

THE ORDER OF CERTAIN CLASSES OF FUNCTIONS DEFINED IN THE UNIT DISK

D. C. RUNG

1. Introduction¹⁾

Let D denote the open unit disk in the complex plane and let C be the boundary of D . If, for a given complex-valued function $f(z)$ defined in D , the existence of a subset M of C is known, with the linear measure of M equal to 2π , as well as an estimate on the growth of $|f(z)|$ on sequences in D which tends to a point of M , then such a result will be called a "statistical" result on order. This terminology is due to Lelong-Ferrand [3].

Such statistical-type results are known, for example, if the function $f(z)$ is the derivative of a univalent, holomorphic function (Seidel and Walsh [5], p. 141.); or if $f(z)$ is holomorphic in D and omits two values there (Rung [4], p. 330). Both of these results depend upon first estimating the order of a holomorphic function $g(z)$ for which

$$(1.0) \quad \iint_D |g(z)|^2 dx dy < \infty.$$

In sections 3, 4, and 5 of this paper we replace the function $|g(z)|$ in (1.0) by several arbitrary real-valued functions defined in D and obtain statistical type results for these functions.

We conclude, in Section 6, by presenting examples of functions exhibiting this behavior.

2. Terminology

For $z_0 \in D \cup C$ and $r > 0$ set $D(z_0, r) = \{z \in D \mid |z - z_0| < r\}$. We proceed to introduce an outer measure on the plane. For $r \geq 0$ let $h(r)$ be a real valued, non-decreasing, continuous function with $h(0) = 0$, $h(r) > 0$ for $r > 0$, and $h(\infty) > 1$.

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DEFINITION 1. For a given set E in the plane and fixed $\rho > 0$ let $\{H(z_j, r_j)\}$ denote any denumerable family of open disks in the plane with center z_j and radius r_j , $r_j < \rho$, which cover E . If A_ρ is the inf $\left\{ \sum_{j=1}^{\infty} h(r_j) \mid \bigcup_{j=1}^{\infty} H(z_j, r_j) \supset E, r_j < \rho \right\}$ define the h -measure of E to be $h^*(E) = \lim_{\rho \rightarrow 0} A_\rho$.

Remark 1. In the case $h(r) = r^k$, $0 < k < 2$, this defines on the plane the usual k -dimensional outer measure.

3. Order of functions summable on D

The following results depend upon a theorem of the author [4, p. 324], which is closely related to a result of Lelong-Ferrand [3, pp. 20-23]. For completeness we state this theorem without proof.

THEOREM A. Let $U(z)$ be a real-valued, non-negative, measurable function defined in D such that,

$$\iint_D U(z) dx dy < \infty, \quad z = x + iy.$$

Then

$$\lim_{r \rightarrow 0} \left[\frac{1}{h(r)} \iint_{D(e^{i\theta}, r)} U(z) dx dy \right] = 0,$$

except for at most a set of $e^{i\theta}$ of h -measure 0.

Remark 2. All integrals are to be considered as Lebesgue integrals.

In the following theorem an estimate is obtained on the order of such summable $U(z)$ on certain sequences in D .

THEOREM 1. Let $U(z)$ satisfy the hypotheses of Theorem A. Then for every point of C , except possibly for a subset S of C of h -measure 0, the following behavior occurs. Let $\{z_n\}$ be any sequence in D tending to a point $e^{i\theta}$ of C not in S . For any fixed t , $0 < t < 1$, there exists a sequence of measurable sets $\{M_n(t)\}$ such that

- i) $M_n(t) \subset D(z_n, (1 - |z_n|)t)$;
- ii) $M_n(t)$ has positive two dimensional Lebesgue measure;
- iii) if $\{\zeta_n\}$ is a sequence with

$$\zeta_n \in M_n(t), \quad n = 1, 2, \dots,$$

$$\lim_{n \rightarrow \infty} \frac{U(\zeta_n)(1 - |\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0,$$

Remark 3. It is obvious that $\zeta_n \in M_n(t)$ implies $\zeta_n \rightarrow e^{i\theta}$ as $n \rightarrow \infty$.

Remark 4. In this theorem, and in the sequel, we assume $h(r)$ also satisfies

$$(3.0) \quad h(\alpha r) \leq K_\alpha h(r),$$

$0 \leq r \leq \infty$, $\alpha > 0$, and K_α is a positive constant depending only on α . This property is not essential for this group of theorems and the necessary changes, if (3.0) is not assumed, will be obvious. For example, $h(r) = r^k$ satisfies this property.

