\mathbb{P}$-Baire. Then $1_{\mathbb{P}} \Vdash \mathrm{p}[T] = \mathrm{p}[U]$.*

The following notation is therefore well defined.

Definition 2.7. For a poset $\mathbb{P}$, a $\mathbb{P}$-Baire set of reals A, and a V-generic filter $G \subseteq \mathbb{P}$, taking any $\mathbb{P}$-absolutely complementing pair of trees $(T, \tilde{T})$ witnessing that A is $\mathbb{P}$-Baire, the *canonical expansion of A to $V[G]$* is

$$A^{V[G]} = \mathrm{p}[T]^{V[G]}.$$

Remark 2.8. The canonical expansion $A^{V[G]}$ depends only on the model $V[G]$ and not the generic filter G.

We will sometimes use the following local version of universal Baireness, following Steel [23] and Larson [10].

Definition 2.9. Let κ be a cardinal.

- A pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$ is *κ-absolutely complementing* if it is $\mathbb{P}$-absolutely complementing for every (wellordered) poset $\mathbb{P}$ of cardinality less than κ.
- A set of reals A is *κ-universally Baire* if $A = \mathrm{p}[T]$ for some κ-absolutely complementing pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$.

We write uB_κ for the pointclass $\{A \subseteq \omega^\omega : A \text{ is } \kappa\text{-universally Baire}\}$.

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The reader should be warned that because our definition says ‘cardinality less than κ’ rather than ‘cardinality less than or equal to κ’, what we call κ^+-universally Baire is equivalent in ZFC to what Feng, Magidor and Woodin call κ-universally Baire [2, Theorem 2.1].

One way to produce κ-absolutely complementing trees is the Martin–Solovay construction from a system of κ-complete measures (which we briefly review in Section 5; see also Section 1.3 of [10]). Another way, assuming the Axiom of Choice, is to amalgamate $\mathbb{P}$-absolutely complementing trees for various posets $\mathbb{P}$. This approach yields the following result, whose well-known proof we include here for the reader’s convenience.

Lemma 2.10. *Assume* ZFC*, let κ be a cardinal, and let A be a set of reals. If A is $\mathbb{P}$-Baire for every poset $\mathbb{P}$ of cardinality less than κ, then A is κ-universally Baire.*

Proof. Let $\eta = 2^{<\kappa}$ and let $\langle \mathbb{P}_\alpha : \alpha < \eta \rangle$ enumerate the set of all posets on cardinals less than κ. For each $\alpha < \eta$, choose a $\mathbb{P}_\alpha$-absolutely complementing pair of trees $(T_\alpha, \tilde{T}_\alpha)$ witnessing that A is $\mathbb{P}_\alpha$-Baire. Consider the tree T on $\omega \times \mathrm{Ord}$ defined by

$$T = \{(s, (\alpha^\frown t) \restriction |s|) : \alpha < \eta \text{ and } (s,t) \in T_\alpha\}.$$

This construction immediately implies that $\mathrm{p}[T] = \bigcup_{\alpha<\eta} \mathrm{p}[T_\alpha]$ in every outer model of V. Similarly, letting

$$\tilde{T} = \{(s, (\alpha^\frown t) \restriction |s|) : \alpha < \eta \text{ and } (s,t) \in \tilde{T}_\alpha\},$$

$\mathrm{p}[\tilde{T}] = \bigcup_{\alpha<\eta} \mathrm{p}[\tilde{T}_\alpha]$ in every outer model of V. In V, we have $\mathrm{p}[T] = A$ and $\mathrm{p}[\tilde{T}] = \omega^\omega \setminus A$. It follows immediately from the construction of T and $\tilde{T}$ that the union of their projections is ω^ω in any forcing extension by a poset of cardinality less than κ. The absoluteness of the wellfoundedness of the tree

$$\{(s,t,u) : (s,t) \in T \wedge (s,u) \in \tilde{T}\}$$

implies that the projections of T and $\tilde{T}$ are still disjoint in any such extension. That is, the pair $(T, \tilde{T})$ is κ-absolutely complementing. □

It follows that, assuming the Axiom of Choice, if λ is a limit cardinal and A is κ-universally Baire for every $\kappa < \lambda$, then A is λ-universally Baire.

3. Semiscales

In this section, we work in ZF, except where noted otherwise. A *norm* on a set of reals A is an ordinal-valued function on A. A *prewellordering* of A is a binary relation on A that is transitive and total, and whose corresponding strict relation is wellfounded. A norm φ on a set A induces the prewellording $\{(x,y) : \varphi(x) \le \varphi(y)\}$, and a prewellordering $\le$ induces the norm sending $x \in A$ to its $\le$-rank.

Definition 3.1. Let A be a set of reals, and let $\vec{\varphi} = \langle \varphi_n : n \in \omega \rangle$ be a sequence of norms on A.

1. For a sequence $\langle x_i : i < \omega \rangle$ of reals in A and a real y, we say that $\langle x_i : i < \omega \rangle$ *converges to y via $\vec{\varphi}$* if x converges to y, and for every $n < \omega$, the sequence $\langle \varphi_n(x_i) : i < \omega \rangle$ is eventually constant.
2. We say that $\vec{\varphi}$ is a *semiscale* on A if for every sequence $\langle x_i : i < \omega \rangle$ of reals in A and every real y, if $\langle x_i : i < \omega \rangle$ converges to y via $\vec{\varphi}$, then $y \in A$.

Remark 3.2. The condition that $\langle x_i : i < \omega \rangle$ converges to y via $\vec{\varphi}$ in Part (2) of Definition 3.1 can equivalently be replaced with the stronger condition that $x_i \restriction i = y \restriction i$ for all $i < \omega$, and $\varphi_n(x_i) = \varphi_n(x_j)$ whenever $n \le i < j$. This can be seen by thinning a counterexample satisfying the weaker condition to produce one satisfying the stronger one.

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If A is a Suslin set of reals, then there is a semiscale on A. More specifically, given a tree T on $\omega \times \mathrm{Ord}$ such that $p[T] = A$, we define the corresponding *leftmost branch semiscale* $\vec{\varphi}^T$ on A by letting $\varphi^T_n(x) = \mathrm{lb}(T_x)(n)$. That is, for each $n < \omega$, $\varphi^T_n(x) = \xi_n$, where $\langle \xi_0, \ldots, \xi_n\rangle \in \mathrm{Ord}^{n+1}$ is lexicographically least such that $(x, f) \in [T]$ for some $f \in \mathrm{Ord}^\omega$ extending $\langle \xi_0, \ldots, \xi_n\rangle$. It is straightforward to verify that this construction gives a semiscale on A.

Conversely, if there is a semiscale on A and $\mathsf{CC}_{\mathbb{R}}$ holds, then A is Suslin ($\mathsf{CC}_{\mathbb{R}}$, or *countable choice for reals*, is the restriction of the Axiom of Choice to countable sets of nonempty sets of real numbers; it holds in the models considered in this paper and in fact follows from AD by a theorem of Mycielski). To see this, given a semiscale $\vec{\varphi}$ on A, define the *tree of* $\vec{\varphi}$ by

$$T_{\vec{\varphi}} = \{(x \restriction i, \langle \varphi_0(x), \ldots, \varphi_{i-1}(x)\rangle) : x \in A,\ i \in \omega\}.$$

Clearly, $A \subseteq \mathrm{p}[T_{\vec{\varphi}}]$. Conversely, suppose that $y \in \mathrm{p}[T_{\vec{\varphi}}]$, and fix an $f \in \mathrm{Ord}^\omega$ such that $(y, f) \in [T_{\vec{\varphi}}]$. For each $i < \omega$, choose a real x_i witnessing $(y \restriction i, f \restriction i) \in T_{\vec{\varphi}}$, so that $x_i \in A$, $y \restriction i = x_i \restriction i$ and $f \restriction i = \langle \varphi_0(x_i), \ldots, \varphi_{i-1}(x_i)\rangle$. Then the sequence $\langle x_i : i < \omega\rangle$ converges to y via $\vec{\varphi}$, so $y \in A$. This shows that $A = \mathrm{p}[T_{\vec{\varphi}}]$.

It follows that if $\mathsf{CC}_{\mathbb{R}}$ holds, then the Suslin sets of reals are exactly the sets of reals that admit semiscales. We do not know if this can be proved in ZF. The following lemma can be used in some situations to show that $A = \mathrm{p}[T_{\vec{\varphi}}]$ without assuming $\mathsf{CC}_{\mathbb{R}}$. The lemma is not useful for showing that A is Suslin because that is one of the hypotheses. Rather, it is useful when we want to represent A as the projection of a tree that is definable from a given semiscale $\vec{\varphi}$.

Lemma 3.3. *Let A be a set of reals, and let $\vec{\varphi}$ be a semiscale on A. Assume that the set A and the relation*

$$E = \{(\bar{n}, x, y) : n < \omega \text{ and } x, y \in A \text{ and } \varphi_n(x) = \varphi_n(y)\}$$

are both Suslin, where $\bar{n}$ denotes the constant function from ω to $\{n\}$. Then $\mathrm{p}[T_{\vec{\varphi}}] = A$.

Proof. Clearly, $A \subseteq \mathrm{p}[T_{\vec{\varphi}}]$ as before. Conversely, let $y \in \mathrm{p}[T_{\vec{\varphi}}]$. Fix $f \in \mathrm{Ord}^\omega$ such that $(y, f) \in [T_{\vec{\varphi}}]$. Take a tree T_A on $\omega \times \mathrm{Ord}$ witnessing that A is Suslin, and take a tree T_E on $\omega \times \omega \times \omega \times \mathrm{Ord}$ witnessing that E is Suslin.

For $N < \omega$, define a *full N-witness* to be an object of the form

$$(\langle x_i : i < N\rangle, \langle g_i : i < N\rangle, \langle h_{n,i,j} : n \le i \le j < N\rangle)$$

such that

- for all $i < N$, $x_i \in \omega^\omega$ witnesses that $(y \restriction i, f \restriction i) \in T_{\vec{\varphi}}$, meaning that $x_i \in A$, $y \restriction i = x_i \restriction i$ and $f \restriction i = \langle \varphi_0(x_i), \ldots, \varphi_{i-1}(x_i)\rangle$;
- for all $i < N$, $g_i \in \mathrm{Ord}^\omega$ witnesses that $x_i \in A$ in the sense that $(x_i, g_i) \in [T_A]$;
- for all $n \le i \le j < N$, $h_{n,i,j} \in \mathrm{Ord}^\omega$ witnesses $\varphi_n(x_i) = \varphi_n(x_j)$ in the sense that $(\bar{n}, x_i, x_j, h_{n,i,j}) \in [T_E]$.

We define a *partial N-witness* to be an object that can be obtained by restricting some full N-witness as above to N to produce the following:

$$(\langle x_i \restriction N : i < N\rangle, \langle g_i \restriction N : i < N\rangle, \langle h_{n,i,j} \restriction N : n \le i \le j < N\rangle).$$

Note that every full N-witness can be extended to a full $(N+1)$-witness by choosing any $x_N \in \omega^\omega$ witnessing $(y \restriction N, f \restriction N) \in T_{\vec{\varphi}}$ and then choosing g_N and $h_{n,i,N}$ appropriately for all n and i such that $n \le i \le N$.

It follows that every partial N-witness can be extended to a partial $(N+1)$-witness: first extend it to a full N-witness, extend that to a full $(N+1)$-witness, and then restrict that to a partial $(N+1)$-witness. By *extending* a partial N-witness to a partial $(N+1)$-witness, we mean that its sequences of

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length N are extended to sequences of length $N+1$, and new sequences of length $N+1$ of the form $x_N \restriction (N+1)$, $g_N \restriction (N+1)$ and $h_{n,i,N} \restriction (N+1)$ are added.

The advantage of partial witnesses (as compared with full witnesses) is that they are essentially finite sequences of ordinals, so the Axiom of Choice is not required to choose among them. We may therefore define a sequence $\langle \sigma_N : N < \omega \rangle$ where for each $N < \omega$, σ_N is an N-witness and σ_{N+1} extends σ_N. Taking $\bigcup_{N<\omega} \sigma_N$, by which we really mean taking the union in each coordinate separately, we obtain an object

$$(\langle x_i : i < \omega \rangle, \langle g_i : i < \omega \rangle, \langle h_{n,i,j} : n \leq i \leq j < \omega \rangle).$$

Then for all $i < N$, we have $x_i \in A$ as witnessed by $g_i \in \mathrm{Ord}^\omega$. Furthermore, the sequence $\langle x_i : i < \omega \rangle$ converges to y since $y \restriction i = x_i \restriction i$ for all $i < \omega$. We might not have $\varphi_n(x_i) = f(n)$ for all n and i such that $n \leq i < \omega$ because this property of full witnesses might be lost by passing to partial witnesses and taking unions. However, we do have $\varphi_n(x_i) = \varphi_n(x_j)$ for all n, i and j such that $n \leq i \leq j < \omega$ because this is witnessed by $h_{n,i,j} \in \mathrm{Ord}^\omega$. Therefore, the sequence $\langle x_i : i < \omega \rangle$ converges to y via $\vec{\varphi}$. Since $\vec{\varphi}$ is a semiscale on A, it follows that $y \in A$, as desired. □

Let A be a set of reals and let $\vec{\varphi}$ be a semiscale on A. The *canonical code* for $\vec{\varphi}$ is the set R consisting of those triples $(\bar{n}, x, y) \in \omega^\omega \times A \times A$ for which $\varphi_n(x) \leq \varphi_n(y)$, where again $\bar{n}$ denotes the constant sequence $(n, n, n, \ldots)$. When this holds, we say that R codes $\vec{\varphi}$.

For any ternary relation $R \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ and any $n \in \omega$, define

$$R_n = \{(x, y) \in \omega^\omega \times \omega^\omega : (\bar{n}, x, y) \in R\}.$$

Note that a set $R \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ codes a semiscale if and only if each R_n is a prewellordering of A, and for every sequence $\langle x_i : n \in \omega \rangle$ of points in A converging to a point $y \in \omega^\omega$, if for each n in ω there is an $m \in \omega$ such that $x_i R_n x_j$ for all $i, j \in \omega \setminus m$, then y is in A.

The following result is immediate from the definitions.

Lemma 3.4. *Let $A \subseteq \omega^\omega$, and let $R \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ code a semiscale on A. Let $\mathcal{M}$ be an inner model of* ZF. *If $R \cap \mathcal{M} \in \mathcal{M}$, then $A \cap \mathcal{M} \in \mathcal{M}$ and $\mathcal{M}$ satisfies '$R \cap \mathcal{M}$ codes a semiscale on $A \cap M$'.*

The following result is proved by standard arguments, but since we do not know of a reference, we include a proof here. Higher-level arguments for similar absoluteness results are given in Section 9.

