Nagoya Math. J., **254** (2024), 462–487
DOI 10.1017/nmj.2023.33

LOCAL SECTIONS OF ARITHMETIC FUNDAMENTAL GROUPS OF p-ADIC CURVES

MOHAMED SAÏDI

Abstract. We investigate *sections* of the arithmetic fundamental group $\pi_1(X)$ where X is either a *smooth affinoid p-adic curve*, or a *formal germ of a p-adic curve*, and prove that they can be lifted (unconditionally) to sections of cuspidally abelian Galois groups. As a consequence, if X admits a compactification Y, and the exact sequence of $\pi_1(X)$ *splits*, then $\mathrm{index}(Y) = 1$. We also exhibit a necessary and sufficient condition for a section of $\pi_1(X)$ to arise from a *rational point* of Y. One of the key ingredients in our investigation is the fact, we prove in this paper in case X is affinoid, that the Picard group of X is *finite*.

§0. Introduction/Main results

This paper is motivated by the p-adic analog of the anabelian Grothendieck section conjecture.

Let $p \geq 2$ be a prime number, let $k/\mathbb{Q}_p$ be a finite extension, and let Y be a proper, smooth, and geometrically connected hyperbolic k-curve. The arithmetic fundamental group $\pi_1(Y)$ of Y projects onto the Galois group $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ of k. A k-rational point $x : \mathrm{Spec}\, k \to Y$ gives rise, by functoriality of fundamental groups, to a section $s_x : G_k \to \pi_1(Y)$ of the projection $\pi_1(Y) \twoheadrightarrow G_k$. We shall refer to such a section s_x as *geometric*.

QUESTION A. *Is every section of the projection $\pi_1(Y) \twoheadrightarrow G_k$ geometric?*

In [22, Theorem 2 in the Introduction], we established two necessary and sufficient conditions for a group-theoretic section of the projection $\pi_1(Y) \twoheadrightarrow G_k$ to be geometric. In [13], Hoshi constructed a group-theoretic section $G_k \to \pi_1(Y)^{(p)}$ of the projection $\pi_1(Y)^{(p)} \twoheadrightarrow G_k$ for a specific example Y, where $\pi_1(Y)^{(p)}$ is the geometrically pro-p quotient of $\pi_1(Y)$, which is *not geometric* (i.e., does not arise from a scheme morphism $x : \mathrm{Spec}\, k \to Y$). The author is not aware of any example of a Y as above and a group-theoretic section of the projection $\pi_1(Y) \twoheadrightarrow G_k$ which is not geometric.

Let X be either a geometrically connected *affinoid* subspace of Y^{rig}, the rigid analytic curve associated with Y, or a *formal germ* of Y, meaning $X = \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_{\mathcal{O}_k} k)$ is geometrically connected, where $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is the completion of the local ring $\mathcal{O}_{\mathcal{Y},y}$ of a model $\mathcal{Y}$ of Y over the ring of valuation $\mathcal{O}_k$ of k at a closed point $y \in \mathcal{Y}^{\mathrm{cl}}$ (cf. Notations). Let $\pi_1(X)$ be the étale fundamental group of X which sits in the exact sequence (cf. Notations)

$$1 \to \pi_1(X)^{\mathrm{geo}} \to \pi_1(X) \to G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k) \to 1.$$

A section $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$ induces a section $s_Y : G_k \to \pi_1(Y)$ of the projection $\pi_1(Y) \twoheadrightarrow G_k$ (cf. Notations, diagram (0.1)) which we shall refer to as a

Received August 3, 2022. Revised August 31, 2023. Accepted October 29, 2023.
2020 Mathematics subject classification: Primary 14H25.
Keywords: sections of arithmetic fundamental groups, p-adic curves, rational points.

local section of the projection $\pi_1(Y) \twoheadrightarrow G_k$. A geometric section is necessarily a local section as one easily verifies. This prompts the following question, which motivates our study in this paper of local sections of arithmetic fundamental groups of p-adic curves.

QUESTION B. *Is every local section of the projection $\pi_1(Y) \twoheadrightarrow G_k$ geometric?*

Motivated by Questions A and B, we investigate sections of arithmetic fundamental groups of affinoid k-curves and formal p-adic germs of curves.