Proof. Theorem A yields

$$(3.1) \quad \lim_{r \rightarrow 0} \left[\frac{1}{h(r)} \iint_{D(e^{i\theta}, r)} U(z) dx dy \right] = 0,$$

except possibly for a subset of C of h -measure 0. Let $e^{i\theta}$ be a point at which (3.1) holds and suppose $\{z_n\}$ is any sequence in D tending to $e^{i\theta}$. Further choose an arbitrary t , $0 < t < 1$, which remains fixed during the course of the proof.

For $\zeta \in D(z_n, (1 - |z_n|)t)$, an easy calculation gives

$$(3.2) \quad (1 - t)|z_n - e^{i\theta}| < |\zeta - e^{i\theta}| < 2|z_n - e^{i\theta}|,$$

and

$$(3.3) \quad (1 - t)(1 - |z_n|) < 1 - |\zeta| < (1 + t)(1 - |z_n|).$$

For the remainder of the proof set

$$D(z_n, (1 - |z_n|)t) \equiv D_n, \quad n = 1, 2, \dots$$

Since $U(z) \geq 0$, the right side of (3.2) gives

$$(3.4) \quad \iint_{D_n} U(z) dx dy \leq \iint_{D(e^{i\theta}, 2|z_n - e^{i\theta}|)} U(z) dx dy.$$

Theorem A, together with (3.0) and (3.4), enable us to conclude

$$(3.5) \quad \lim_{n \rightarrow \infty} \left[\frac{1}{h(|z_n - e^{i\theta}|)} \iint_{D_n} U(z) dx dy \right] = 0.$$

The existence of the sets $M_n(t)$ is now demonstrated. Fix a positive integer n and for this value of n let H denote the set of all points $\zeta = u + iv$ contained in D_n for which

$$(3.6) \quad U(\xi)\pi(1 - |z_n|)^2 t^2 > \iint_{D_n} U(z) dx dy.$$

Since $U(z)$ is a measurable function H is a measurable set. Further if the measure of H were equal to $\pi(1 - |z_n|)^2 t^2$ integrating both sides of (3.6) over H would give

$$\iint_H U(\xi) dudv > \iint_{D_n} U(z) dx dy = \iint_H U(z) dx dy$$

which is impossible. Hence the measure of H is less than $\pi(1 - |z_n|)^2 t^2$. Setting $M_n(t)$ equal to the complement of H relative to D_n we have, for $\zeta_n \in M_n(t)$, $n = 1, 2, \dots$

$$(3.7) \quad U(\zeta_n)\pi(1 - |z_n|)^2 t^2 \leq \iint_{D_n} U(z) dx dy.$$

We remark that the sequence of sets $\{M_n(t)\}$ depend upon the function $U(z)$, the sequence $\{z_n\}$, and the value t . In the sequel, if we introduce a function $U(z)$, a sequence $\{z_n\}$ and a value t , $0 < t < 1$, $\{M_n(t)\}$ will always represent the above sequence of sets.

The proof is nearly complete since combining (3.5) and (3.7) gives

$$(3.8) \quad \lim_{n \rightarrow \infty} \frac{U(\zeta_n)(1 - |z_n|)^2}{h(|z_n - e^{i\theta}|)} = 0.$$

However (3.8) may be revised to give

$$\lim_{n \rightarrow \infty} \frac{U(\zeta_n)(1 - |\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0,$$

if we refer to the right side of both (3.2) and (3.3) together with (3.0). Since this limit holds at every point $e^{i\theta}$ at which (3.1) is valid, the proof of Theorem 1 is complete.

If we restrict the sequence $\{z_n\}$ to approach $e^{i\theta}$ within some Stolz domain Theorem 1 can be reformulated. To this end let $S(e^{i\theta}, \alpha)$, $0 < \alpha < \frac{\pi}{2}$, denote the symmetric Stolz domain at $e^{i\theta}$ of opening 2α .