Lemma 3.5. *Let $A \subseteq \omega^\omega$, and let $R \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ code a semiscale on A. Let $\mathbb{P}$ be a poset, and let $G \subseteq \mathbb{P}$ be a V-generic filter on $\mathbb{P}$. If A and R are $\mathbb{P}$-Baire, then $V[G]$ satisfies '$R^{V[G]}$ codes a semiscale on $A^{V[G]}$'.*

Proof. Let $(T_A, T_{\neg A})$ be a pair of trees on $\omega \times \mathrm{Ord}$ witnessing that A is $\mathbb{P}$-Baire, and let $(T_R, T_{\neg R})$ be a pair of trees on $\omega \times \omega \times \omega \times \omega \times \mathrm{Ord}$ witnessing that R is $\mathbb{P}$-Baire. The desired conclusion will follow from the fact that the wellfoundedness of various trees defined from these four trees is absolute from V to $V[G]$.

First, we show that $R_n^{V[G]}$ is a prewellordering of $A^{V[G]}$ for each $n \in \omega$. Let $n \in \omega$. We have $R_n \subseteq A \times A$, so the trees

$$\{(s, t, u, v) : (\bar{n}, s, t, u) \in T_R \text{ and } (s, v) \in T_{\neg A}\}$$

and

$$\{(s, t, u, v) : (\bar{n}, s, t, u) \in T_R \text{ and } (t, v) \in T_{\neg A}\}$$

on $\omega \times \omega \times \mathrm{Ord} \times \mathrm{Ord}$ are wellfounded, and therefore, $R_n^{V[G]} \subseteq A^{V[G]} \times A^{V[G]}$ (here, $\bar{n}$ refers to the constant sequence of length $|s|$ with value n).

Since R_n is transitive, the tree on $\omega \times \omega \times \omega \times \mathrm{Ord} \times \mathrm{Ord} \times \mathrm{Ord}$ consisting of all tuples (s, t, u, v, w, x) such that

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- $(\bar{n}, s, t, v) \in T_R$,
- $(\bar{n}, t, u, w) \in T_R$ and
- $(\bar{n}, s, u, x) \in T_{\neg R}$

is wellfounded. It follows that $R_n^{V[G]}$ is transitive.

Since R_n is total on A, the tree on $\omega \times \omega \times \mathrm{Ord} \times \mathrm{Ord} \times \mathrm{Ord} \times \mathrm{Ord}$ consisting of all tuples (s, t, u, v, w, x) such that

- $(s, u) \in T_A$
- $(t, v) \in T_A$
- $(\bar{n}, s, t, w) \in T_{\neg R}$, and
- $(\bar{n}, t, s, x) \in T_{\neg R}$

is wellfounded. It follows that $R_n^{V[G]}$ is total on $A^{V[G]}$.

For the remaining parts, which are the wellfoundedness of each $R_n^{V[G]}$ and the semiscale condition, it will help to introduce the following notation. Fix a recursive bijection $\omega \times \omega \to \omega$ denoted by $(i, j) \mapsto \langle i, j \rangle$. Given $s \in \omega^{<\omega}$ and integers i, j, define the finite sequence $s_{i,j}$ to be $\langle s(\langle i, 0 \rangle), \ldots, s(\langle i, j-1 \rangle) \rangle$ if all necessary values of s are defined, and undefined otherwise.

The strict part of R_n is wellfounded, so R_n has no infinite strictly decreasing sequences. Then the tree on $\omega \times \mathrm{Ord} \times \mathrm{Ord}$ consisting of all tuples (s, t, u) such that for all $i, j \in \omega$ such that $s_{i,j}$ (and hence also $t_{i,j}$ and $u_{i,j}$) and $s_{i+1,j}$ are defined,

- $(\bar{n}, s_{i+1,j}, s_{i,j}, t_{i,j}) \in T_R$ and
- $(\bar{n}, s_{i,j}, s_{i+1,j}, u_{i,j}) \in T_{\neg R}$

is wellfounded, so there is no strictly $R_n^{V[K]}$-decreasing infinite sequence in any generic extension $V[K]$ (we apply this fact with $K = (G, H)$ in the next paragraph).

Because we are not assuming the Axiom of Choice, the previous paragraph does not immediately show that $R_n^{V[G]}$ is wellfounded. To show that it is, let S be a nonempty subset of $A^{V[G]}$ in $V[G]$. Let $\mathbb{Q}$ be the poset of strictly $R_n^{V[G]}$-decreasing finite sequences of reals in S, ordered by reverse inclusion, and take a $V[G]$ generic filter $H \subseteq \mathbb{Q}$. Because the tree on $\omega \times \mathrm{Ord} \times \mathrm{Ord}$ defined above is still wellfounded in $V[G][H]$, the sequence $\bigcup H$ cannot be infinite, so it has a last element, which must be an $R_n^{V[G]}$-minimal element of S.

Lastly, we prove the semiscale condition. Because R codes a semiscale on A, the tree on $\omega \times \omega \times \mathrm{Ord} \times \mathrm{Ord} \times \mathrm{Ord}$ consisting of all tuples (c, s, t, u, v) such that for all $n, i, j \in \omega$ such that $c(n) \leq \min(i, j)$ and $s_{i,j}$ (and hence also $t_{i,j}$ and $u_{i,j}$) and $s_{i+i,j}$ are defined,

- $(\bar{n}, s_{i,j}, s_{i+1,j}, t_{i,j}) \in T_R$,
- $(\bar{n}, s_{i+1,j}, s_{i,j}, u_{i,j}) \in T_R$,
- $s_{i,c(n)} = s_{i+1,c(n)}$ and
- $(s_{i,n}, v \restriction n) \in T_{\neg A}$

is wellfounded (here, $c(n)$ indicates a point such that the reals $\bigcup_j s_{i,j}$ for $i > c(n)$ agree both in the n-th norm and also for their first n digits; we could have done without c by using Remark 3.2). This, along with the fact that $R_n^{V[G]}$ is a prewellordering of $A^{V[G]}$ for each $n \in \omega$, implies that $R^{V[G]}$ is a semiscale on $A^{V[G]}$ in $V[G]$. □

The following definition is used several times in the next section.

Definition 3.6. A set $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ has the *semiscale property* if every member of Γ has a semiscale which is coded by a continuous preimage of a member of Γ.

See Kechris and Moschovakis [8] or Moschovakis [15] for more on semiscales.

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4. Canonical models for universal Baireness

In this section, we continue to work in ZF. We show that if $\Gamma \subseteq \mathrm{uB}$ and Γ is selfdual and has the semiscale property, then there is an inclusion-minimal inner model containing $\mathbb{R}$ and Γ and satisfying the statement that every member of Γ is universally Baire. In the context of our main theorem, this minimal model has no sets of reals other than those in Γ and therefore satisfies the statement that every set of reals is universally Baire.

Definition 4.1. We let F be the class of all quadruples $(A, \mathbb{P}, p, \dot{x})$ such that

- $A \in \mathrm{uB}$, $\mathbb{P}$ is a poset, $p \in \mathbb{P}$, $\dot{x}$ is a $\mathbb{P}$-name for a real, and
- p forces $\dot{x}$ to be in the canonical expansion of A.

For sets $\Gamma \subseteq \mathrm{uB}$, we define a local version of F that tells us how to compute the canonical extensions of sets in Γ only:

$$F\restriction\Gamma = \{(A, \mathbb{P}, p, \dot{x}) \in F : A \in \Gamma\}.$$

We will eventually (in Lemma 4.3) show that the desired minimal model is $L^{F\restriction\Gamma}(\Gamma, \mathbb{R})$, by which we mean the model constructed from the transitive set

$$V_\omega \cup \mathbb{R} \cup \Gamma \cup \{\Gamma\}$$

relative to the predicate $F\restriction\Gamma$. First, we show that the sets in Γ are universally Baire in this model when Γ is selfdual and has the semiscale property. In the proof, it will be important that the model knows how to expand not only the sets in Γ but also the relations coding semiscales on these sets and their complements.

Theorem 4.2. *If $\Gamma \subseteq \mathrm{uB}$ is selfdual and has the semiscale property, then $L^{F\restriction\Gamma}(\Gamma, \mathbb{R}) \models \Gamma \subseteq \mathrm{uB}$.*

Proof. Let $\mathcal{M}$ denote $L^{F\restriction\Gamma}(\mathbb{R}, \Gamma)$, and fix $A \in \Gamma$. Let $\vec{\varphi}_+$ and $\vec{\varphi}_-$ be semiscales on A and $\omega^\omega \setminus A$, respectively, such that the corresponding codes

$$R_+ = \{(\bar{n}, x, y) \in \omega^\omega \times A \times A : \varphi_{+,n}(x) \leq \varphi_{+,n}(y)\}$$
and
$$R_- = \{(\bar{n}, x, y) \in \omega^\omega \times (\omega^\omega \setminus A) \times (\omega^\omega \setminus A) : \varphi_{-,n}(x) \leq \varphi_{-,n}(y)\}$$

are in Γ.

Consider an arbitrary set $Z \in \mathcal{M}$. By Remark 2.3, it suffices to find a $\mathrm{Col}(\omega, Z)$-absolutely complementing pair of trees for A in $\mathcal{M}$. To do this, we will fix a V-generic filter $H \subseteq \mathrm{Col}(\omega, Z)$, find a complementing pair of trees for A in $\mathcal{M}[H]$, and then show that these trees exist in $\mathcal{M}$ and are as desired.

Fixing H, we have by Lemma 3.5 that the expanded relations $R_+^{V[H]}$ and $R_-^{V[H]}$ code semiscales on the expanded sets $A^{V[H]}$ and $(\omega^\omega \setminus A)^{V[H]}$, respectively. Letting ρ_+ and ρ_- be the standard $\mathrm{Col}(\omega, Z)$-names for the canonical expansions of the relations R_+ and R_-, respectively, we see that the restrictions $\rho_+ \cap \mathcal{M}$ and $\rho_- \cap \mathcal{M}$ are in $\mathcal{M}$, as they are coded into the predicate $F\restriction\Gamma$. Therefore, the restricted relations $R_+^{V[H]} \cap \mathcal{M}[H]$ and $R_-^{V[H]} \cap \mathcal{M}[H]$ are in the model $\mathcal{M}[H]$. These restricted relations code semiscales $\vec{\psi}_+, \vec{\psi}_- \in \mathcal{M}[H]$ on the sets $A^{V[H]} \cap \mathcal{M}[H]$ and $(\omega^\omega \setminus A)^{V[H]} \cap \mathcal{M}[H]$, respectively, by Lemma 3.4. In $\mathcal{M}[H]$, let S_+ and S_- be respectively the trees of the semiscales $\vec{\psi}_+$ and $\vec{\psi}_-$. It suffices to see that these trees are in $\mathcal{M}$ and that they project to complements in any forcing extension of $\mathcal{M}$ by $\mathrm{Col}(\omega, Z)$.

These two claims are proved by the same symmetry argument, showing that S_+ and S_- do not depend on the generic filter H. More precisely, suppose toward a contradiction that there exist conditions $p, p' \in \mathrm{Col}(\omega, Z)$, with $p \in H$, which force contradictory statements about the membership of some sequence σ (from $\mathcal{M}$) in either of the trees S_+ and S_-, noting that these trees are definable in the

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$\mathrm{Col}(\omega, Z)$-extension of $\mathcal{M}$ from the realizations of $\rho_+ \cap \mathcal{M}$ and $\rho_- \cap \mathcal{M}$. By the symmetry of $\mathrm{Col}(\omega, Z)$, there exists a V-generic filter $H' \subseteq \mathrm{Col}(\omega, Z)$, with $p' \in H'$, such that $\bigcup H$ and $\bigcup H'$ differ by only finitely many coordinates, so that $V[H] = V[H']$ and $\mathcal{M}[H] = \mathcal{M}[H']$. We have then (by Remark 2.8) that $A^{V[H]} = A^{V[H']}$, $(\omega^\omega \setminus A)^{V[H]} = (\omega^\omega \setminus A)^{V[H']}$, $R_+^{V[H]} = R_+^{V[H']}$ and $R_-^{V[H]} = R_-^{V[H']}$, from which it follows that the corresponding versions of S_+ and S_- in $\mathcal{M}[H]$ and $\mathcal{M}[H']$ are the same, giving a contradiction. The same argument shows that no condition p' can force that S_+ and S_- fail to project to complements in a $\mathrm{Col}(\omega, Z)$-extension of $\mathcal{M}$, since they do project to complements in $\mathcal{M}[H]$. □

Next, we show that the definition of $F{\restriction}\Gamma$ is absolute to any inner model in which all sets in Γ are universally Baire. It follows that any such inner model must contain $L^{F{\restriction}\Gamma}(\Gamma, \mathbb{R})$.

Lemma 4.3. *Let* $\Gamma \subseteq \mathrm{uB}$. *Let* $\mathcal{M}$ *be a model of* ZF *such that*

$$\mathbb{R} \cup \Gamma \cup \{\Gamma\} \cup \mathrm{Ord} \subseteq \mathcal{M}$$

and $\mathcal{M} \models \Gamma \subseteq \mathrm{uB}$. *Then* $(F{\restriction}\Gamma)^{\mathcal{M}} = (F{\restriction}\Gamma) \cap \mathcal{M}$ *and* $L^{F{\restriction}\Gamma}(\Gamma, \mathbb{R}) \subseteq \mathcal{M}$.

Proof. To show that $(F{\restriction}\Gamma)^{\mathcal{M}} = (F{\restriction}\Gamma) \cap \mathcal{M}$, fix $A \in \Gamma$ and let $\mathbb{P}$ be a poset in $\mathcal{M}$. Let $p \in \mathbb{P}$, and let $\dot{x} \in \mathcal{M}$ be a $\mathbb{P}$-name for a real. We claim that

$$\mathcal{M} \models (A, \mathbb{P}, p, \dot{x}) \in F{\restriction}\Gamma \iff (A, \mathbb{P}, p, \dot{x}) \in F{\restriction}\Gamma.$$

Take a pair of trees $(T, \tilde{T}) \in \mathcal{M}$ witnessing that A is $\mathbb{P}$-Baire in $\mathcal{M}$ and take a pair of trees $(U, \tilde{U})$ witnessing that A is $\mathbb{P}$-Baire in V. Because $\mathrm{p}[T] = \mathrm{p}[U] = A$ and $\mathrm{p}[\tilde{T}] = \mathrm{p}[\tilde{U}] = \omega^\omega \setminus A$ in V and the pair $(U, \tilde{U})$ is $\mathbb{P}$-absolutely complementing, Lemma 2.5 gives

$$1_{\mathbb{P}} \Vdash (\mathrm{p}[T] \subseteq \mathrm{p}[U] \text{ and } \mathrm{p}[\tilde{T}] \subseteq \mathrm{p}[\tilde{U}]).$$

We want to show that the following statements are equivalent for a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{x}$ in $\mathcal{M}$:

1. $\mathcal{M} \models p \Vdash \dot{x} \in \mathrm{p}[T]$;
2. $p \Vdash \dot{x} \in \mathrm{p}[U]$.

To see this, assume first that (1) is true and take a V-generic filter $G \subseteq \mathbb{P}$ containing p. Because G is also $\mathcal{M}$-generic, we have

$$\dot{x}_G \in \mathrm{p}[T]^{M[G]} \subseteq \mathrm{p}[T]^{V[G]} \subseteq \mathrm{p}[U]^{V[G]}.$$

This shows that (2) is true.