Let X be either a smooth and geometrically connected *k-affinoid* curve or a *formal p-adic germ* (cf. Notations for precise definitions). Let $\pi_1(X)^{\text{geo,ab}}$ be the maximal abelian quotient of $\pi_1(X)^{\text{geo}}$, and let $\pi_1(X)^{(\text{ab})}$ be the *geometrically abelian* quotient of $\pi_1(X)$ which sits in the exact sequence

$$1 \to \pi_1(X)^{\text{geo,ab}} \to \pi_1(X)^{(\text{ab})} \to G_k \to 1.$$

Similarly, let $G_X \overset{\text{def}}{=} \text{Gal}(\overline{L}/L)$ be the absolute Galois group of the function field L of X (see Notations for the definition of L) which sits in the exact sequence (cf. §1)

$$1 \to G_X^{\text{geo}} \to G_X \to G_k \to 1.$$

Let $G_X^{\text{geo,ab}}$ be the maximal abelian quotient of G_X^{geo}, and let $G_X^{(\text{ab})}$ be the *geometrically abelian* quotient of G_X which sits in the exact sequence

$$1 \to G_X^{\text{geo,ab}} \to G_X^{(\text{ab})} \to G_k \to 1.$$

We have an exact sequence

$$1 \to \widetilde{\mathcal{H}}_X \to G_X^{(\text{ab})} \to \pi_1(X)^{(\text{ab})} \to 1,$$

where $\widetilde{\mathcal{H}}_X \overset{\text{def}}{=} \text{Ker}[G_X^{(\text{ab})} \twoheadrightarrow \pi_1(X)^{(\text{ab})}]$. In §1, we investigate the structure of the G_k-module $\widetilde{\mathcal{H}}_X$. We prove in Proposition 1.4 that $\widetilde{\mathcal{H}}_X$ is (canonically) isomorphic to $\prod_{x \in X^{\text{cl}}} \text{Ind}_{k(x)}^{k} \hat{\mathbb{Z}}(1)$ where the product is over all closed points of X and $k(x)$ is the residue field at x.

The Galois group G_X sits in an exact sequence

$$1 \to \mathcal{H}_X \to G_X \to \pi_1(X) \to 1,$$

where $\mathcal{H}_X \overset{\text{def}}{=} \text{Ker}[G_X \twoheadrightarrow \pi_1(X)]$. Let $\mathcal{H}_X^{\text{ab}}$ be the maximal abelian quotient of $\mathcal{H}_X$, and let $G_X^{(\text{c}-\text{ab})}$ be the *geometrically cuspidally abelian* quotient of G_X which sits in the exact sequence

$$1 \to \mathcal{H}_X^{\text{ab}} \to G_X^{(\text{c}-\text{ab})} \to \pi_1(X) \to 1.$$

In §2, we investigate, in the framework of the theory of *cuspidalization* of sections of arithmetic fundamental groups (cf. [19], [22]), sections $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$. Let Y be a *k-compactification* of X, and let $s_Y : G_k \to \pi_1(Y)$ be the induced *local section* of the projection $\pi_1(Y) \twoheadrightarrow G_k$ (cf. Notations for precise definitions and the diagram (0.1) therein). One of our main results is the following (cf. Theorems 2.4 and 3.1(ii)).

THEOREM A (Lifting of sections to cuspidally abelian Galois groups). *Let $s : G_k \to \pi_1(X)$ be a section of the projection $\pi_1(X) \twoheadrightarrow G_k$. The followings hold.*

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(i) *There exists a section* $s^{c-\mathrm{ab}}: G_k \to G_X^{(c-\mathrm{ab})}$ *of the projection* $G_X^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lifts the section* s*, that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{s^{c-\mathrm{ab}}} & G_X^{(c-\mathrm{ab})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X) \end{array} \tag{0.1}$$

where the right vertical map is the natural projection $G_X^{(c-\mathrm{ab})} \twoheadrightarrow \pi_1(X)$*. In particular, the set of sections of the projection* $G_X^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lift the section* s *is non-empty, and is (up to conjugation by elements of* $\mathcal{H}_X^{\mathrm{ab}}$*) a torsor under* $H^1(G_k, \mathcal{H}_X^{\mathrm{ab}})$*.*

(ii) *Assume* Y *is hyperbolic. Then the section* $s_Y : G_k \to \pi_1(Y)$ *induced by* s *is uniformly orthogonal to* Pic *in the sense of [19, Definition 1.4.1].*

The section s is uniformly orthogonal to Pic (as in (ii) above) means that the retraction map $s^* : H^2(\pi_1(Y), \hat{\mathbb{Z}}(1)) \xrightarrow{\sim} H^2_{\mathrm{et}}(Y, \hat{\mathbb{Z}}(1)) \to H^2(G_k, \hat{\mathbb{Z}}(1))$, which is induced by the section s, annihilates the Picard part of $H^2_{\mathrm{et}}(Y, \hat{\mathbb{Z}}(1))$, and similarly for every neighborhood $Y' \to Y$ of the section s.