COROLLARY 1. *Let $U(z)$ satisfy the hypotheses of Theorem 1, and let $\{z_n\}$ be a sequence in D tending to a point $e^{i\theta}$ but with $z_n \in S(e^{i\theta}, \alpha)$, $n = 1, 2, \dots$, for some $0 < \alpha < \frac{\pi}{2}$. Then for any fixed $0 < t < 1$, and any sequence $\{\zeta_n\}$, $\zeta_n \in M_n(t)$,*

$n = 1, 2, \dots,$

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{U(\zeta_n)(1 - |\zeta_n|)^2}{h(1 - |\zeta_n|)} = 0$$

except for at most a subset of C of h -measure 0.

Proof. An argument involving elementary geometry shows that, for any sufficiently small $\varepsilon > 0$, there exists a positive integer N_0 , depending on ε , such that for $n > N_0$

$$(3.10) \quad D_n \subset S(e^{i\theta}, \alpha + \arcsin(t \cos \alpha) + \varepsilon).$$

This implies that for $n > N_0$ all ζ_n lie in the above Stolz domain, and, as is well known, then satisfy

$$1 \leq \frac{|\zeta_n - e^{i\theta}|}{1 - |\zeta_n|} \leq C,$$

for suitable constant C . Again referring to (3.0) as well as to the monotonicity of $h(r)$ we see that Theorem 1 can be restated to give Corollary 1.

Remark 5. I am indebted to Professor W. Seidel for indicating (3.10).

Remark 6. Setting $h(r) = r$, which defines on C the ordinary outer linear measure, the conclusion of Corollary 1 now reads

$$\lim_{n \rightarrow \infty} U(\zeta_n)(1 - |\zeta_n|) = 0,$$

and the exceptional subset of C has linear measure 0.

The question arises as to whether any estimate can be obtained for such summable $U(z)$ on the original sequence $\{z_n\}$. Several sufficient conditions are discussed in 4 and 5.

4. Sequentially subharmonic functions

If we return to the proof of Theorem 1 we see that (3.7) relates the values of $U(z)$ at certain points in D_n to the value of the integral of $U(z)$ over D_n . With this in mind we give

DEFINITION 2. Let $U(z)$ be a real valued, non-negative measurable function defined in D . We say $U(z)$ is sequentially subharmonic in D if, for each sequence $\{z_n\}$ in D and for at least one value of t , $0 < t < 1$, there exists a positive constant K (which is a function of both the sequence and the value t) such that

$$(4.0) \quad U(z_n)\pi(1 - |z_n|)^2 t^2 K \leq \iint_{D(z_n, (1 - |z_n|)t)} U(z) dx dy,$$

for $n = 1, 2, \dots$

Remark 7. If $G(z)$ is a positive subharmonic function in D then it is sequentially subharmonic in D since K can be chosen identically 1 for each sequence and each $0 < t < 1$.

The conclusion of Theorem 1 can be revised so that the sequence $\{\zeta_n\}$ is replaced by the original sequence $\{z_n\}$ if (3.7) is replaced by (4.0). This gives

THEOREM 2. *Let $U(z)$ be sequentially subharmonic in D and suppose also*

$$\iint_n U(z) dx dy < \infty.$$

If $\{z_n\}$ is a sequence in D which tends to a point $e^{i\theta}$ we have

$$\lim_{n \rightarrow \infty} \frac{U(z_n)(1 - |z_n|)^2}{h(|z_n - e^{i\theta}|)} = 0$$

except possibly for a set of $e^{i\theta}$ of h -measure 0.

COROLLARY 2. *Let the hypotheses of Theorem 2 be satisfied and in addition suppose the sequence $\{z_n\}$ approaches $e^{i\theta}$ within some Stolz domain at $e^{i\theta}$. Then*

$$\lim_{n \rightarrow \infty} \frac{U(z_n)(1 - |z_n|)^2}{h(1 - |z_n|)} = 0$$

except possibly for a set of $e^{i\theta}$ of h -measure 0.

Remark 8. If $V(z)$ is a positive subharmonic function in D then Theorem 2 applies to the function $U(z) = V^p(z)$, $p \geq 1$, since $V^p(z)$ is also subharmonic. In the case $0 < p < 1$, $V^p(z)$ is still subharmonic provided $\log V(z)$ is. This generalizes a result of Gehring [1, p. 77].

5. Complex-valued functions summable over D

Let $\phi(z)$ denote a complex-valued function defined in D . For any two points a, b of D set $\rho(a, b)$ equal to the non-euclidean (hyperbolic) distance between a and b , i.e. $\rho(a, b) = 1/2 \log \frac{|1 - a\bar{b}| + |a - b|}{|1 - a\bar{b}| - |a - b|}$.