Now assume that (1) is false, and take a condition $q \leq p$ such that

$$\mathcal{M} \models q \Vdash \dot{x} \notin \mathrm{p}[T].$$

Take a filter $G \subseteq \mathbb{P}$ containing q that is V-generic, and hence also M-generic. Then we have

$$\begin{aligned} \dot{x}_G &\in (\omega^\omega \setminus \mathrm{p}[T])^{\mathcal{M}[G]} \\ &= \mathrm{p}[\tilde{T}]^{\mathcal{M}[G]} \\ &\subseteq \mathrm{p}[\tilde{T}]^{V[G]} \\ &\subseteq \mathrm{p}[\tilde{U}]^{V[G]} \\ &= (\omega^\omega \setminus \mathrm{p}[U])^{V[G]}. \end{aligned}$$

This shows that (2) is false.

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To see that $L^{F\restriction\Gamma}(\Gamma,\mathbb{R}) \subseteq \mathcal{M}$, consider the model $(L^{F\restriction\Gamma}(\Gamma,\mathbb{R}))^{\mathcal{M}} = L^{(F\restriction\Gamma)^{\mathcal{M}}}(\Gamma,\mathbb{R})$. It is contained in $\mathcal{M}$, and its construction agrees with that of the model $L^{F\restriction\Gamma}(\Gamma,\mathbb{R})$ at every stage because $(F\restriction\Gamma)^{\mathcal{M}} = (F\restriction\Gamma)\cap\mathcal{M}$. □

It follows that $L^{F\restriction\Gamma}(\Gamma,\mathbb{R})$ sees the definition of F and therefore sees its own construction.

Corollary 4.4. *Let* $\Gamma \subseteq \mathrm{uB}$. *If* Γ *is selfdual and has the semiscale property, then the model* $L^{F\restriction\Gamma}(\Gamma,\mathbb{R})$ *satisfies the statement* $V = L^{F\restriction\Gamma}(\Gamma,\mathbb{R})$.

5. A model of ZF + AD$^+$ + 'every set of reals is universally Baire'

In this section, we present our main theorem and outline its proof. First, we recall some standard notation from the proof of Woodin's Derived Model Theorem, as presented in Steel [23]. We let ScS denote the pointclass consisting of those $A \subseteq \omega^\omega$ such that A and $\omega^\omega \setminus A$ are both Suslin (i.e., such that A is Suslin and co-Suslin).

Definition 5.1. Let λ be a limit of inaccessible cardinals, and let $G \subseteq \mathrm{Col}(\omega,<\lambda)$ be a V-generic filter. We define

- $\mathbb{R}^*_G$ to be $\bigcup_{\xi<\lambda}\mathbb{R}^{V[G\restriction\xi]}$.
- HC^*_G to be $\bigcup_{\xi<\lambda}\mathrm{HC}^{V[G\restriction\xi]}$, where HC denotes the collection of hereditarily countable sets.
- Hom^*_G to be the pointclass consisting of all sets of the form $\mathrm{p}[S]\cap\mathbb{R}^*_G$, for some λ-absolutely complementing pairs of trees (S,T) appearing in a model $V[G\restriction\xi]$ with $\xi<\lambda$.

We define the symmetric extension $V(\mathbb{R}^*_G)$ to be $\mathrm{HOD}^{V[G]}_{V\cup\mathbb{R}^*_G\cup\{\mathbb{R}^*_G\}}$. By a theorem of Gödel proved in Section 5.2 of [20], $V(\mathbb{R}^*_G)$ is a model of ZF. By the definition of the model, $\mathbb{R}^*_G \subseteq \mathbb{R}^{V(\mathbb{R}^*_G)}$. The reverse containment $\mathbb{R}^*_G \subseteq \mathbb{R}^{V(\mathbb{R}^*_G)}$ follows from the homogeneity of $\mathrm{Col}(\omega,<\lambda)$ (by a standard argument which appears, for instance, on page 307 of [23]): each member x of $\mathbb{R}^{V(\mathbb{R}^*_G)}$ is definable in $V[G]$ from $\{\mathbb{R}^*_G\}$ and parameters existing in some model $V[G\restriction\xi]$ with $\xi<\lambda$, and the homogeneity of the remainder of $\mathrm{Col}(\omega,<\lambda)$ after ξ implies that x exists already in $V[G\restriction\xi]$.

A similar homogeneity argument shows that $\mathrm{ScS}^{V(\mathbb{R}^*_G)} \subseteq \mathrm{Hom}^*_G$, since any pair of trees witnessing membership in $\mathrm{ScS}^{V(\mathbb{R}^*_G)}$ would exist in some such $V[G\restriction\xi]$ (being definable in $V[G]$ from $\{\mathbb{R}^*_G\}$ and parameters in such a model), and would have to be λ-absolutely complementing in order to project to complements in $V(\mathbb{R}^*_G)$. Finally, $\mathrm{Hom}^*_G \subseteq \mathrm{ScS}^{V(\mathbb{R}^*_G)}$ since $V(\mathbb{R}^*_G)$ contains each model $V[G\restriction\xi]\,(\xi<\lambda)$.

With these definitions in hand, we can state the main theorem of this paper.

Theorem 5.2. *Let* λ *be a limit of Woodin cardinals and a limit of strong cardinals, let* $G \subseteq \mathrm{Col}(\omega,<\lambda)$ *be a* V*-generic filter, and let* $\mathcal{M}$ *be the model*

$$L^F(\mathbb{R}^*_G,\mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)},$$

where $L^F(\mathbb{R}^*_G,\mathrm{Hom}^*_G)$ *denotes the class of all sets constructible from the transitive set* $V_\omega\cup\mathbb{R}^*_G\cup\mathrm{Hom}^*_G$ *relative to the predicate* F*. Then the following hold.*

1. $\mathcal{M}\models\mathsf{AD}_{\mathbb{R}}$.
2. $\mathcal{P}(\mathbb{R})^{\mathcal{M}} = \mathrm{Hom}^*_G$
3. $\mathcal{M}\models$ *'every set of reals is universally Baire'.*
4. $\mathcal{M}\models V = \mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$.

Moreover, $\mathcal{M}$ *has the following minimality property: if* $\mathcal{M}' \subseteq V(\mathbb{R}^*_G)$ *is a model of* ZF + *'every set of reals is universally Baire' such that*

$$\mathbb{R}^*_G\cup\mathrm{Hom}^*_G\cup\mathrm{Ord}\subseteq\mathcal{M}',$$

then $\mathcal{M}\subseteq\mathcal{M}'$.

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Remark 5.3. By results of Moschovakis and Woodin (see Theorem 0.3 of [11]), the theory $\mathsf{ZF} + \mathsf{AD} +$ all sets of reals are Suslin implies AD^+ (whose definition we give below). So AD^+ holds in the model $\mathcal{M}$ from our Main Theorem. By results of Martin and Woodin (Theorem 13.1 of [11]), the theory $\mathsf{ZF} + \mathsf{AD} +$ all sets of reals are Suslin implies $\mathsf{AD}_{\mathbb{R}}$. So part (1) of the main theorem follows from part (3) plus the fact that $\mathcal{M} \models \mathsf{AD}$. We will in fact prove that $\mathcal{M} \models \mathsf{AD}^+$ on the way to proving part (3) of the main theorem.

Remark 5.4. A theorem of Woodin from the 1980s (Theorem 7.1 of [23]) says, using the notation above, that if λ is a limit of Woodin cardinals, then $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G) \models \mathsf{AD}^+$ and Hom^*_G is $\mathrm{ScS}^{L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)}$. The model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$ is sometimes referred to as the old derived model. This is to contrast it with the (larger) new derived model $D(V, \lambda, G)$; that is, $L(\Gamma_+, \mathbb{R}^*_G)$, where Γ_+ is the set of $B \subseteq \mathbb{R}^*_G$ in $V(\mathbb{R}^*_G)$ for which $L(B, \mathbb{R}^*_G) \models \mathsf{AD}^+$. Theorem 31 of [24] says that $D(V, \lambda, G) \models \mathsf{AD}^+$, again under the assumption that λ is a limit of Woodin cardinals. The hypotheses of our main theorem imply (via Theorem 9.1) that the old and new derived models are equal (see Remark 9.2).

Before starting the proof of the main theorem, we review some material on homogeneously Suslin sets of reals (which is presented in more detail in [10, 23]). A *tree of measures* (on an ordinal γ, the choice of which will usually not concern us) is a family $\langle \mu_s : s \in \omega^{<\omega} \rangle$ of measures such that each measure μ_s concentrates on the set $\gamma^{|s|}$ and such that μ_t projects to μ_s whenever t extends s, meaning that for each $A \in \mu_s$, the set of $\sigma \in \gamma^{|t|}$ with $\sigma \restriction |s| \in A$ is in μ_t. Given a tree of measures $\vec{\mu} = \langle \mu_s : s \in \omega^{<\omega} \rangle$, for every real x, we let $\vec{\mu}_x$ denote the tower (i.e., projecting sequence) of measures $\langle \mu_{x \restriction n} : n < \omega \rangle$. We define the set of reals $\mathbf{S}_{\vec{\mu}}$ to be the set of $x \in \omega^\omega$ for which $\vec{\mu}_x$ is wellfounded (i.e., for which the direct limit of the $\mu_{x\restriction n}$-ultrapowers of V (for $n \in \omega$) induced by the projection maps is wellfounded). Given a cardinal κ, a set of reals A is said to be *κ-homogeneous* if $A = \mathbf{S}_{\vec{\mu}}$ for some tree $\vec{\mu}$ of κ-complete measures.

By an observation of Woodin (see Steel [23, Proposition 2.5]), every tree $\vec{\mu}$ of κ-complete measures on γ is a κ-homogeneity system for some tree T on $\omega \times \gamma$, meaning that each measure μ_s concentrates on the set $T \cap \gamma^{|s|}$, and for every real $x \in \mathrm{p}[T]$, the tower $\vec{\mu}_x$ is wellfounded. For every real $x \notin \mathrm{p}[T]$ then the tower $\vec{\mu}_x$ must be ill-founded, so $S_{\vec{\mu}} = \mathrm{p}[T]$. So all κ-homogeneous sets of reals are Suslin, and they are sometimes called *κ-homogeneously Suslin* to emphasize this property. In the weakest nontrivial case, when $\kappa = \omega_1$, we say that A is *homogeneously Suslin*.

By the Levy-Solovay theorem, for every generic extension $V[g]$ of V by a poset of size less than κ, every κ-complete $\mu \in V$ induces a corresponding measure $\hat{\mu}$ in $V[g]$, and the ultrapower of the ordinals by μ in V agrees with the ultrapower of the ordinals by $\hat{\mu}$ in $V[g]$. (Henceforth, we will denote both measures by μ where it will not cause confusion.) So we can also define the set $\mathbf{S}_{\vec{\mu}}$ in small generic extensions, and we have $(\mathbf{S}_{\vec{\mu}})^{V[g]} \cap V = (\mathbf{S}_{\vec{\mu}})^V$.

Given a tree $\vec{\mu}$ of measures, the Martin–Solovay tree $\mathrm{ms}(\vec{\mu})$ is the tree of attempts to build a real x and a sequence of ordinals witnessing the ill-foundedness of the corresponding tower $\vec{\mu}_x$ (see [23, 10] for a precise definition.) The Martin–Solovay tree has the key property that

$$\mathbf{S}_{\vec{\mu}} = \omega^\omega \setminus \mathrm{p}[\mathrm{ms}(\vec{\mu})].$$

If the measures in $\vec{\mu}$ are κ-complete, then definition of the Martin–Solovay tree is absolute to generic extensions via posets of cardinality less than κ, so the equality $\mathbf{S}_{\vec{\mu}} = \omega^\omega \setminus \mathrm{p}[\mathrm{ms}(\vec{\mu})]$ continues to hold in such extensions. It follows that for any tree T carrying a homogeneity system $\vec{\mu}$ consisting of κ-complete measures, the pair $(T, \mathrm{ms}(\vec{\mu}))$ is κ-absolutely complementing. This implies that every κ-homogeneous set of reals $A = \mathbf{S}_{\vec{\mu}}$ is κ-universally Baire, and that whenever $V[g]$ is a generic extension of V by a poset of size less than κ, we have $A^{V[g]} = (\mathbf{S}_{\vec{\mu}})^{V[g]}$. In other words, the canonical extension of A given by κ-homogeneity agrees with that given by κ-universal Baireness.

Another property of the Martin–Solovay tree that we will use is the fact that its definition from $\vec{\mu}$ is continuous: for every $n < \omega$, the subtree $\mathrm{ms}(\vec{\mu}) \restriction n$ consisting of the first n levels of $\mathrm{ms}(\vec{\mu})$ is determined by finitely many measures from $\vec{\mu}$.

Remark 5.5. If κ is a cardinal and δ_0 and δ_1 are Woodin cardinals with $\kappa < \delta_0 < \delta_1$, then every δ_1^+-universally Baire set of reals is κ-homogeneously Suslin. This follows by combining a theorem

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of Woodin (see Larson [10, Theorem 3.3.8] or Steel [23, Theorem 4.1]) with a theorem of Martin and Steel [12]. In particular, if λ is a limit of Woodin cardinals, then every λ-universally Baire set of reals is $<\lambda$-homogenously Suslin (i.e., κ-homogeneous for all $\kappa < \lambda$; we write $\mathrm{Hom}_{<\lambda}$ for the set of $<\lambda$-homgeneously Suslin sets of reals). As outlined above, the reverse inclusion follows from the Martin–Solovay construction.

We now proceed to give the proof of the main theorem, assuming three facts that will be proved in later sections. As in the proof of the Derived Model Theorem, part (1) will be obtained as an immediate consequence of a Σ^2_1 reflection property of the model $\mathcal{M}$, as follows.

Lemma 5.6 (8.1). *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter, and let $\mathcal{M}$ be the model $L^F(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)}$. For every sentence φ, if there is a set of reals $A \in \mathcal{M}$ such that $(\mathrm{HC}^*_G, \in, A) \models \varphi$, then there is a set of reals $A \in \mathrm{Hom}^V_{<\lambda}$ such that $(\mathrm{HC}^V, \in, A) \models \varphi$.*

Taking Lemma 8.1 for granted, we immediately get that $\mathcal{M} \models \mathsf{AD}$, since $<\lambda$-homogeneously Suslin sets of reals are determined whenever λ is greater than a measurable cardinal, a fact which was essentially shown by Martin in the course of proving determinacy for $\boldsymbol{\Pi}^1_1$ sets of reals from a measurable cardinal (see Theorem 33.31 of [4] or Exercise 32.2 of [5]).

Lemma 8.1 also implies that $\mathcal{M} \models \mathsf{AD}^+$. One way to see this is to note that AD^+ can be reformulated as a Π^2_1 statement that is true in the pointclass $\mathrm{Hom}^V_{<\lambda}$. In more (standard) detail, the axiom AD^+ is the conjunction of the following three statements:

- ○ $\mathsf{DC}_{\mathbb{R}}$
- ○ Every set of reals is ∞-Borel.
- ○ $<\Theta$-Determinacy.