Theorem A(ii) implies that local sections of arithmetic fundamental groups of hyperbolic p-adic curves satisfy condition (i) in [22, Theorem 2 in the Introduction]. In this sense, local sections are close to being geometric. Establishing Theorem A(ii) was one of the main motivations for the author to investigate local sections of arithmetic fundamental groups of p-adic curves. Apart from local sections, and geometric sections, the author is not aware (for the time being) of any examples of group-theoretic sections of arithmetic fundamental groups of hyperbolic p-adic curves which are orthogonal to Pic.

As a consequence of Theorem A, and an observation of Esnault and Wittenberg on geometrically abelian sections of p-adic curves, we deduce the following (cf. Theorem 2.5).

Theorem B. *Assume that* X *admits a* k*-compactification* Y *(cf. Notations). If the projection* $\pi_1(X) \twoheadrightarrow G_k$ *splits, then* $\mathrm{index}(Y) = 1$*.*

Theorem B asserts that the existence of local sections of arithmetic fundamental groups of p-adic curves implies the existence of degree 1 rational divisors. The link between sections of geometrically abelian Galois groups and the existence of degree 1 rational divisors has been investigated in [5].

In §3, we assume that X admits a k-compactification Y (cf. Notations). Let $\Pi_Y[X]$ be the étale fundamental group which classifies finite covers $Y' \to Y$ which only ramify at points of Y *not in* X (cf. 3.3, as well as Notations for the meaning of *not in* X). A section $s : G_k \to \pi_1(X)$ of the projection $\pi_1(X) \twoheadrightarrow G_k$ induces naturally a section $s^\dagger : G_k \to \Pi_Y[X]$ of the projection $\Pi_Y[X] \twoheadrightarrow G_k$. We say that the section s is *geometric* (relative to Y) if the image $s^\dagger(G_k)$ is contained in a decomposition group $D_x \subset \Pi_Y[X]$ associated with a rational point $x \in Y(k)$ (cf. Definition 3.3.2). Further, we say that s is *admissible* (relative to Y) (cf. Definition 3.5.1) if for every open subgroup $H \subset \Pi_Y[X]$ with $s^\dagger(G_k) \subset H$, corresponding to (a possibly ramified) cover $Y' \to Y$, the following holds. Let $G_{Y'}^{(1/p^2-\mathrm{sol})}$ be the *geometrically cuspidally* $1/p^2$*-solvable Galois group* of Y': that is, the maximal quotient $G_{Y'} \twoheadrightarrow H \twoheadrightarrow \pi_1(Y')$ of the absolute Galois group $G_{Y'}$ of Y' such that $\mathrm{Ker}[H \twoheadrightarrow \pi_1(Y')]$ is abelian annihilated by p^2 (cf. [22, 3.1]). There exists a section $\tilde{s}_{Y'} : G_k \to G_{Y'}^{(1/p^2-\mathrm{sol})}$ of the

https://doi.org/10.1017/nmj.2023.33 Published online by Cambridge University Press

projection $G_{Y'}^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ (such a section exists unconditionally [see discussion in 3.5]) satisfying the following property:

For each open subgroup $F \subset G_{Y'}^{(1/p^2-\mathrm{sol})}$ *with* $\tilde{s}_{Y'}(G_k) \subset F$, *corresponding to a (possibly ramified) cover* $Y'' \to Y'$ *with* Y'' *geometrically connected, the class of* $\mathrm{Pic}^1_{Y''}$ *in* $H^1(G_k, \mathrm{Pic}^0_{Y''})$ *is divisible by* p.

Our main result in §3 is the following (cf. Theorem 3.5.2).

THEOREM C. *The section* $s: G_k \to \pi_1(X)$ *is geometric (relative to* Y*) if and only if* s *is admissible (relative to* Y*).*

One of the key ingredients used in the proofs of the above results is the fact that $\mathrm{Pic}(X)$ is *finite*. In the case where X is a *formal p-adic germ*, this is established in [22, Proposition 5.4], as a consequence of a result of Shuji Saito (cf. [22, Proposition 5.4]). In case X is *affinoid*, this is proven in §4 (cf. Proposition 4.1) and may be of interest independently of the topics discussed in this paper. More precisely, we prove the following.

THEOREM D (Picard groups of affinoid p-adic curves). *Let k be a p-adic local field (i.e.,* $k/\mathbb{Q}_p$ *is a finite extension), and let* $X = \mathrm{Sp}(A)$ *be a smooth and geometrically connected k-affinoid curve. Then the Picard group* $\mathrm{Pic}(X)$ *is finite.*

Finally, in §5, we prove (cf. Proposition 5.1) a compactification result for two-dimensional complete local p-adic rings which is used in the proofs of Propositions 1.2 and 2.2.