DEFINITION 3. *Let $\{z_n\}$ be a sequence in D which tends to a point of C . A complex-valued function $\phi(z)$ defined in D is said to be close along $\{z_n\}$ if there*

exists a pair of positive real numbers (δ, M) such that if $\rho(z, z_n) < \delta$ then $|\phi(z) - \phi(z_n)| < M$, for each $n = 1, 2, \dots$. If $\phi(z)$ is close along $\{z_n\}$ for all $\{z_n\}$ such that $\lim_{n \rightarrow \infty} z_n = e^{i\theta}$, $e^{i\theta} \in C$ (respectively $e^{i\theta} \in B$, B a subset of C with $h^*(B) = h^*(C)$) then we say $\phi(z)$ is close along all sequences (respectively close along almost all sequences in the h -measure). When $h(r) = r$ we omit the phrase "in the h -measure."

For $z_0 \in D$ and $\delta > 0$ set $N(z_0, \delta) = \{z \mid \rho(z_0, z) < \delta\}$. This set of points is known to be an open Euclidean disk. Thus let z' denote the center and $(1 - |z'|)t'$ the radius (both in the Euclidean geometry) of $N(z, \delta)$. We now indicate a connection between the non-Euclidean radius δ and the value t' .

LEMMA 1. Given the non-Euclidean disk $N(z, \delta)$ and its corresponding Euclidean representation $D(z', (1 - |z'|)t')$ then

$$t' = \frac{K(1 + |z|)}{1 + K^2|z|}.$$

Hence as $|z| \rightarrow 1$, $t' \rightarrow \frac{2K}{1 + K^2}$. $K = \frac{e^{2\delta} - 1}{e^{2\delta} + 1}$.

Proof. If $z = re^{i\theta}$, then the point z_1 on the boundary of $N(z, \delta)$ closest to z in the Euclidean sense, is of the form $z_1 = r_1e^{i\theta}$, $r_1 > r$; the point z_2 furthest from z is $z_2 = r_2e^{i\theta}$, $r_2 < r$. The point z' is also on the radius to $e^{i\theta}$ thus $z' = r'e^{i\theta}$, $r_2 < r' < r < r_1$. If we put $|z_1 - z| = (1 - |z|)t_1$ and $|z - z_2| = (1 - |z|)t_2$, an elementary calculation gives $t_1 = \frac{K(1 + |z|)}{1 + K|z|}$, and $t_2 = \frac{K(1 + |z|)}{1 - K|z|}$. Thus the Euclidean radius of $N(z, \delta)$ is $\frac{|z_1 - z_2|}{2} = (1 - |z|)\left(\frac{t_1 + t_2}{2}\right)$, and the Euclidean center $z' = \frac{z_2 + z_1}{2}$. Finally to find t' note that $(1 - |z'|)t' = (1 - |z|)\left(\frac{t_1 + t_2}{2}\right)$ and a straightforward calculation gives the value t' in the Lemma.

This enables us to view a sequence of non-Euclidean disks $N(z_n, \delta)$, with $\lim_{n \rightarrow \infty} |z_n| = 1$, as a sequence of Euclidean disks $D(z'_n(1 - |z'_n|)t'_n)$ where for $n > N_0$, $0 < \underline{t} \leq t'_n \leq \bar{t} < 1$, with N_0 , \underline{t} and \bar{t} determined by Lemma 1.

THEOREM 3. Let $\phi(z)$ be a complex-valued measurable function defined in D which is close along almost all sequences in the h -measure and also, with $z = x + iy$,

$$\iint_D |\phi(z)| dx dy < \infty.$$

Then, if $z_n \rightarrow e^{i\theta}$ as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{|\phi(z_n)|(1 - |z_n|)^2}{h(|z_n - e^{i\theta}|)} = 0,$$

except for at most a set of $e^{i\theta}$ of h -measure 0.

Proof. Set $U(z) = |\phi(z)|$ and let S denote the exceptional set of Theorem 1 for this $U(z)$. If B is the set of all $e^{i\theta}$ relative to which $\phi(z)$ is close along sequences, $h^*(B) = h^*(C)$; hence setting $S_1 = C - B$, $h^*(S_1) = 0$, and $h^*(S_2) = 0$ where $S_2 = S \cup S_1$.