The axiom $\mathsf{DC}_{\mathbb{R}}$ says that every tree on $\mathbb{R}$ without terminal nodes has an infinite branch. Using a pairing function on $\mathbb{R}$, we can represent such a tree T with the set A of reals coding sequences in T. Since homogeneously Suslin sets are Suslin, they can be uniformized. It follows that if A is homogeneously Suslin, then there is a function f picking for each $x \in A$ a $y \in A$ coding a proper extension in T of the node coded by x. Starting with any $x \in A$ and coding the sequence $\langle f^i(x) : i \in \omega\rangle$ with a real, we get that an infinite branch through A exists in $(\mathrm{HC}, \in, A)$. This shows that $\mathsf{DC}_{\mathbb{R}}$ holds (in V) for all trees on $\mathbb{R}$ coded by homogeneously Suslin trees. Lemma 8.1 then implies that it holds for all trees on $\mathbb{R}$ in $\mathcal{M}$.

Similarly, AD implies that $<\Theta$-Determinacy holds for games with Suslin, co-Suslin payoff (see Corollary 7.3 of [11]). Since $<\lambda$-homogeneously Suslin sets are Suslin and co-Suslin (being $<\lambda$-universally Baire by the remarks above), and the games referred to in the definition of $<\Theta$-Determinacy can be coded by sets of reals, Lemma 8.1 then implies that $<\Theta$-Determinacy holds in $\mathcal{M}$. Finally, the arguments in Section 8.2 of [11] show that if a set $A \subseteq \omega^\omega$ is ∞-Borel, then A has an ∞-Borel code coded by a set of reals which is $\boldsymbol{\Sigma}^1_1$ in a prewellordering which is Wadge below either A or its complement (i.e., one coded in the structure $(\mathrm{HC}, \in, A)$). Every homogeneously Suslin set A is Suslin and therefore has an ∞-Borel code definable over $(\mathrm{HC}, \in A)$.

The proof of the Σ^2_1 reflection property (i.e., Lemma 8.1) is the most technically demanding part of this paper. It will be given in Section 8 after some background information and preliminary work are presented in Sections 6 and 7, respectively. Our proof will be similar to that given by Steel [22] for the model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$. Roughly speaking, the proof can be adapted to the model $\mathcal{M}$ because the information added by the predicate F is canonical.

Next, we outline the proof of part (2) of the main theorem. It suffices to prove the inclusion $\mathcal{P}(\mathbb{R})^{\mathcal{M}} \subseteq \mathrm{Hom}^*_G$, since the reverse inclusion holds by the definition of the model $\mathcal{M}$. Let A be a set of reals in $\mathcal{M}$. Note that AD^+ holds in the model $L(A, \mathbb{R}^*_G)$ since it holds in $\mathcal{M}$, and due to its equivalent Π^2_1 reformulation, AD^+ is downward absolute to transitive inner models with the same reals (Theorem 8.22 of [11]). Therefore, A is in the new derived model $D(V, \lambda, G)$ (which was defined in Remark 5.4). Since λ is a limit of $<\lambda$-strong cardinals, $D(V, \lambda, G)$ satisfies the statement that every set of reals is Suslin, by

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a theorem of Woodin which appears as Theorem 9.1 below. Therefore, $A \in \mathrm{ScS}^{D(V,\lambda,G)} \subseteq \mathrm{ScS}^{V(\mathbb{R}^*_G)} = \mathrm{Hom}^*_G$, so we have $A \in \mathrm{Hom}^*_G$.

To prove part (3) of the main theorem, we begin with the observation that every set in Hom^*_G is $<$Ord-universally Baire in the symmetric model $V(\mathbb{R}^*_G)$. This follows from the assumption that λ is a limit of strong cardinals by a standard argument: a pair of trees (S,T) for a Hom^*_G set appears $V[G\restriction\xi]$ for some $\xi < \lambda$, where it is λ-absolutely complementing. Since λ is a limit of strong cardinals in V, there is a strong cardinal $\kappa < \lambda$ in $V[G\restriction\xi]$. For any cardinal χ, then there is an elementary embedding $j\colon V \to M$ with critical point κ and $j(\kappa) > \chi$ and $V_\chi \subseteq M$. Then $(j(S), j(T))$ is a χ-complementing pair in $V(\mathbb{R}^*_G)$, and $\mathrm{p}[S] = \mathrm{p}[j(S)]$.

It then follows that sets in Hom^*_G are (fully) universally Baire in $V(\mathbb{R}^*_G)$, by the following lemma.

Lemma 5.7. *Let λ be a cardinal, and let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then in the symmetric model $V(\mathbb{R}^*_G)$, every $<$Ord-universally Baire set of reals is universally Baire. Moreover, this holds locally: for every set Z, there is a cardinal η such that every η^+-absolutely complementing pair of trees on $\omega \times \mathrm{Ord}$ is $\mathrm{Col}(\omega, Z)$-absolutely complementing.*

Proof. Fix Z, let $\eta = \max(\lambda, |Z|^{V[G]})$, and let (S,T) be an η^+-absolutely complementing pair of trees in $V(\mathbb{R}^*_G)$. Then, as above, because S and T are subsets of the ground model, a standard homogeneity argument shows that $(S,T) \in V[G\restriction\xi]$ for some ordinal $\xi < \lambda$. The pair (S,T) is η^+-absolutely complementing in the model $V[G\restriction\xi]$ also because this property is downward absolute (since $\lambda \leq \eta$ the cardinal successor of η is the same in all models under consideration).

Since $\eta \geq \lambda$, every generic extension of $V[G]$ by the poset $\mathrm{Col}(\omega,\eta)$ is also a generic extension of $V[G\restriction\xi]$ by the poset $\mathrm{Col}(\omega,\eta)$. It follows that the pair (S,T) is η^+-absolutely complementing in $V[G]$. Since $\eta \geq |Z|^{V[G]}$, this implies that the pair (S,T) is $\mathrm{Col}(\omega, Z)$-absolutely complementing in $V[G]$. This property is downward absolute, so the pair (S,T) is $\mathrm{Col}(\omega, Z)$-absolutely complementing in $V(\mathbb{R}^*_G)$, as desired. □

To prove part (3) of the main theorem, it remains to show that the universal Baireness of Hom^*_G sets in $V(\mathbb{R}^*_G)$ is absorbed by the model $\mathcal{M}$ via the F predicate. This would follow from Theorem 4.2 if we knew that each member of Hom^*_G carried a semiscale coded by a set in Hom^*_G. This fact is proved in Section 9, giving the following lemma, finishing the proof of part (3), and part (1), by Remark 5.3.

Lemma 5.8 (9.9). *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, and let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then every set of reals in Hom^*_G is universally Baire in $L^F(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)}$.*

For part (4) of the main theorem, parts (2) and (3), along with Theorem 4.3 (with $\Gamma = \mathrm{Hom}^*_G$) imply that the restriction of F to $\mathcal{M}$ is the same as the predicate F defined in $\mathcal{M}$. As in Corollary 4.4, it follows that $\mathcal{M}$ sees its own construction (in particular, $\mathcal{M} = L^F(\mathbb{R}, \mathcal{P}(\mathbb{R}))^{\mathcal{M}}$), so every element of $\mathcal{M}$ is ordinal-definable in $\mathcal{M}$ from a member of $\mathcal{P}(\mathbb{R})^{\mathcal{M}}$.

For the last part of the theorem (the minimality property of $\mathcal{M}$), note that if $\mathcal{M}' \subseteq V(\mathbb{R}^*_G)$ contains Hom^*_G and satisfies the statement that all sets of reals are universally Baire (and thus Suslin), then, since $\mathrm{Hom}^*_G = \mathrm{ScS}^{V(\mathbb{R}^*_G)}$, $\mathrm{Hom}^*_G = \mathcal{P}(\mathbb{R})^{\mathcal{M}'}$, and in particular Hom^*_G, is an element of $\mathcal{M}'$. It follows from Lemma 4.3 then that $\mathcal{M} \subseteq \mathcal{M}'$.

To finish the proof of the main theorem, then, it remains only to prove Lemmas 8.1 and 9.9, and Theorem 9.1.

6. ℝ-genericity iterations

As in Steel's stationary-tower-free proof [22] of Σ^2_1 reflection for the model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$, our proof of Σ^2_1 reflection for the model $\mathcal{M}$ of the main theorem will make fundamental use of the genericity iterations of Neeman [17], repeated ω many times to produce an ℝ-genericity iteration. We refer the

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reader also to Neeman [18] for a thorough account of the use of $\mathbb{R}$-genericity iterations to prove AD in $L(\mathbb{R}^*_G)$ by a similar reflection argument.

The definitions we give below are more or less standard; we include them here to set out the notation that we will use in the rest of the paper. In the following definition, 'sufficient fragment' can be taken to mean sufficiently large for the definition (e.g., Woodin cardinals and generic extensions) to make sense. In practice, P_0 will be the transitive collapse of a suitably large rank initial segment of the universe.

Definition 6.1. Let P_0 be a model of a sufficient fragment of ZFC, and let $\bar{\lambda}$ be a limit of Woodin cardinals of P_0. An $\mathbb{R}$-*genericity iteration of* P_0 *at* $\bar{\lambda}$ is a sequence

$$\langle P_j, i_{jk}, x_\ell, \delta_\ell, g_\ell : j \leq k \leq \omega, \ell < \omega \rangle,$$

existing in a generic extension of V by $\mathrm{Col}(\omega, \mathbb{R})$, such that

1. P_j is in V, for each $j < \omega$,
2. i_{jk} is an elementary embedding from P_j to P_k, for all $j \leq k \leq \omega$,
3. i_{jk} is an element of V, whenever $j \leq k < \omega$,
4. $x_\ell \in \mathbb{R}^V$, for all $\ell < \omega$,
5. each δ_ℓ is a Woodin cardinal below $i_{0,\ell}(\bar{\lambda})$ in P_ℓ,
6. for all $\ell < \omega$, $g_\ell \in V$ is a $P_{\ell+1}$-generic filter for the poset $\mathrm{Col}(\omega, i_{\ell,\ell+1}(\delta_\ell))$ such that $x_\ell \in P_{\ell+1}[g_\ell]$,
7. $i_{kp} \circ i_{jk} = i_{jp}$ whenever $j \leq k \leq p \leq \omega$,
8. P_ω and the maps $i_{j,\omega}$ $(j < \omega)$ are obtained as direct limits from the maps i_{jk} $(j \leq k < \omega)$,
9. $i_{j,j+1}(\delta_j) < \delta_{j+1}$ for all $j < \omega$,
10. for all $j < k < \omega$, the map $i_{j+1,k+1}$ has critical point above $i_{j,j+1}(\delta_j)$,
11. $\{x_j : x < \omega\} = \mathbb{R}^V$,
12. $i_{0,\omega}(\bar{\lambda}) = \sup\{i_{j,\omega}(\delta_j) : j < \omega\}$ and
13. there is a P_ω-generic filter $g \subseteq \mathrm{Col}(\omega, {<}i_{0,\omega}(\bar{\lambda}))$ such that

$$\mathbb{R}^V = \bigcup\{\mathbb{R} \cap P_\omega[g \restriction \xi] : \xi < i_{0,\omega}(\bar{\lambda})\}.$$

Parallel to the notation in Section 5, in the context of item (13) above, we write $P_\omega(\mathbb{R}^V)$ for

$$\mathrm{HOD}^{P_\omega[g]}_{P_\omega \cup \mathbb{R}^V \cup \{\mathbb{R}^V\}}.$$

By item (10), each embedding $i_{j+1,k+1}$ extends to an elementary embedding from $P_{j+1}[g_j]$ to $P_{k+1}[g_j]$, which we call $i^*_{j+1,k+1}$.

Remark 6.2. Each partial order $\mathrm{Col}(\omega, i_{\ell,\ell+1}(\delta_\ell))$ as in item (6) above is forcing-equivalent to the corresponding partial order

$$\mathrm{Col}(\omega, (i_{\ell-1,\ell+1}(\delta_{\ell-1}), i_{\ell,\ell+1}(\delta_\ell)))$$

in the case $\ell > 0$, and $\mathrm{Col}(\omega \leq i_{01}(\delta_0))$ in the case $\ell = 0$. In practice (i.e., in the proof of Lemma 6.6), the generic filter g in item (13) is the union of the images of the filters g_ℓ under isomorphisms witnessing these forcing-equivalences (via bookkeeping that we leave to the reader).

To ensure that the symmetric model $P_\omega(\mathbb{R}^V)$ resembles V in some sense, we will use $\mathbb{R}$-genericity iterations of countable hulls P_0 of rank initial segments of V where the iteration maps factor into the uncollapse map of the hull. The notation below suppresses γ, which is implicitly assumed to be associated with the map π_0.

Definition 6.3. Let λ be a limit of Woodin cardinals, let $\gamma > \lambda$ be an ordinal, and let $\pi_0 \colon P_0 \to V_\gamma$ be an elementary embedding of a countable transitive set P_0 into V such that $\lambda \in \mathrm{ran}(\pi_0)$.

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An $\mathbb{R}$-genericity iteration of P_0 at $\pi_0^{-1}(\lambda)$, as in Definition 6.1, is π_0-*realizable* if there are elementary embeddings $\pi_j : P_j \to V_\gamma$ for $j \leq \omega$ that commute with the iteration maps, meaning that $\pi_k \circ i_{jk} = \pi_j$ for $j \leq k \leq \omega$, and such that $\pi_j \in V$ for all $j < \omega$.

For simplicity, we will sometimes abuse notation by referring to a π_0-realizable genericity iteration of P_0 at $\pi_0^{-1}(\lambda)$ as a π_0-realizable genericity iteration of P_0 at λ; this should not cause any confusion because λ is a limit of Woodin cardinals and $\pi_0^{-1}(\lambda)$ is a countable ordinal.

Remark 6.4. If C is a λ-universally Baire set of reals, $\gamma > \lambda$ is a limit ordinal, P_0 is a countable transitive set and $\pi_0 \colon P_0 \to V_\gamma$ is an elementary embedding with $(C, \lambda) \in \mathrm{ran}(\pi_0)$, then for every π_0-realizable genericity iteration $P_0 \to P_\omega$ at λ, we have $C \in P_\omega(\mathbb{R}^V)$, and in particular, C is equal to $\pi_\omega^{-1}(C)^{P_\omega(\mathbb{R}^V)}$, the canonical expansion of $\pi_\omega^{-1}(C)$ in $P_\omega(\mathbb{R}^V)$. To see this, note that $\mathrm{ran}(\pi_0)$ contains a λ-absolutely complementing pair of trees (S, T) for C. Then $\mathrm{p}[\pi_\omega^{-1}(S)] \subseteq \mathrm{p}[S] = C$ and $\mathrm{p}[\pi_\omega^{-1}(T)] \subseteq \mathrm{p}[T]$, so the canonical expansion of $\pi_\omega^{-1}(C)$ in $P_\omega(\mathbb{R}^V)$ is equal to C.