The results in §4 and §5 are used in this paper in §2 and §3; none of the results in §2 and §3 is used in §4 and §5.

In this paper, we worked with full arithmetic fundamental groups. Instead, one could consider a similar setting and work with geometrically pro-p arithmetic fundamental groups and Galois groups as in [22] (where one considers geometrically pro-Σ arithmetic fundamental groups and Galois groups, Σ being a set of primes containing p). In this geometrically pro-p (pro-Σ) setting, one can prove analogs of Theorems A and C.

Notations. The following notations will be used throughout this paper (unless we specify otherwise).

- $p \geq 2$ is a prime number, and k is a *p-adic local field* (i.e., $k/\mathbb{Q}_p$ is a finite extension) with ring of valuation $\mathcal{O}_k$, uniformizer π, and residue field F. Thus, F is a finite field of characteristic p.
- A proper, smooth, and geometrically connected k-curve Y is *hyperbolic* if $\mathrm{genus}(Y) \geq 2$.
- For a profinite group H, we denote by H^{ab} the maximal *abelian* quotient of H.
- Let

$$1 \to H' \to H \xrightarrow{\mathrm{pr}} G \to 1$$

  be an exact sequence of profinite groups. We will refer to a continuous homomorphism $s: G \to H$ such that $\mathrm{pr} \circ s = \mathrm{id}_G$ as a (group-theoretic) *section* of the above sequence, or simply a section of the projection $\mathrm{pr}: H \twoheadrightarrow G$.
- All scheme cohomology groups considered in this paper are étale cohomology groups.

0.1 Affinoid p-adic curves

- $X = \mathrm{Sp}\, A$ is a *smooth* and geometrically connected *affinoid k-curve*. On occasions, we will write, if there is no risk of confusion, $X = \mathrm{Spec}\, A$ for the corresponding affine k-scheme.

- One can embed X into a proper, smooth, and geometrically connected rigid analytic curve $Y^{\mathrm{rig}}: X \hookrightarrow Y^{\mathrm{rig}}$ so that X is an open affinoid subspace of Y^{rig} (cf. [6, 2.6, Corollaire 2]). Write Y for the algebraization of Y^{rig} via the rigid GAGA functor, which is a proper, smooth, and geometrically connected algebraic k-curve. We will refer to X as a *p-adic affinoid curve* (or simply an affinoid) and Y a *k-compactification* of X.

0.2 Formal p-adic germs

- A is a *normal two-dimensional complete local ring* containing $\mathcal{O}_k$ with maximal ideal $\mathfrak{m}_A$ containing π and residue field $F = A/\mathfrak{m}_A$. Write $A_k \overset{\mathrm{def}}{=} A \otimes_{\mathcal{O}_k} k = A[\frac{1}{\pi}]$ and $X \overset{\mathrm{def}}{=} \operatorname{Spec} A_k$. We assume X is geometrically connected and refer to X as a *formal p-adic germ*.
- A (k-)*compactification* of $\operatorname{Spec} A$ is a proper and flat relative $\mathcal{O}_k$-curve $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_k$ with $\mathcal{Y}$ normal, $Y \overset{\mathrm{def}}{=} \mathcal{Y} \times_{\operatorname{Spec} \mathcal{O}_k} \operatorname{Spec} k$ geometrically connected, $y \in \mathcal{Y}^{\mathrm{cl}}$ is a closed point, $\mathcal{O}_{\mathcal{Y},y}$ is the local ring of $\mathcal{Y}$ at y, $\hat{\mathcal{O}}_{\mathcal{Y},y}$ its completion, with an isomorphism $\hat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\sim} A$. We have a natural scheme morphism $X \to Y$. We shall refer to Y as a *k-compactification* of X. In §5, we prove the existence of such a compactification $X \to Y$ after possibly a finite extension of k (cf. Proposition 5.1).

In what follows, X is either an *affinoid p-adic curve* or a *formal p-adic germ*.