Let $\{z_n\}$ be any sequence in D tending to a point $e^{i\theta}$ not in S_2 . Since $\phi(z)$ is close along $\{z_n\}$ there exists a pair of positive real numbers (δ, M) such that $\rho(z, z_n) < \delta$ implies $|\phi(z) - \phi(z_n)| < M$, $n = 1, 2, \dots$

Referring to Lemma 1 we consider the sequence $\{N(z_n, \delta)\}$ as a sequence of Euclidean disks $\{D(z'_n, (1 - |z'_n|)t'_n)\}$. Since $t_n > \underline{t}$, $n > N_0$, we apply Theorem 1 with $t = \underline{t}$.

Thus for any sequence $\{\zeta_n\}$, $\zeta_n \in M_n(t)$, $n = 1, 2, \dots$,

$$(5.0) \quad \lim_{n \rightarrow \infty} \frac{|\phi(\zeta_n)|(1 - |\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0.$$

Since $M_n(t) \subset D(z'_n, (1 - |z'_n|)t')$, $n = 1, 2, \dots$, application of Lemma 1 gives $M_n(t) \subset N(z_n, \delta)$, $n > N_0$. Since $\phi(z)$ is close along $\{z_n\}$.

$$(5.1) \quad |\phi(z_n)| \leq M + |\phi(\zeta_n)|, \quad n > N_0$$

Combining (3.0), (3.2), (3.3) and (5.1)

$$(5.2) \quad 0 \leq \frac{|\phi(z_n)|(1 - |z_n|)^2}{h(|z_n - e^{i\theta}|)} \leq \frac{M(1 - |z_n|)^2}{h(|z_n - e^{i\theta}|)} + \frac{|\phi(\zeta_n)|(1 - |\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)}.$$

Now under the assumption that the h -measure of C is positive (otherwise the statement of the theorem is vacuous)

$$(5.3) \quad \lim_{z \rightarrow e^{i\theta}} \frac{(1 - |z|)^2}{h(|z - e^{i\theta}|)} = 0,$$

for all $e^{i\theta} \in C$. This follows by setting $U(z) \equiv 1$ in Theorem 2 and observing that if (5.3) holds for some $e^{i\theta}$, it holds for all $e^{i\theta}$.

The proof of Theorem 3 is completed by combining (5.0) and (5.3) with

(5.2).

As Corollary 1 follows from Theorem 1 so also does the following Corollary follow from Theorem 3.

COROLLARY 3. *Under the hypotheses of Theorem 3, and supposing also that $\{z_n\}$ tends to $e^{i\theta}$ within some Stolz domain at $e^{i\theta}$,*

$$\lim_{n \rightarrow \infty} \frac{|\phi(z_n)|(1 - |z_n|)^2}{h(1 - |z_n|)} = 0$$

except possibly for a set of $e^{i\theta}$ of h -measure 0.

6. Examples to Theorem 1

The following examples to Theorem 1 are constructed with $h(r) = r$.

EXAMPLE 1. *Given an arbitrary countable subset P of C there exists a function $U_1(z)$ satisfying the hypotheses of Theorem 1 for which the exceptional set S (of Theorem 1) contains P .*

Let $\{e^{i\theta_j}\}$ be some enumeration of the points of P . For each point $e^{i\theta_j} \in P$ we will consider a sequence $\{z_n^{(j)}\}$ tending radially to $e^{i\theta_j}$; a sequence of disks $D(z_n^{(j)}, (1 - |z_n^{(j)}|)t_j)$; and a function $U_1(z)$ which takes the value $\frac{1}{(1 - |z|)^{4/3}}$ in each disk and which is summable over D . If $\zeta_n^{(j)} \in D(z_n^{(j)}, (1 - |z_n^{(j)}|)t_j)$, $n = 1, 2, \dots$

$$\lim_{n \rightarrow \infty} U_1(\zeta_n^{(j)})(1 - |\zeta_n^{(j)}|) = \infty.$$

Referring to Corollary 1 of Theorem 1 we see that the sets $M_n(t_j)$ do not exist for the sequence $\{z_n^{(j)}\}$; and P is therefore a subset of the exceptional set S . We proceed to the details.