Remark 6.5. The notion of $\mathbb{R}$-genericity iteration given by Definition 6.1 is weaker than the usual one because it does not require the elementary embeddings $i_{jk} \colon P_j \to P_k$ to be iteration maps. (We could have called them something like '$\mathbb{R}$-genericity systems' instead.) We do it this way so that the reader unfamiliar with iteration trees can take the following lemma (which is implicit in Steel [22]) as a black box; its proof is the only place where iteration trees will appear in this paper.

The conditions on γ in the hypotheses of Lemma 6.6 are chosen to make V_γ satisfy the theory in Definition 1.1 of [13], with respect to λ. Recall from Remark 5.5 that $\mathrm{Hom}_{<\lambda} = \mathrm{uB}_\lambda$ when λ is a limit of Woodin cardinals. Remark 6.4 shows that the expression $P_\omega(\mathbb{R}^V) \models \psi[C, x]$ in the statement of the lemma makes sense (i.e., that C is in $P_\omega(\mathbb{R}^V)$).

Lemma 6.6. *Let λ be a limit of Woodin cardinals, and let C be a λ-universally Baire set of reals. Let P_0 be a countable transitive set, and let γ be a limit ordinal of cofinality greater than λ. Let $\pi_0 \colon P_0 \to V_\gamma$ be an elementary embedding with $(C, \lambda) \in \mathrm{ran}(\pi_0)$. Suppose that A is a set of reals in V and that there is a binary formula ψ such that for every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ in $V^{\mathrm{Col}(\omega,\mathbb{R})}$ and every real $x \in V$, we have*

$$x \in A \iff P_\omega(\mathbb{R}^V) \models \psi[C, x].$$

Then $A \in \mathrm{Hom}_{<\lambda}$.

Proof. By the wellfoundedness of the Wadge hierarchy on homogeneously Suslin sets (see Theorem 3.3.5 of [10] and the discussion before it), there is minimal $\kappa < \lambda$ such that every κ-homogeneously Suslin set of reals is $<\lambda$-homogeneously Suslin. By the elementarity of π, we have $\kappa \in \mathrm{ran}(\pi_0)$ and $(S, T) \in \mathrm{ran}(\pi_0)$ for some λ-absolutely complementing pair of trees (S, T) for C. Define $(\bar{\lambda}, \bar{\kappa}, \bar{S}, \bar{T}) = \pi_0^{-1}(\lambda, \kappa, S, T)$. Let δ_0 denote the least Woodin cardinal of P_0 above $\bar{\kappa}$.

Let W denote the set of all (reals coding) ordered pairs $(\mathcal{T}, b)$ such that

- $\mathcal{T}$ is a 2^ω-closed iteration tree on P_0 of length ω (meaning that each extender produces a 2^ω-closed ultrapower when applied to the model from which it is chosen),
- $\mathcal{T}$ is above $\bar{\kappa}$ and based on δ_0 (meaning that each extender chosen has critical point above $\bar{\kappa}$ and von Neumann rank below the corresponding image of δ_0),
- b is a cofinal branch of $\mathcal{T}$, and
- the model $\mathcal{M}_b^{\pi_0\mathcal{T}}$ is wellfounded, where $\pi_0\mathcal{T}$ is the lifted tree on V_γ.

As shown by Martin and Steel [13, p. 27], the condition that the model $\mathcal{M}_b^{\pi_0\mathcal{T}}$ is wellfounded implies that the branch b is π_0-realizable (meaning that the branch embedding $i_b^{\mathcal{T}}$ factors into π_0), which in turn implies that the model $\mathcal{M}_b^{\mathcal{T}}$ is wellfounded.

By a result of K. Windszus (see Steel [22, Lemma 1.1]) the set W is κ-homogeneously Suslin and therefore $<\lambda$-homogeneously Suslin by our choice of κ. (We added the condition that $\mathcal{T}$ is based on

δ_0, which is harmless and convenient.) Note that essentially all of the complexity of W comes from condition that $\mathcal{M}_b^{\pi_0\mathcal{T}}$ is wellfounded; the other conditions are arithmetic.

Next, we show (following Steel [22, Claim 2 on p. 7]) that the following statements are equivalent, for all $x \in \mathbb{R}$:

1. $x \in A$.
2. There exist a pair $(\mathcal{T}, b) \in W$ and an $\mathcal{M}_b^{\mathcal{T}}$-generic filter

$$g \subseteq \mathrm{Col}(\omega, i_b^{\mathcal{T}}(\delta_0))$$

such that $x \in \mathcal{M}_b^{\mathcal{T}}[g]$ and $\psi\big[\mathrm{p}[i_b^{\mathcal{T}}(\bar{S})], x\big]$ holds in the symmetric extension of $\mathcal{M}_b^{\mathcal{T}}[g]$ at its limit $i_b^{\mathcal{T}}(\bar{\lambda})$ of Woodin cardinals. (It does not matter which symmetric extension, by the homogeneity of the Levy collapse forcing.)

The two directions of the equivalence of (1) and (2) can be proved by constructing a suitable π_0-realizable $\mathbb{R}$-genericity iteration of P_0 at λ. We prove the forward direction first and then note the changes needed for the reverse direction.

Fix $x \in \mathbb{R}$ (at the end of the proof, we will assume in addition that x is in A), and note that because δ_0 is a Woodin cardinal of P_0, by Theorem 7.16 of Neeman [18], there is a 2^ω-closed iteration tree $\mathcal{T}$ on P_0 that is above $\bar{\kappa}$, based on δ_0, of length ω, such that for every cofinal wellfounded branch b of $\mathcal{T}$, there is a $\mathcal{M}_b^{\mathcal{T}}$-generic filter $g \subseteq \mathrm{Col}(\omega, i_b^{\mathcal{T}}(\delta_0))$ such that $x \in \mathcal{M}_b^{\mathcal{T}}[g]$. By Martin and Steel [13, Corollary 5.7], the 2^ω-closed iteration tree $\pi_0\mathcal{T}$ on V_γ has a cofinal wellfounded branch b. For such a branch b, the pair $(\mathcal{T}, b)$ is in W, and there is a $\mathcal{M}_b^{\mathcal{T}}$-generic filter $g \subseteq \mathrm{Col}(\omega, i_b^{\mathcal{T}}(\delta_0))$ such that $x \in \mathcal{M}_b^{\mathcal{T}}[g]$.

Now let $x_0 = x$, and let $\mathcal{T}_0,\ b_0$ and g_0 be such that $(\mathcal{T}_0, b_0) \in W,\ g_0 \subseteq \mathrm{Col}(\omega, i_b^{\mathcal{T}}(\delta_0))$ is an $\mathcal{M}_{b_0}^{\mathcal{T}_0}$-generic filter, and $x \in \mathcal{M}_{b_0}^{\mathcal{T}_0}[g_0]$. (The previous paragraph shows that such $\mathcal{T}_0,\ b_0$ and g_0 exist.) Let $P_1 = \mathcal{M}_{b_0}^{\mathcal{T}_0}$, and let $i_{0,1} = i_{b_0}^{\mathcal{T}_0}$ be the branch embedding from P_0 to P_1. As noted above, the condition that $\mathcal{M}_{b_0}^{\pi_0\mathcal{T}_0}$ is wellfounded implies that the branch b_0 is π_0-realizable, meaning that there is an elementary embedding $\pi_1 : P_1 \to V_\gamma$ such that $\pi_1 \circ i_{0,1} = \pi_0$.

Now with the model P_1 in place of the model P_0, the map π_1 in place of the map π_0, a Woodin cardinal δ_1 of P_1 such that $i_{0,1}(\delta_0) < \delta_1 < i_{0,1}(\bar{\lambda})$ in place of δ_0, and an arbitrary real x_1, we can repeat the process above, applying the Neeman and Martin–Steel theorems again to get an iteration tree $\mathcal{T}_1$ on P_1 of length ω, above $i_{0,1}(\delta_0)$ and based on δ_1, a π_1-realizable branch b_1 of $\mathcal{T}_1$, and an $\mathcal{M}_{b_1}^{\mathcal{T}_1}$-generic filter $g_1 \subseteq \mathrm{Col}(\omega, i_{b_1}^{\mathcal{T}_1}(\delta_1))$ such that $x_1 \in \mathcal{M}_{b_1}^{\mathcal{T}_1}[g_1]$.

We can repeat this process for ω many stages. (Note that, for the purposes of showing the equivalence of (1) and (2), at a stage $j > 0$, it suffices to use a π_j-realizable branch of $\mathcal{T}_j$ given by Martin and Steel [13, Theorem 3.12]; the requirement that $\mathcal{M}_{b_j}^{\pi_j\mathcal{T}_j}$ is wellfounded was only necessary for stage $j = 0$ to establish that $(\mathcal{T}_0, b_0) \in W$.) Furthermore, given a generic enumeration $\{x_j : j < \omega\}$ of $\mathbb{R}^V$ in $V^{\mathrm{Col}(\omega,\mathbb{R})}$, we do this in such a way to produce a π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_1 \to \cdots P_\omega$. With suitable bookkeeping, one can do this in such a way that the Woodin cardinals $i_{j,\omega}(\delta_j)$ for $j < \omega$ are cofinal in $i_{0,\omega}(\bar{\lambda})$, and the generic filters g_j for $j < \omega$ fit together into a P_ω-generic filter $g \subseteq \mathrm{Col}(\omega, {<}i_{0,\omega}(\bar{\lambda}))$ (as in Remark 6.2) such that $\mathbb{R}^V = \bigcup\{\mathbb{R} \cap P_\omega[g{\restriction}\xi] : \xi < i_{0\omega}(\bar{\lambda})\}$.

To see that (1) implies (2), it remains to observe that the condition $x \in A$ is equivalent to the condition $P_\omega(\mathbb{R}^V) \models \psi\big[i_{0,\omega}(\bar{S}), x\big]$ by the hypothesis of the lemma and the fact that $C = \mathrm{p}[i_{0,\omega}(\bar{S})]^{P_\omega(\mathbb{R}^V)}$, as in Remark 6.4. This in turn is equivalent to the condition that $\psi\big[\mathrm{p}[i_{0,1}(\bar{S})], x\big]$ holds in the symmetric extension of $P_1[g_0]$ at its limit $i_{0,1}(\bar{\lambda})$ of Woodin cardinals (i.e., that the first step of our iteration witnessed (2)) by the fact that the iteration map $i_{1,\omega} : P_1 \to P_\omega$ extends to an elementary embedding $i^*_{1,\omega} : P_1[g_0] \to P_\omega[g_0]$.

The proof that (2) implies (1) is the same, except that the first step of the genericity iteration is given by assuming (2), and the two equivalences in the previous paragraph are applied in reverse.

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This characterization of the set A given by the equivalence of (1) and (2) shows that A is a projection of a $<\lambda$-homogeneously Suslin set (essentially all of whose complexity comes from W) – that is, that A is $<\lambda$-weakly homogeneously Suslin. It follows that λ is $<\lambda$-homogeneously Suslin by the main theorem from the proof of projective determinacy (Martin and Steel [12, Theorem 5.11]; see also Theorem 3.3.13 of [10]). □

7. Absoluteness of the *F* predicate

Although our main theorem is concerned only with the predicate F as it is defined in the symmetric model $V(\mathbb{R}^*_G)$, the proof of Σ^2_1 reflection will need to consider the predicate as it is defined in other models (in particular V and $\mathbb{R}$-genericity iterates of countable hulls of rank initial segments of V) and show that the definition satisfies a certain absoluteness property between these models (as in Lemma 7.5 below).

First, we will need a lemma that shows that, in the presence of Woodin cardinals, the universally Baire sets added generically over a countable model that embeds into V are closely related to universally Baire sets in V.

In the presence of Woodin cardinals, the pointclass of (sufficiently) homogeneously Suslin sets is closed under complementation in a strong sense. That is, if δ is a Woodin cardinal, Y is a set of δ^+-complete measures with $|Y| < \delta$, and $\kappa < \delta$ is a cardinal, then by Steel [22, Lemma 2.1], there is a 'tower-flipping' function f that associates to every finite tower $\langle \rho_0, \ldots, \rho_{n-1}\rangle$ of measures from Y a length-n tower $f(\langle \rho_0, \ldots, \rho_{n-1}\rangle)$ of κ-complete measures with the following properties. First, f respects extensions of finite towers, so that to every infinite tower $\vec{\rho}$ of measures from Y, it continuously associates an infinite tower $\bigcup_{n<\omega} f(\vec{\rho}\restriction n)$ of κ-complete measures. Second, this associated tower $\bigcup_{n<\omega} f(\vec{\rho}\restriction n)$ is wellfounded if and only if the given tower $\vec{\rho}$ is ill-founded. Third, this property of the tower-flipping function continues to hold in all generic extensions by posets of size less than κ.

The following lemma shows that in the presence of a Woodin cardinal, the Martin–Solovay construction can be used to produce absolutely complementing trees of a nice form, which we will use in Lemma 7.3 to prove an absoluteness result for the F predicate. A similar idea appears in the proof of Steel [22, Theorem 2.2, Subclaim 1.1].

For a tree T and a natural number n, we let $T\restriction n$ be the subset of T consisting of all nodes of length less than n.

Lemma 7.1. *Let P be a countable transitive set, let γ be a limit ordinal, and let $\pi\colon P \to V_\gamma$ be an elementary embedding. Let $\bar{\alpha}$ be a cardinal of P, and suppose that $\bar{\delta}$ is a Woodin cardinal of P with $\bar{\alpha} < \bar{\delta}$. Let $g \subseteq \mathrm{Col}(\omega, \bar{\alpha})$ be a P-generic filter in V, and let A be a $\bar{\delta}^+$-homogeneously Suslin set of reals in $P[g]$.*

Then for each cardinal $\bar{\kappa}$ of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta}$, there is a $\bar{\kappa}$-absolutely complementing pair of trees $(\bar{S}, \bar{T}) \in P[g]$ for A such that, for every $n < \omega$, the restrictions $\bar{S}\restriction n$ and $\bar{T}\restriction n$ to finite levels are in P, and, letting $S = \bigcup_{n<\omega} \pi(\bar{S}\restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T}\restriction n)$, the pair $(S, T) \in V_\gamma$ is $\pi(\bar{\kappa})$-absolutely complementing.

Proof. Let $A = \mathbf{S}_{\vec{\mu}}$ where $\vec{\mu}$ is a tree of $\bar{\delta}^+$-complete measures in $P[g]$. By the Levy–Solovay theorem, each measure μ_s is induced by the $\bar{\delta}^+$-complete measure $\mu_s \cap P \in P$, which we will also denote by μ_s. In P, there is a set Y consisting of $\bar{\delta}^+$-complete measures such that $|Y|^P = \bar{\alpha}$ every member of $\vec{\mu}$ is induced by a member of Y.