For a fixed $e^{i\theta_j} \in P$ consider a Stolz domain $S(e^{i\theta_j}, \alpha_j)$ of opening $2\alpha_j$, $0 < \alpha_j < \frac{\pi}{2}$, where $\{\alpha_j\}$ is any decreasing sequence of positive numbers satisfying

$$(6.0) \quad \sum_{j=1}^{\infty} \alpha_j < \infty.$$

Let $\{z_n^{(j)}\}$ be a sequence approaching $e^{i\theta_j}$ radially with $|z_n^{(j)}| = 1 - \frac{1}{n^2}$, $n = 1, 2, \dots$. If $t_j = \sin \alpha_j$ the disks $D(z_n^{(j)}, (1 - |z_n^{(j)}|)t_j)$, $n = 1, 2, \dots$, are easily seen to lie inside $S(e^{i\theta_j}, \alpha_j)$ (see figure 1). Hence-forth we use $D_n^{(j)}$ to denote these disks.

New define

$$U_1(z) = \begin{cases} \frac{1}{(1-|z|)^{4/3}}, & z \in \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} D_n^{(j)}; \\ 0, & z \in D - \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} D_n^{(j)}. \end{cases}$$

In order to compute the double integral of $U(z)$ over D some estimates of the integral of $U(z)$ over each $D_n^{(j)}$ are required.

Let $\theta_n^{(j)}$ be that positive angle formed by the radius to e^{ip_j} and the line segment from the origin tangent to the circumference of $D_n^{(j)}$ (See figure 1.).

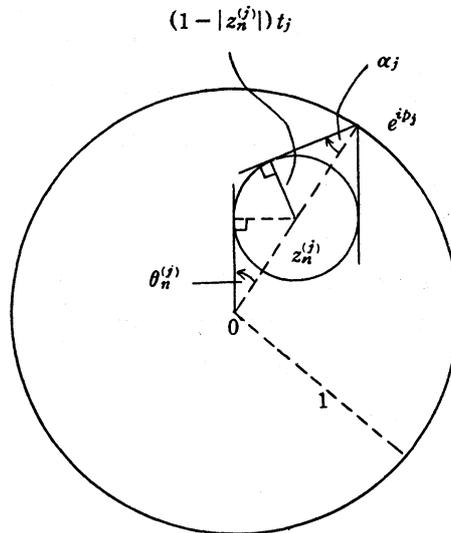


FIGURE 1

Since

$$(6.1) \quad \sin \theta_n^{(j)} = \frac{(1 - |z_n^{(j)}|) t_j}{|z_n^{(j)}|} = \frac{t_j}{n^2 - 1}.$$

and

$$(6.2) \quad \theta_n^{(j)} < \frac{\pi}{2} \sin \theta_n^{(j)},$$

then for $n \geq 2$, and all j ,

$$(6.3) \quad \theta_n^{(j)} < \frac{\pi}{2} \frac{t_j}{n^2 - 1} < \frac{\pi t_j}{n^2}.$$

Since the quadrilateral

$$Q_n^{(j)} = \left\{ r e^{i\theta} \mid p_j - \theta_n^{(j)} \leq \theta \leq p_j + \theta_n^{(j)}, 1 - \left(\frac{1 + t_j}{n^2} \right) \leq r \leq 1 - \left[\frac{1 - t_j}{n^2} \right] \right\}$$

contains $D_n^{(j)}$ for each value n and j , setting $z = re^{i\theta}$

$$\begin{aligned}
 (6.5) \quad \iint_{D_n^{(j)}} U_1(z) r dr d\theta &= \iint_{D_n^{(j)}} \frac{1}{(1-r)^{4/3}} r dr d\theta \\
 &\leq \iint_{Q_n^{(j)}} \frac{r}{(1-r)^{4/3}} dr d\theta \\
 &< \frac{6 \theta_n^{(j)} n^{2/3}}{(1-t_1)^{1/3}}.
 \end{aligned}$$

By (6.3)

$$(6.6) \quad \frac{\theta_n^{(j)} n^{2/3}}{(1-t_1)^{1/3}} < \frac{\pi^{t_j}}{n^{4/3} (1-t_1)^{1/3}}.$$

Thus (6.5) and (6.6) combine to give

$$\begin{aligned}
 \iint_D U_1(z) r dr d\theta &\leq \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \iint_{D_n^{(j)}} U_1(z) r dr d\theta \\
 &< \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{6 \pi t_j}{(1-t_1)^{1/3} n^{4/3}} \\
 &= \frac{6 \pi}{(1-t_1)^{1/3}} \sum_{j=1}^{\infty} t_j \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}.
 \end{aligned}$$

That this expression is finite follows from (6.0); hence $U(z)$ is summable over D .