Given a cardinal $\bar{\kappa}$ of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta}$ (as in the statement of the lemma), let $f \in P$ be a tower-flipping function that continuously associates to every tower of measures from Y a tower of $\bar{\kappa}$-complete measures. In $P[g]$, define the tree of $\bar{\kappa}$-complete measures $\vec{\nu} = \langle \nu_s : s \in \omega^{<\omega}\rangle$ that is continuously associated to $\vec{\mu}$ by f in the sense that $\langle \nu_{s\restriction n} : n \leq |s|\rangle = f(\langle \mu_{s\restriction n} : n \leq |s|\rangle)$ for every finite sequence $s \in \omega^{<\omega}$. So in $P[g]$, we have $\mathbf{S}_{\vec{\nu}} = \omega^\omega \setminus \mathbf{S}_{\vec{\mu}}$ by the tower-flipping property of f. Moreover, $\pi(f)$ is a tower-flipping function in V_γ by the elementarity of π, so we have $\mathbf{S}_{\pi``\vec{\nu}} = \omega^\omega \setminus \mathbf{S}_{\pi``\vec{\mu}}$ in V_γ.

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In $P[g]$, define the Martin–Solovay trees $\bar{S} = \mathrm{ms}(\vec{\mu})$ and $\bar{T} = \mathrm{ms}(\vec{\nu})$, and in V_γ, define the Martin–Solovay trees $S = \mathrm{ms}(\pi``\vec{\mu})$ and $T = \mathrm{ms}(\pi``\vec{\nu})$. Because for each $n \in \omega$ the first n levels of the Martin–Solovay tree are determined by finitely many measures (in the ground model, by the absoluteness of the construction of the Martin-Solovay tree between P and $P[g]$ for $\bar{\delta}^+$-complete measures), we have $(\bar{S}\restriction n, \bar{T}\restriction n) \in P$ for all $n < \omega$. For the same reason, we have $S = \bigcup_{n<\omega} \pi(\bar{S}\restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T}\restriction n)$.

In the model $P[g]$, the pair $(\bar{S}, \bar{T})$ is $\bar{\kappa}$-absolutely complementing: this follows from the fact that $\mathrm{p}[\bar{S}] = \omega^\omega \setminus \mathrm{p}[\bar{T}]$ (in $P[g]$) and the fact that $\bar{S}$ and $\bar{T}$, being Martin–Solovay trees, are each $\bar{\kappa}$-absolutely complemented. A similar argument shows that in V, the pair (S, T) is $\pi(\bar{\kappa})$-absolutely complementing, as desired. □

Using a strong cardinal, we can strengthen the previous lemma to give any desired degree of absolute complementation.

Lemma 7.2. *Let P be a countable transitive set, let γ be a limit ordinal, and let $\pi\colon P \to V_\gamma$ be an elementary embedding. Let $\bar{\alpha}$ be a cardinal of P, and let $\bar{\delta}$ be a Woodin cardinal of P with $\bar{\alpha} < \bar{\delta}$. Let $g \subseteq \mathrm{Col}(\omega, \bar{\alpha})$ be a P-generic filter in V, and let A be a $\bar{\delta}^+$-homogeneously Suslin set of reals in $P[g]$.*

Let $\bar{\kappa}$ and $\bar{\eta}$ be cardinals of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta} < \bar{\eta}$ such that $\bar{\kappa}$ is $(\bar{\eta}+1)$-strong in P. Then there is an $\bar{\eta}$-absolutely complementing pair of trees $(\bar{S}, \bar{T}) \in P[g]$ for A such that for every $n < \omega$, the restrictions $\bar{S}\restriction n$ and $\bar{T}\restriction n$ to finite levels are in P, and letting $S = \bigcup_{n<\omega} \pi(\bar{S}\restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T}\restriction n)$, the pair $(S, T) \in V_\gamma$ is $\pi(\bar{\eta})$-absolutely complementing.

Proof. Let $(\kappa, \eta) = \pi(\bar{\kappa}, \bar{\eta})$. By Lemma 7.1, there is a $\bar{\kappa}$-absolutely complementing pair of trees $(\bar{S}_0, \bar{T}_0)$ in $P[g]$ for A such that for every $n < \omega$, the restrictions $\bar{S}_0\restriction n$ and $\bar{T}_0\restriction n$ to finite levels are in P, and letting $S_0 = \bigcup_{n<\omega} \pi(\bar{S}_0\restriction n)$ and $T_0 = \bigcup_{n<\omega} \pi(\bar{T}_0\restriction n)$, the pair $(S_0, T_0) \in V_\gamma$ is κ-absolutely complementing.

In P, let $\bar{E}$ be an extender witnessing that $\bar{\kappa}$ is $(\bar{\eta}+1)$-strong. The ultrapower map $j_{\bar{E}}\colon P \to \mathrm{Ult}(P, \bar{E})$ extends canonically to a map $P[g] \to \mathrm{Ult}(P[g], \bar{E})$, which we will also denote by $j_{\bar{E}}$. In V_γ, the extender $E = \pi(\bar{E})$ witnesses that κ is $(\eta+1)$-strong, and we have an ultrapower map $j_E : V_\gamma \to \mathrm{Ult}(V_\gamma, E)$ with $j_E(\kappa) > \eta$ and $V_{\eta+1} \subseteq \mathrm{Ult}(V_\gamma, E)$.

Letting $\bar{S} = j_{\bar{E}}(\bar{S}_0)$ and $\bar{T} = j_{\bar{E}}(\bar{T}_0)$, the pair of trees $(\bar{S}, \bar{T})$ is an $\bar{\eta}$-absolutely complementing pair for A in $P[g]$. Similarly, letting $S = j_E(S_0)$ and $T = j_E(T_0)$, the pair of trees $(S, T) \in V_\gamma$ is η-absolutely complementing. Because $\pi \circ j_{\bar{E}} = j_E \circ \pi$, we have $S = \bigcup_{n<\omega} \pi(\bar{S}\restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T}\restriction n)$, as desired. □

Applying Lemma 7.2 to $\mathbb{R}$-genericity iterations yields the following result. Part (1) below can also be derived as an immediate consequence of Steel [22, Subclaim 1.1], which applies more generally to any limit λ of Woodin cardinals. However, in our situation, we can use the fact that λ is also a limit of strong cardinals to give a proof of part (1) using Lemma 7.2 instead. Part (2

Lemma 7.3. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, and let γ be a limit ordinal of cofinality greater than λ. Let P_0 be a countable transitive set, and let $\pi_0\colon P_0 \to V_\gamma$ be an elementary embedding with $\lambda \in \mathrm{ran}(\pi_0)$. Let P_ω be obtained from P_0 by a π_0-realizable $\mathbb{R}$-genericity iteration at λ, and let A be a set of reals in $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$. Then the following hold.*

1. $A \in \mathrm{uB}^V$
2. *For every set $Z \in P_\omega(\mathbb{R}^V) \cap V$, every condition $p \in \mathrm{Col}(\omega, Z)$, and every $\mathrm{Col}(\omega, Z)$-name $\dot{x} \in P_\omega(\mathbb{R}^V) \cap V$ for a real, the statement 'p forces $\dot{x}$ to be in the canonical expansion of A' is absolute between $P_\omega(\mathbb{R}^V)$ and V.*

Proof. In terms of the notation for $\mathbb{R}$-genericity iterations given in (and just after) Definition 6.1, the set $A \in \mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ is represented by a pair of absolutely $\pi_\omega^{-1}(\lambda)$-complementing trees in $P_\omega(\mathbb{R}^V)$, which, by the usual homogeneity argument, exist in $P_\omega[g_{j_0}]$, for some $j_0 < \omega$. Since P_ω is the direct limit of $\langle P_i : i < \omega\rangle$, these trees can be taken to be in the range of $i^*_{j,\omega}$ (the canonical extension of $i_{j,\omega}$ to $P_j[g_{j-1}]$) for some positive $j < \omega$. Otherwise stated, A comes from a set $A_j \in \mathrm{uB}^{P_j[g_{j-1}]}$ appearing at stage j, in the sense that $A = i^*_{j,\omega}(A_j)^{P_\omega(\mathbb{R}^V)}$.

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To prove part (1), we apply Lemma 7.2 with $P = P_j$, $\gamma = \gamma$, $\pi = \pi_j$, $g = g_{j-1}$, $A = A_j$, $\bar{\alpha} = i_{j-1,j}(\delta_{j-1})$, $\bar{\kappa}$ equal to the least strong cardinal of P_j above $i_{j-1,j}(\delta_{j-1})$, $\bar{\delta}$ equal to the least Woodin cardinal of P_j above $\bar{\kappa}$, and $\bar{\eta} = \pi_j^{-1}(\lambda)$. This gives us a pair of $\pi_j^{-1}(\lambda)$-absolutely complementing trees $(\bar{S}, \bar{T}) \in P_j[g_{j-1}]$ for A_j such that for every $n < \omega$, the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels are in P_j, and letting $S = \bigcup_{n<\omega} \pi_j(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi_j(\bar{T} \restriction n)$, the pair (S, T) is λ-absolutely complementing in V.

Since $\pi_\omega \circ i_{j,\omega} = \pi_j$, $\pi_\omega(i_{j,\omega}(\bar{S} \restriction n)) = \pi_j(\bar{S} \restriction n)$ and $\pi_\omega(i_{j,\omega}(\bar{T} \restriction n)) = \pi_j(\bar{T} \restriction n)$ for each $n \in \omega$, so $S = \pi_\omega(i^*_{j,\omega}(\bar{S}))$ and $T = \pi_\omega(i^*_{j,\omega}(\bar{T}))$. This implies that $\pi_\omega``(i^*_{j,\omega}(\bar{S})) \subseteq S$ and $\pi_\omega``(i^*_{j,\omega}(\bar{T})) \subseteq T$. It follows that the realization map π_ω takes branches of $i^*_{j,\omega}(\bar{S})$ and $i^*_{j,\omega}(\bar{T})$ pointwise to branches of S and T, respectively, so the projections of these trees satisfy the inclusions $\mathrm{p}[i^*_{j,\omega}(\bar{S})] \subseteq \mathrm{p}[S]$ and $\mathrm{p}[i^*_{j,\omega}(\bar{T})] \subseteq \mathrm{p}[T]$.

Because $A = \mathrm{p}[i^*_{j,\omega}(\bar{S})] = \omega^\omega \setminus \mathrm{p}[i^*_{j,\omega}(\bar{T})]$ in $P_\omega(\mathbb{R}^V)$, these two inclusions imply that $A = \mathrm{p}[S] = \omega^\omega \setminus \mathrm{p}[T]$ in V. Therefore, A is in uB^V_λ, which is equal to uB^V because there is a strong cardinal less than λ.

For part (2), note that our genericity iteration was formed in some generic extension of V_γ by the poset $\mathrm{Col}(\omega, \mathbb{R})$, and the resulting model $P_\omega(\mathbb{R}^V)$ is countable there. In particular, the set Z is countable there. So in a generic extension of V_γ by the poset $\mathrm{Col}(\omega, \mathbb{R}) \times \mathrm{Col}(\omega, \omega)$, which is forcing-equivalent to $\mathrm{Col}(\omega, \mathbb{R})$, we can find a filter $H \subseteq \mathrm{Col}(\omega, Z)$ that contains the condition p and is both V-generic and $P_\omega(\mathbb{R}^V)$-generic. Let $x = \dot{x}_H$. We want to see that the two ways of expanding the set A, given by $A \in \mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and $A \in \mathrm{uB}^V$, respectively, agree on whether x is a member. This would follow from Lemma 2.5 if $P_\omega(\mathbb{R}^V) \in V$, but it seems that the condition $P_\omega(\mathbb{R}^V) \in V$ may fail in general, which is why we seem to need Lemma 7.2.

By Lemma 5.7, in the model $P_\omega(\mathbb{R}^V)$, there is an ordinal $\eta_\omega \geq \pi_\omega^{-1}(\lambda)$ such that every η_ω-absolutely complementing set of trees is Z-absolutely complementing. Increasing j (from the first paragraph of the proof of the lemma) if necessary, we may assume that $\eta_\omega = i_{j,\omega}(\bar{\eta})$ for some ordinal $\bar{\eta} \in P_j$.

Now we can apply Lemma 7.2 as in the proof of part (1), except that now $\bar{\eta}$ may be strictly larger than $\pi_j^{-1}(\lambda)$, to get a pair of $\bar{\eta}$-absolutely complementing trees $(\bar{S}, \bar{T}) \in P_j[g_{j-1}]$ for A_j such that for every $n < \omega$, the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels are in P_j, and letting $S = \bigcup_{n<\omega} \pi_j(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi_j(\bar{T} \restriction n)$ in V_γ, the pair (S, T) is η-absolutely complementing where $\eta = \pi_j(\bar{\eta})$. In particular, the pair (S, T) is $\mathbb{R}$-absolutely complementing. Moreover, the pair $(S_\omega, T_\omega) = (i^*_{j,\omega}(\bar{S}), i^*_{j,\omega}(\bar{T}))$ is η_ω-absolutely complementing in $P_\omega[g_{j-1}]$, so also in $P_\omega(\mathbb{R}^V)$. It follows from our choice of η_ω that (S_ω, T_ω) is Z-absolutely complementing in $P_\omega(\mathbb{R}^V)$.

We can use the pair of trees (S_ω, T_ω) to decide membership of x in the canonical expansion of A from the point of view of $P_\omega(\mathbb{R}^V)$, and use the pair of trees (S, T) to decide membership of x in the canonical expansion of A from the point of view of V. As in the proof of part (1), we get the same answer in both cases because the realization map π_ω takes branches of S_ω pointwise to branches of S and takes branches of T_ω pointwise to branches of T. □

The first part of Lemma 7.3 implies (assuming its hypotheses) that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ is a Wadge-initial segment of uB^V. In conjunction with the second part, this implies that

$$L^F(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)} \subseteq L^F(\mathbb{R}, \mathrm{uB})^V.$$

Lemma 7.5 below is a stronger statement that implies this inclusion, phrased in terms of the following definition.

Definition 7.4. Given an ordinal α, we let $\mathrm{uB} \restriction \alpha$ denote the collection of universally Baire sets of Wadge rank less than α, and let F_α be the class of all quadruples $(A, \mathbb{P}, p, \dot{x})$ in F for which $A \in \mathrm{uB} \restriction \alpha$.

If α is at least the Wadge rank of the pointclass uB, then we simply have $\mathrm{uB} \restriction \alpha = \mathrm{uB}$. Using these definitions, we can state an immediate consequence of Lemma 7.3:

Lemma 7.5. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, and let γ be a limit ordinal of cofinality greater than λ. Let P_0 be a countable transitive set, and let $\pi_0 : P_0 \to V_\gamma$ be an elementary embedding with $\lambda \in \operatorname{ran}(\pi_0)$. Let P_ω be obtained from P_0 by a π_0-realizable $\mathbb{R}$-genericity iteration at λ. Then we have*

$$L^F(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)} = L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V,$$

where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and β is the ordinal height of the model P_ω.

8. Σ_1^2 reflection

In this section, we prove the following Σ_1^2 reflection result, following Steel's stationary-tower-free proof [22] of Σ_1^2 reflection for the model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$.

Lemma 8.1. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter, and let $\mathcal{M}$ be the model $L^F(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)}$. For every sentence φ, if there is a set of reals $A \in \mathcal{M}$ such that $(\mathrm{HC}^*_G, \in, A) \models \varphi$, then there is a set of reals $A \in \mathrm{Hom}^V_{<\lambda}$ such that $(\mathrm{HC}^V, \in, A) \models \varphi$.*

Recall that $\mathrm{Hom}^*_G = \mathrm{uB}^{V(\mathbb{R}^*_G)}$, as λ is a limit of strong cardinals, so we have an alternative characterization of $\mathcal{M}$ that will be useful in this section:

$$\mathcal{M} = L^F(\mathbb{R}^*_G, \mathrm{uB})^{V(\mathbb{R}^*_G)}.$$

Supposing that the model $\mathcal{M}$ has a φ-witness, the idea of the proof is to take a countable hull of some V_γ containing a φ-witness, and then to do an $\mathbb{R}$-genericity iteration of the hull to get a φ-witness that is a subset of $\mathbb{R}^V$. More specifically, we start by fixing a limit ordinal γ of cofinality greater than λ, so that every set of reals in $\mathcal{M}$ is in $L^F_\gamma(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V_\gamma(\mathbb{R}^*_G)}$. Considering an $\mathbb{R}$-genericity iteration of the transitive collapse P_0 of an elementary substructure of V_γ, the elementarity of the corresponding map $\pi_\omega : P_\omega \to V_\gamma$ (and the definability of forcing) gives a φ-witness in the model $L^F(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)}$, which is equal to the model $L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$ by Lemma 7.5, where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and β is the ordinal height of the model P_ω. In particular, there is a φ-witness in the model $L^F(\mathbb{R}, \mathrm{uB})^V$.

We will show that the least φ-witness arising in this manner is in $\mathrm{Hom}_{<\lambda}$. To do this, we will consider different countable hulls P_0 and different genericity iterations of these hulls giving rise to possibly different models of the form $L^F(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)}$. We will see that the least φ-witness will be present in all of these models and will admit a uniform definition there in the sense of Lemma 6.6.

We consider two cases involving two different notions of least φ-witness. The cases are exhaustive but not necessarily mutually exclusive; if they overlap, either one can be used.

Case 1. For cofinally many elementary countable hulls $\pi_0 : P_0 \to V_\gamma$ with $\lambda \in \operatorname{ran}(\pi_0)$, every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ has the property that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} = \mathrm{uB}^V$.

In this case, let β be the least ordinal such that the model $L^F_\beta(\mathbb{R}, \mathrm{uB})^V$ has a φ-witness (which is less than γ, as noted above). Take a set of reals $C \in \mathrm{uB}^V$ such that this model contains a φ-witness that is ordinal-definable from the parameter C and the predicate F, and let $A \in L^F_\beta(\mathbb{R}, \mathrm{uB})^V$ be the least such φ-witness in the canonical well-ordering of sets ordinal definable over $L^F_\beta(\mathbb{R}, \mathrm{uB})^V$ from C and F.

Take a countable hull $\pi_0 : P_0 \to V_\gamma$ such that $(C, \lambda) \in \operatorname{ran}(\pi_0)$, and for every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ, we have $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} = \mathrm{uB}^V$. For every such $\mathbb{R}$-genericity iteration, we have $\beta \in P_\omega$ by the minimality of β; otherwise, the model $L^F(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)}$ would not be tall enough to reach a φ-witness (by Lemma 7.5), violating the elementarity of π_ω. Moreover, the set C is in $P_\omega(\mathbb{R}^V)$ because it is the canonical expansion of the set $\pi_\omega^{-1}(C) \in P_\omega$, as in Remark 6.4.

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For every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ, the ordinal β is definable in $P_\omega(\mathbb{R}^V)$ as the least ordinal such that the model $L^F_\beta(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)}$ contains a φ-witness, and in turn, the least φ-witness A is definable from C in the model $L^F_\beta(\mathbb{R}, \mathrm{uB})^{P_\omega(\mathbb{R}^V)}$ by the same definition as in the model $L^F_\beta(\mathbb{R}, \mathrm{uB})^V$, which is the same model. This implies that $A \in \mathrm{Hom}^V_{<\lambda}$ by Lemma 6.6 as desired.

Case 2. For cofinally many sufficiently elementary countable hulls $\pi_0 \colon P_0 \to V_\gamma$ with $\lambda \in \mathrm{ran}(\pi_0)$, there is a π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ such that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$.

In this case, we will need the following claim, which says that we can put an upper bound on the pointclass necessary to construct a φ-witness.

Claim 8.2. *Under the assumption of Case 2, there is a set of reals $D \in \mathrm{uB}^V$ such that, letting D^* denote the canonical expansion $D^{V(\mathbb{R}^*_G)}$ and letting α^* denote the Wadge rank of D^* in $V(\mathbb{R}^*_G)$, the model*

$$L^{F_{\alpha^*}}(\mathbb{R}^*_G, \mathrm{uB}\restriction\alpha^*)^{V(\mathbb{R}^*_G)}$$

contains a φ-witness.

Proof. Take an elementary countable hull $\pi_0 \colon P_0 \to V_\gamma$ with $\lambda \in \mathrm{ran}(\pi_0)$, and take a π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ such that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$. Then by Lemma 7.5 and the elementarity of the realization map $\pi_\omega \colon P_\omega \to V_\gamma$, there is a φ-witness in the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$, where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and $\beta = \mathrm{Ord}^{P_\omega}$.

Take a set of reals $D \in \mathrm{uB}^V$ of Wadge rank α, take a sufficiently elementary countable hull $\pi'_0 \colon P'_0 \to V_\gamma$ with $(D, \lambda) \in \mathrm{ran}(\pi'_0)$, and take a π'_0-realizable $\mathbb{R}$-genericity iteration $P'_0 \to P'_\omega$ at λ such that $\mathrm{uB}^{P'_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$.

The model $P'_\omega(\mathbb{R}^V)$ sees the set D as the canonical expansion of $\pi_\omega^{-1}(D)$, and it sees the pointclass $\mathrm{uB}^V\restriction\alpha$ as its collection of uB sets of Wadge rank less than that of D. If we have $\mathrm{Ord}^{P_\omega} \leq \mathrm{Ord}^{P'_\omega}$, then the model $P'_\omega(\mathbb{R}^V)$ is tall enough to see that a φ-witness can be built from $\mathrm{uB}^V\restriction\alpha$ using the F predicate, and the claim follows from the elementarity of the realization map $\pi'_\omega \colon P'_\omega \to V_\gamma$.

However, if $\mathrm{Ord}^{P_\omega} > \mathrm{Ord}^{P'_\omega}$, then we can repeat this procedure with the hull $\pi'_0 \colon P'_0 \to V_\gamma$ in place of the hull $\pi_0 \colon P_0 \to V_\gamma$. (This is why we made sure to take the second hull with $\mathrm{uB}^{P'_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$ also.) Repeating this procedure for ω many steps, either at some step, we get a set $D \in \mathrm{uB}^V$ satisfying the claim, or else we get an infinite decreasing sequence of ordinals $\mathrm{Ord}^{P_\omega} > \mathrm{Ord}^{P'_\omega} > \mathrm{Ord}^{P''_\omega} > \cdots$, a contradiction. □

Now we can proceed as in Case 1. Let $D \in \mathrm{uB}^V$ be a set as in the claim. Taking any countable hull $\pi_0 : P_0 \to V_\gamma$ with $D \in \mathrm{ran}(\pi_0)$ and taking any genericity iterate $P_0 \to P_\omega$ of this hull, by the elementarity of the factor map π_ω and the proof of Lemma 7.5, we see that there is a φ-witness in the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$ for some ordinal $\beta < \gamma$, where α is the Wadge rank of D. Let β be the least ordinal such that there is a φ-witness in the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$.

Take a set of reals $C \in \mathrm{uB}^V\restriction\alpha$ such that the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$ has a φ-witness that is ordinal-definable from the parameter C and the predicate F, and let $A \in L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$ be the least such φ-witness in the canonical well-ordering relative to C.

Take a countable hull $\pi_0 \colon P_0 \to V_\gamma$ with $(\lambda, C, D) \in \mathrm{ran}(\pi_0)$. For every $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$, we have $\beta \in P_\omega$ by the minimality of β; otherwise, the model $L^{F_\alpha}(\mathbb{R}, \mathrm{uB}\restriction\alpha)^{P_\omega(\mathbb{R}^V)}$ would not be tall enough to reach a φ-witness, violating the elementarity of π_ω.

For every such genericity iteration, the ordinal α is definable from the set D in $P_\omega(\mathbb{R}^V)$ as its Wadge rank, the ordinal β is definable from α in $P_\omega(\mathbb{R}^V)$ as the least ordinal such that the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^{P_\omega(\mathbb{R}^V)}$ contains a φ-witness, and in turn, the least φ-witness A is definable from C in the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^{P_\omega(\mathbb{R}^V)}$ by the same definition as in the model $L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB}\restriction\alpha)^V$,which is the same model. This implies that $A \in \mathrm{Hom}_{<\lambda}$ by Lemma 6.6 as desired.

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9. Suslin sets in the derived model

This section is mostly an exposition of the following theorem regarding the derived model $D(V, \lambda, G)$, which was defined in Remark 5.4. A proof of the corresponding theorem (also due to Woodin) for the old dervied model $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$ can be found in Steel [23, Section 9].

Theorem 9.1 (Woodin). *Let λ be a limit of Woodin cardinals and of $<\lambda$-strong cardinals. Let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then the derived model $D(V, \lambda, G)$ satisfies the statement that every set of reals is Suslin.*

Remark 9.2. As shown at the beginning of Section 5, Hom^*_G is equal to the set of subsets of $\mathbb{R}^*_G$ in $V(\mathbb{R}^*_G)$ which are Suslin and co-Suslin in $V(\mathbb{R}^*_G)$. It follows from Theorem 9.1 then that (assuming its hypotheses) every set of reals in $D(V, \lambda, G)$ is in Hom^*_G, so $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G) = D(V, \lambda, G)$.

The proof we give for Theorem 9.1 is based on one by Steel,[2] with a couple of modifications. One modification is that we separate the argument into two parts. The first part is Lemma 9.5 below, which uses only the assumption that λ is a limit of $<\lambda$-strong cardinals, and which may be of independent interest. The remainder of the argument will use (in addition to the conclusion of Lemma 9.5) only the assumption that λ is a limit of Woodin cardinals. This large cardinal hypothesis implies that $\mathrm{Hom}^*_G \subseteq \Gamma_+$ (by the version of the derived model theorem which appears as Theorem 7.1 in [23]) and that $D(V, \lambda, G)$ satisfies AD^+, which is the first part of Theorem 31 of [24].

The second modification is that we consider Suslin-norms (as defined below) rather than Suslin wellfounded relations as in Steel's proof. This seems to be necessary in order to split up the proof as we have done.

Given a norm φ on a set of reals A, the *code* of φ is the pair of binary relations $(R_\leq, R_<)$ defined by

$$R_\leq = \{(x, y) \in \omega^\omega \times \omega^\omega : x \in A \wedge (y \in A \rightarrow \varphi(x) \leq \varphi(y))\}$$
$$R_< = \{(x, y) \in \omega^\omega \times \omega^\omega : x \in A \wedge (y \in A \rightarrow \varphi(x) < \varphi(y))\}.$$

Note that the property of being a code for some norm on A can be expressed as the conjunction of the following conditions:

- $R_\leq \cap (A \times A)$ is a prewellordering on A,
- $R_< \cap (A \times A)$ is the strict part of this prewellordering,
- $(A \times \neg A) \subseteq R_\leq \subseteq (A \times \omega^\omega)$ and
- $(A \times \neg A) \subseteq R_< \subseteq (A \times \omega^\omega)$.

Two norms on a set A are said to be *equivalent* if their codes are the same. Every norm is equivalent to a unique *regular norm*, a norm whose range is an ordinal. An *initial segment* of a set A under a norm φ on A is a set of the form $\{x \in A : \varphi(x) < \xi\}$ for some ordinal ξ.

We will need the following fact about extending norms. The proof is standard and is left to the reader.

Lemma 9.3. *Let φ be a regular norm on a set of reals A, and let φ' be a regular norm on a set of reals A'. Let $(R_\leq, R_<)$ and $(R'_\leq, R'_<)$ denote the pairs of relations coding φ and φ', respectively. If $A \subseteq A'$, $R_\leq \subseteq R'_\leq$ and $R_< \subseteq R'_<$, then the set A is an initial segment of the set A' under its norm φ', and we have $\varphi = \varphi' \restriction A$.*

If Γ is a pointclass, a Γ-*norm* on a set of reals $A \in \Gamma$ is a norm φ on A with the property that its code $(R_\leq, R_<)$ is in Γ, meaning that both relations $R_\leq$ and $R_<$ are continuous preimages of sets in Γ. This is equivalent to the usual definition [15, p. 153] if Γ is an adequate pointclass. All of the pointclasses that we will consider are adequate.

Remark 9.4. For pointclasses Γ closed under unions, intersections and continuous preimages (as all the pointclasses we consider are), if φ' is a Γ-norm on a set $A' \in \Gamma$, then every proper initial segment A

[2] unpublished, but see pp. 27–28 of Martin Zeman's notes on Steel's lectures at the 2010 Conference on the Core Model Induction and Hod Mice at WWU Münster; http://wwwmath.uni-muenster.de/logik/Personen/rds/notes_by_zeman_1.pdf

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of A' under φ' is in $\Delta = \Gamma \cap \check{\Gamma}$, and the corresponding restricted norm $\varphi' \restriction A$ is a Δ-norm (see Remark 4.13 of [11], for instance).

By a *Suslin-norm*, we mean a Γ-norm where Γ is the pointclass of Suslin sets. Given a cardinal λ, a uB_λ*-norm* is a Γ-norm where Γ is the pointclass of λ-universally Baire sets.

For a pointset R, a $\mathrm{pos}\Sigma^1_1(R)$-*statement* is one of the form

$$\exists y\,\big(Q(y)\ \ \&\ \ \forall n<\omega\,(y)_n \in R\big),$$

where Q is a Σ^1_1 formula and we think of a real y as coding a sequence of reals $(y)_0, (y)_1, (y)_2, \ldots$ in some recursive way (see, for instance, Moschovakis [15, Lemma 7D.7] or page 21 of [11]). For pointsets $R_1, \ldots, R_n$, we define $\mathrm{pos}\Sigma^1_1(R_1, \ldots, R_n)$ as $\mathrm{pos}\Sigma^1_1(R)$, where R is the disjoint union of the sets $R_1, \ldots, R_n$.

If R is a binary relation, then the statement that R is ill-founded can be expressed by a $\mathrm{pos}\Sigma^1_1(R)$ statement, and the statement that R is not a prewellordering can be expressed by a $\mathrm{pos}\Sigma^1_1(R, \neg R)$ statement. In addition, one can say (quite trivially) that $A \cap B \neq \emptyset$ with a $\mathrm{pos}\Sigma^1_1(A, B)$ statement.