To conclude, we see that for any fixed value j , and any sequence $\{\zeta_n^{(j)}\}$, $\zeta_n^{(j)} \in D_n^{(j)}$, $n = 1, 2, \dots$, $U_1(\zeta_j)(1 - |\zeta_j|) \rightarrow \infty$ as $n \rightarrow \infty$. Thus the sets $M_n^{(t)}$ do not exist for the sequence $\{\zeta_n^{(j)}\}$ and any $t \leq t_j$. Since this behavior is true for all values j the exceptional set for $U_1(z)$ contains the set of points e^{it_j} , $j = 1, 2, \dots$.

Our next example concerns the behavior of a function $U_2(z)$ whose integral over D diverges in some specified manner but for each $e^{i\theta} \in C$, there is a sequence tending to $e^{i\theta}$ and a value t , $0 < t < 1$, for which the sets $M_n^{(t)}$ fail to exist.

For $0 \leq r < 1$ let $\Psi(r)$ denote any real-valued function such that

$$(6.7) \quad \begin{aligned}
 &\text{I) } \Psi(r) \text{ is a non-decreasing function of } r, 0 \leq r < 1; \\
 &\text{II) } \Psi(0) = 0, \lim_{r \rightarrow 1} \Psi(r) = \infty.
 \end{aligned}$$

EXAMPLE 2. Let $\Psi(r)$ be an arbitrary function satisfying (6.7). Then there

exists a real-valued, non-negative, measurable function $U_2(z)$ defined in D with the property that the double integral of $U_2(z)$ over D is infinite but, setting $z = re^{i\theta}$,

$$\int_0^\rho \int_0^{2\pi} U_2(z) r d\theta dr < \Psi(\rho)$$

for $0 \leq \rho < 1$. Further for any $\theta \in [0, 2\pi]$ there is a sequence $\{z_j\}$ in D which tends to $e^{i\theta}$ such that, if $\zeta_j \in D(z_j, (1 - |z_j|)1/8)$, then

$$\lim_{j \rightarrow \infty} U_2(\zeta_j)(1 - |\zeta_j|) = \infty.$$

Define a sequence of concentric, disjoint rings in D as follows: let a sequence of positive numbers $\{n_j\}$ be chosen so that

$$(6.8) \quad n_{j+1} \geq 2 n_j,$$

and

$$(6.9) \quad 2 \pi j^2 \log 2 \leq \Psi\left(1 - \frac{1}{n_j}\right), \quad j = 1, 2, \dots$$

That such a sequence exists follows from the monotonicity of $\Psi(r)$. Next let R_j be the ring $\left\{z \in D \mid 1 - \frac{1}{n_j} \leq |z| < 1 - \frac{1}{2 n_j}\right\}$ $j = 1, 2, \dots$. Note that these rings are disjoint by (6.8). Lastly select any function $\Psi^*(r)$ satisfying (6.7) and in addition

$$(6.10) \quad 1 \leq \Psi^*\left(1 - \frac{1}{2 n_j}\right) \leq j \quad j = 1, 2, \dots$$

The desired function is

$$U_2(z) = \begin{cases} \frac{\Psi^*(|z|)}{1 - |z|}, & z \in \bigcup_{j=1}^{\infty} R_j; \\ 0, & z \in D - \bigcup_{j=1}^{\infty} R_j. \end{cases}$$

To demonstrate that the integral of $U_2(z)$ over D diverges in the proper fashion fix a value $r_0, 0 < r_0 < 1$, and let j_0 be chosen so that

$$(6.11) \quad 1 - \frac{1}{n_{j_0}} \leq r_0 < 1 - \frac{1}{n_{j_0+1}}.$$

Since $U_2(z)$ vanishes except on the ring R_j , (6.9), (6.10) and (6.11) imply

$$\int_0^{r_0} \int_0^{2\pi} U_2(z) r d\theta dr \leq \sum_{j=1}^{j_0} \iint_{R_j} \frac{\Psi^*(r) r d\theta dr}{1 - |z|}$$

$$\begin{aligned} &< 2 \pi j_0^2 \log 2 \\ &< \Psi(r_0). \end{aligned}$$

A short calculation yields that the integral of $U_2(z)$ over D is infinite.

Finally to exhibit sequences which have the desired properties consider, for any $0 \leq \theta < 2 \pi$, the sequence $z_j = \left(1 - \frac{3}{4} \frac{1}{n_j}\right) e^{i\theta}$, $j = 1, 2, \dots$. The width of the ring R_j is $\frac{1}{2 n_j}$, while the diameter of $D(z_j, (1 - |z_j|)1/8)$ is $\frac{3}{16} \frac{1}{n_j}$; and since the point z_j is equidistant from the boundary circles of R_j

$$D(z_j, (1 - |z_j|)1/8) \subset R_j, \quad j = 1, 2, \dots$$

From the definition of $U_2(z)$ and $\Psi^*(|z|)$ if $\zeta_j \in D(z_j, (1 - |z_j|)1/8)$, $j = 1, 2, \dots$, then

$$\lim_{j \rightarrow \infty} U_2(\zeta_j)(1 - |\zeta_j|) = \infty.$$

The last example indicates that the rate of growth demonstrated in Theorem 1 cannot be improved.

EXAMPLE 3. Let $\Psi(r)$ be any function satisfying (6.7). Then there exists a real valued, nonnegative, measurable function $U_3(z)$ defined in D for which

$$\iint_D U_3(z) dx dy < \infty ;$$

further for any $0 \leq \theta < 2 \pi$ there is a sequence $\{z_j\}$ in D tending radially to $e^{i\theta}$ with the property that if $\zeta_j \in D(z_j, (1 - |z_j|)1/8)$, $j = 1, 2, \dots$, then

$$\lim_{j \rightarrow \infty} U_3(\zeta_j)(1 - |\zeta_j|)\Psi(|\zeta_j|) = \infty.$$

Let $\Psi^*(r)$ be any function satisfying (6.7) and such that as $r \rightarrow 1$, $\Psi(r)/\Psi^*(r) \rightarrow \infty$. As before, define a sequence of concentric, disjoint rings R_j in D by first specifying a sequence of positive integers $\{n_j\}$ with

$$(6.12) \quad \begin{aligned} \text{I) } &\Psi^*\left(1 - \frac{1}{n_j}\right) \geq j^2; \\ \text{II) } &n_{j+1} \geq 2 n_j \end{aligned}$$

for $j = 1, 2, \dots$;

then set $R_j = \left\{z \mid 1 - \frac{1}{n_j} \leq |z| < 1 - \frac{1}{2 n_j}\right\}$, $j = 1, 2, \dots$. Define

$$U_3(z) = \begin{cases} \frac{1}{(1-|z|)\Psi^*(|z|)}, & z \in \bigcup_{j=1}^{\infty} R_j; \\ 0, & z \in D - \bigcup_{j=1}^{\infty} R_j. \end{cases}$$

The finiteness of the double integral of $U_3(z)$ over D follows from

$$\begin{aligned} \iint_D U_3(r) r dr d\theta &= \sum_{j=1}^{\infty} \iint_{R_j} \frac{r dr d\theta}{(1-r)\Psi^*(|z|)} \\ &< \sum_{j=1}^{\infty} \frac{2\pi \log 2}{\Psi^*\left(1 - \frac{1}{n_j}\right)}. \end{aligned}$$

By (6.12) the last term is less than or equal to

$$\sum_{j=1}^{\infty} \frac{2\pi \log 2}{j^2}.$$

For arbitrary $e^{i\theta}$ let $z_j = \left(1 - \frac{3}{4} \frac{1}{n_j}\right) e^{i\theta}$, $j = 1, 2, \dots$. As in the preceding example $D(z_j, (1 - |z_j|)1/8) \subset R_j$ for all values j . Thus if $\zeta_j \in D(z_j, (1 - |z_j|)1/8)$

$$\lim_{j \rightarrow \infty} U_3(\zeta_j)(1 - |\zeta_j|)\Psi^*(|\zeta_j|) = 1,$$

then the definition of $\Psi^*(r)$ gives

$$\lim_{j \rightarrow \infty} U_3(\zeta_j)(1 - |\zeta_j|)\Psi(|\zeta_j|) = \infty.$$

This completes the proof of example 3.

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Pennsylvania State University
University Park, Pennsylvania