For any trees $T_1, \ldots, T_n$, by $\mathrm{pos}\Sigma^1_1(\mathrm{p}[T_1], \ldots, \mathrm{p}[T_n])$ *generic absoluteness*, we mean the fact that, for any generic extension $V[g]$, any given $\mathrm{pos}\Sigma^1_1$ statement about the sets $\mathrm{p}[T_1]^{V[g]}, \ldots, \mathrm{p}[T_n]^{V[g]}$ holds in $V[g]$ if and only if the corresponding $\mathrm{pos}\Sigma^1_1$ statement about the sets $\mathrm{p}[T_1], \ldots, \mathrm{p}[T_n]$ holds in V. This fact is easily proved using the absoluteness of wellfoundedness for a certain tree built from $T_1, \ldots, T_n$ and a tree for a $\mathrm{pos}\Sigma^1_1$ set. (There is nothing special about the case of generic extensions versus arbitrary outer models here, but it is the only case we will need.) This generic absoluteness property encompasses many of the common absoluteness arguments involving trees. It can be used to show, for example, that if R is a universally Baire wellfounded relation (or prewellordering), then the canonical expansion $R^{V[g]}$ is a universally Baire wellfounded relation (or prewellordering). It can also be used to show that the canonical expansions of universally Baire sets are unique (which we did in Lemma 2.6).

The following lemma is the first part of the proof of Theorem 9.1, as outlined above.

Lemma 9.5. *Let λ be a limit of $<\lambda$-strong cardinals, and let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then in the symmetric extension $V(\mathbb{R}^*_G)$, every Suslin set that admits a Suslin-norm is co-Suslin.*

Proof. Let A be a Suslin set in $V(\mathbb{R}^*_G)$ that admits a regular Suslin-norm φ, and let $R_\leq$ and $R_<$ be the Suslin relations coding φ. Let T, $T_\leq$, and $T_<$ be trees in $V(\mathbb{R}^*_G)$ projecting to A, $R_\leq$ and $R_<$ respectively. By the usual homogeneity argument T, $T_\leq$ and $T_<$ are in $V[G \restriction \xi]$ for some $\xi < \lambda$. Let $\kappa < \lambda$ be a $<\lambda$-strong cardinal in $V[G \restriction \xi]$.

In any generic extension of $V[G \restriction \xi]$ by a poset of size less than λ, the pair of relations $(\mathrm{p}[T_\leq], \mathrm{p}[T_<])$ codes a norm on the set $\mathrm{p}[T]$, since (for instance) the properties of $(A, R_\leq, R_<)$ in $V(\mathbb{R}^*_G)$ that ensure coding a norm are preserved under restriction to the reals of a smaller universe.

Let $\alpha = 2^{2^\kappa}$. Define the set $A_0 = \mathrm{p}[T]^{V[G\restriction\alpha]}$ and the pair of relations $(R^0_\leq, R^0_<) = (\mathrm{p}[T_\leq], \mathrm{p}[T_<])^{V[G\restriction\alpha]}$. Then the pair $(R^0_\leq, R^0_<)$ codes a norm φ_0 on the set A_0.

Claim 9.6. In $V[G \restriction \alpha]$, the set A_0 is λ-universally Baire and the norm φ_0 on A_0 is a uB_λ-norm.

Proof. Fix $\eta < \lambda$. We will show that the set A_0 is η-universally Baire and the norm φ_0 on A_0 is a uB_η-norm. By increasing η, we may assume that $\alpha < \eta$ and $|V_\eta| = \eta$. Take an elementary embedding $j\colon V[G\restriction\xi] \to M$ with critical point κ, where M is transitive, $M^\omega \subseteq M$, $V_\eta[G\restriction\xi] \subseteq M$ and $\eta < j(\kappa)$.

Let $A_\eta = \mathrm{p}[j(T)]^{V[G\restriction\alpha]}$, $R^\eta_\leq = \mathrm{p}[j(T_\leq)]^{V[G\restriction\alpha]}$ and $R^\eta_< = \mathrm{p}[j(T_<)]^{V[G\restriction\alpha]}$. Then A_η, $R^\eta_\leq$ and $R^\eta_<$ are η-universally Baire in $V[G\restriction\alpha]$ by a theorem of Woodin (see Steel [23, Theorem 4.5].)

In any generic extension of M by a poset of size less than $j(\lambda)$, the pair of relations $(\mathrm{p}[j(T_\leq)], \mathrm{p}[j(T_<)])$ codes a norm on the set $\mathrm{p}[j(T)]$, by the elementarity of j. Since $\alpha < j(\lambda)$ and the models $M[G\restriction\alpha]$ and $V[G\restriction\alpha]$ have the same reals, it follows that the pair $(R^\eta_\leq, R^\eta_<)$ codes a regular norm φ_η on the set A_η. We have then that φ_η is a uB_η-norm on A_η in $V[G\restriction\alpha]$.

Using j to map branches of trees pointwise, we get $A_0 \subseteq A_\eta$, $R^0_\leq \subseteq R^\eta_\leq$, and $R^0_< \subseteq R^\eta_<$. By Lemma 9.3, these inclusions imply that the set A_0 is an initial segment of the set A_η under its norm φ_η,

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and that the norm φ_0 on A_0 is the restriction of φ_η to this initial segment. By Remark 9.4, in $V[G\restriction\alpha]$, the set A_0 is η-universally Baire and the norm φ_0 on A_0 is a uB_η-norm. □

Now we complete the proof of Lemma 9.5. By the claim, we may fix λ-absolutely complementing pairs of trees $(T^*, \tilde{T})$, $(T^*_\leq, \tilde{T}_\leq)$, and $(T^*_<, \tilde{T}_<)$ in $V[G\restriction\alpha]$ for A_0, $R^0_\leq$, and $R^0_<$, respectively, and define the canonical expansions $A^* = \mathrm{p}[T^*]^{V(\mathbb{R}^*_G)}$, $R^*_\leq = \mathrm{p}[T^*_\leq]^{V(\mathbb{R}^*_G)}$ and $R^*_< = \mathrm{p}[T^*_<]^{V(\mathbb{R}^*_G)}$.

Because the pair of relations $(R^0_\leq, R^0_<)$ codes a norm on the set A_0, it follows that the pair of relations $(R^*_\leq, R^*_<)$ codes a regular norm φ^* on the set A^* by generic absoluteness for $\Sigma^1_1(\mathrm{p}[T^*], \mathrm{p}[\tilde{T}], \mathrm{p}[T^*_\leq], \mathrm{p}[\tilde{T}_\leq], \mathrm{p}[T^*_<], \mathrm{p}[\tilde{T}_<])$.

We have $A \subseteq A^*$ by Lemma 2.5, which can be seen as an instance of $\Sigma^1_1(\mathrm{p}[T], \mathrm{p}[\tilde{T}])$ generic absoluteness. Similarly, $R_\leq \subseteq R^*_\leq$ and $R_< \subseteq R^*_<$. By Lemma 9.3 again, these inclusions imply that the set A is an initial segment of the set A^* under the norm φ^*. In $V(\mathbb{R}^*_G)$, the norm φ^* is an ScS-norm where ScS denotes the pointclass of Suslin co-Suslin sets, so its inital segment A is a Suslin co-Suslin set as in Remark 9.4, as desired. □

Remark 9.7. In [21], a pointclass Γ consisting of universally Baire sets of reals is defined to be *productive* if it is closed under complements and projections, and satisfies that statement that, for all $A \subseteq \omega^\omega \times \omega^\omega$ in (or continuously reducible to a member of) Γ, in all set-generic forcing extensions $V[H]$, the sets $\{y : \exists x \in \omega^\omega (x, y) \in A^{V[H]}\}$ and $(\{y \in \omega^\omega : \exists x \in \omega^\omega (x, y) \in A\})^{V[H]}$ are equivalent. An argument very similar to the proof of Lemma 9.5 shows that the collection of universally Baire sets of reals is productive in $V(\mathbb{R}^*_G)$, assuming the hypotheses of the lemma. We conjecture that this fact (and in fact a stronger fact) already follows from just the theory AD^+ + 'All sets of reals are universally Baire'.

More specifically, we conjecture that if V is a model of AD^++ 'All sets of reals are universally Baire', then for every set of reals A, in all set generic extensions $V[G * H]$, there is an elementary embedding

$$j : L(A^{V[G]}, \mathbb{R}^{V[G]}) \to L\left(A^{V[G*H]}, \mathbb{R}^{V[G*H]}\right)$$

with $j(A^{V[G]}) = A^{V[G*H]}$.

Before proceeding with the proof of Theorem 9.1, we make a remark about the necessity of assuming in Lemma 9.5 that the Suslin set of reals A admits a Suslin-norm.

Remark 9.8. If λ is a Woodin cardinal, then Lemma 9.5 holds even without the hypothesis that the set has a Suslin-norm: every Suslin set in $V(\mathbb{R}^*_G)$ is co-Suslin by a theorem of Woodin (see Larson [10, Theorem 1.5.12] or Steel [23, Theorem 4.4]). However, in our application, we do not want to make the assumption that λ is a Woodin cardinal; together with the assumption that it is a limit of Woodin cardinals, this would be significantly stronger than the hypothesis of the main theorem.

However, if λ is singular, then the hypothesis of the existence of a Suslin-norm in Lemma 9.5 is necessary: in $V(\mathbb{R}^*_G)$, there is a Suslin set of reals that is not co-Suslin. An example was pointed out to us by Woodin (in an email on November 26, 2014). We generalize this example to show that if ZF holds and ω_1 is singular (as in the symmetric model obtained by the Levy collapse at a singular cardinal), then there is a Suslin set of reals that is not co-Suslin.

Take a sequence of countable ordinals $\langle \alpha_n : n < \omega \rangle$ that is cofinal in ω_1, and let A be the set of all reals of the form $n^\frown x$, where $n < \omega$ and x is a real coding a wellordering of ω of order type at most α_n. This set A is Suslin, as witnessed by the tree of attempts to build a real $n^\frown x$ and a function $f : \omega \to \alpha_n$ such that x codes a linear ordering on ω and f is strictly increasing with respect to this linear ordering.

Assume toward a contradiction that the set A is co-Suslin. The set of all reals $n^\frown x$ where $n < \omega$ and x is a real coding *any* wellordering of ω is also Suslin, so the intersection of this set with the complement of A is Suslin.

In other words, the set of all reals $n^\frown x$ where $n < \omega$ and x is a real coding a wellordering of ω of order type greater than α_n is Suslin. Then by considering leftmost branches we can choose, for each $n < \omega$, a real x_n coding a wellordering of ω of order type greater than α_n. This is a contradiction because it allows us to define a wellordering of ω of order type ω_1.

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Proof of Theorem 9.1. Working in the derived model $D(V, \lambda, G)$, suppose toward a contradiction that not every set of reals is Suslin. Since AD^+ holds, the set of Suslin cardinals is closed in Θ (see Ketchersid [9] or Larson [11] for a proof) so there is a largest Suslin cardinal κ (by Corollary 6.19 of [11]). The pointclass of κ-Suslin sets (equivalently, of all Suslin sets) is non-selfdual, and it has the prewellordering property, meaning that every Suslin set has a Suslin-norm (see Jackson [3, Lemma 3.6] and Remark 6.6 of [11]). So we can fix a complete Suslin set A of $D(V, \lambda, G)$ and note that a Suslin-norm on A exists.

By Lemma 9.5, the set A is Suslin and co-Suslin in $V(\mathbb{R}^*_G)$, which means that A is in Hom^*_G. We can fix then a $\xi < \lambda$ and λ-absolutely complementing trees T_A and $\tilde{T}_A$ on $\omega \times \mathrm{Ord}$ in $V[G\restriction\xi]$ such that $A = \mathrm{p}[T_A]^{V(\mathbb{R}^*_G)}$. Letting $A_0 = \mathrm{p}[T_A]^{V[G\restriction\xi]}$, we have then that $A = A_0^{V(\mathbb{R}^*_G)}$. This means that $V[G\restriction\xi] \models \omega^\omega \setminus A_0 \in \mathrm{uB}_\lambda$ as well, so the set $\omega^\omega \setminus A_0$ has a uB_λ-semiscale $\vec{\varphi}$ in $V[G\restriction\xi]$ by Steel [23, Theorem 5.3]. (This result uses the fact that $\mathrm{uB}_\lambda = \mathrm{Hom}_{<\lambda}$ because λ is a limit of Woodin cardinals. Also, the semiscale obtained by Steel is a scale, but we do not need the additional lower semicontinuity property of scales here.)

As in Section 3, to say that $\vec{\varphi}$ is a uB_λ-semiscale means that its component norms φ_n are uB_λ-norms, uniformly in n. (In fact, the uniformity is automatic here because the pointclass uB_λ is closed under countable unions.) More precisely, the relation R_0 coding the semiscale $\vec{\varphi}$, defined by

$$R_0 = \left\{(\bar{n}, x, y) \in \omega^\omega \times (\omega^\omega \setminus A_0) \times (\omega^\omega \setminus A_0) : \varphi_n(x) \leq \varphi_n(y)\right\},$$

is λ-universally Baire, where $\bar{n}$ indicates the constant function from ω to n. Fix trees T_R and $\tilde{T}_R$ in $V[G\restriction\xi]$ witnessing this, and let R denote the expanded relation $R_0^{V(\mathbb{R}^*_G)}$.

The expanded relation R codes a semiscale on the set $\omega^\omega \setminus A$, which was our universal co-Suslin set of the derived model. One way to see this is to use the fact that $(\mathrm{HC}^{V[G\restriction\xi]}; \in, A_0, R_0) \prec (\mathrm{HC}^*_G; \in, A, R)$ because λ is a limit of Woodin cardinals (see Steel [23, Lemma 7.3].) Another way is to apply $\mathrm{pos}\Sigma^1_1(\mathrm{p}[T_A], \mathrm{p}[\tilde{T}_A], \mathrm{p}[T_R], \mathrm{p}[\tilde{T}_R])$ generic absoluteness.

Because the relation R is in Hom^*_G, it is in the derived model $D(V, \lambda, G)$. In $D(V, \lambda, G)$, the tree of the semiscale coded by R witnesses that the set $\omega^\omega \setminus A$ is Suslin (i.e., that the set A is co-Suslin), giving a contradiction. □

Finally, we establish the last remaining component required for the proof of the main theorem:

Lemma 9.9. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, and let $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be a V-generic filter. Then every set of reals in Hom^*_G is universally Baire in $L^F(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)}$.*

Proof. Let $A \in \mathrm{Hom}^*_G$, which means that $\omega^\omega \setminus A \in \mathrm{Hom}^*_G$ as well. As in the proof of Theorem 9.1, it follows from Steel [23, Theorem 5.3] (using the fact that λ is a limit of Woodin cardinals) that A and its complement admit Hom^*_G-semiscales. By Lemma 5.7 and the discussion preceding it, the fact that λ is a limit of strong cardinals implies that every set in Hom^*_G is universally Baire in $V(\mathbb{R}^*_G)$. So in $V(\mathbb{R}^*_G)$, the set A is universally Baire and there are uB-semiscales on A and $\omega^\omega \setminus A$. By Theorem 4.2, then, with $\Gamma = \mathrm{Hom}^*_G$, A is universally Baire in $L^F(\mathbb{R}^*_G, \mathrm{Hom}^*_G)^{V(\mathbb{R}^*_G)}$. □

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