- We say that X is *hyperbolic* if there exists a finite extension k'/k such that $X_{k'} \overset{\mathrm{def}}{=} \operatorname{Spec}(A \otimes_k k')$ (resp. $X_{k'} \overset{\mathrm{def}}{=} \operatorname{Sp}(A \otimes_k k')$ if X is affinoid) possesses a k'-compactification Y with Y hyperbolic. There exist a finite extension k'/k and a finite geometric étale cover $X' \to X_{k'}$ with X' geometrically connected and hyperbolic. This is Proposition 5.3 in case X is a *formal p-adic germ* and follows from [21. Theorem A] in case X is *affinoid*.
- η is a fixed choice of a geometric point of X with values in its generic point. Thus, η determines algebraic closures $\bar{k}$, $\overline{L}$, of k, and $L \overset{\mathrm{def}}{=} \operatorname{Fr}(A)$, respectively. We have an exact sequence of fundamental groups

$$1 \longrightarrow \pi_1(X,\eta)^{\mathrm{geo}} \longrightarrow \pi_1(X,\eta) \longrightarrow G_k \longrightarrow 1,$$

  where $\pi_1(X,\eta)$ is the étale fundamental group of X with geometric point η (cf. [21, 2.1] for more details on the definition of $\pi_1(X,\eta)$ in case X is an affinoid), $\pi_1(X,\eta)^{\mathrm{geo}} \overset{\mathrm{def}}{=} \operatorname{Ker}[\pi_1(X,\eta) \twoheadrightarrow G_k]$, and $G_k \overset{\mathrm{def}}{=} \operatorname{Gal}(\bar{k}/k)$ is the absolute Galois group of k.

In what follows, Y is a *k-compactification* of X.

- We have a commutative diagram of exact sequences of arithmetic fundamental groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X,\eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(Y_{\bar{k}},\bar{\eta}) & \longrightarrow & \pi_1(Y,\eta) & \longrightarrow & G_k & \longrightarrow & 1, \end{array} \tag{0.2}$$

  where $\pi_1(Y,\eta)$ (resp. $\pi_1(Y_{\bar{k}},\bar{\eta})$) is the étale fundamental group of Y (resp. $Y_{\bar{k}} \overset{\mathrm{def}}{=} Y \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$) with geometric point η (resp. $\bar{\eta}$ which is induced by η). In case X is an *affinoid* (resp. a *formal p-adic germ*), the middle vertical map is induced by the rigid analytic morphism $X \to Y^{\mathrm{rig}}$ and the rigid GAGA functor (resp. the scheme morphism $X \to Y$).

- We write X^{cl} (resp. Y^{cl}) for the set of closed points of X (resp. Y). For a closed point x of X (resp. Y), we write $k(x)$ for the residue field at x. Thus, $k(x)$ is a finite extension of k.
- We say that $x \in Y^{\mathrm{cl}}$ is *not* in X if x is not in the image of the scheme morphism $X \to Y$ if X is a *formal p-adic germ* or $x \notin X^{\mathrm{cl}}$ in case X is *affinoid*. In case $X = \mathrm{Spec}(\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_k} k)$ is a *formal p-adic germ*, the set of closed points of Y *not* in X is in one-to-one correspondence with the set of closed points of Y which *do not* specialize in y (cf. [16, §10, Proposition 1.40(a)]).

Throughout §§1–3, X will denote either an affinoid p-adic curve or a formal p-adic germ. In §3, we will assume X admits a k-compactification Y which is hyperbolic and fix a choice of such a compactification throughout.

§1. Geometrically abelian arithmetic fundamental groups

In this section, we investigate the structure of various geometrically abelian arithmetic fundamental groups and absolute Galois group associated with X. Let

$$\pi_1(X,\eta)^{(\mathrm{ab})} \overset{\mathrm{def}}{=} \pi_1(X,\eta)/\mathrm{Ker}[\pi_1(X,\eta)^{\mathrm{geo}} \twoheadrightarrow \pi_1(X,\eta)^{\mathrm{geo,ab}}]$$

be the *geometrically abelian* fundamental group of X (here, $\pi_1(X,\eta)^{\mathrm{geo,ab}}$ denotes the maximal abelian quotient of $\pi_1(X,\eta)^{\mathrm{geo}}$).

PROPOSITION 1.1. *We use the above notations. The followings hold.*

(i) *Assume X is an affinoid. For each prime number ℓ, the pro-ℓ-Sylow subgroup of $\pi_1(X,\eta)^{\mathrm{geo,ab}}$ is pro-ℓ abelian free, of infinite rank if $\ell = p$, and finite (computable) rank otherwise (see [21, Theorem A] for the precise value of this rank in case $\ell \neq p$).*

(ii) *Assume X is a formal p-adic germ. For each prime number $\ell \neq p$, the pro-ℓ-Sylow subgroup of $\pi_1(X,\eta)^{\mathrm{geo,ab}}$ is pro-ℓ abelian free of finite computable rank (see [23, Theorem A] for the precise value of this rank).*

Proof. Assertion (i) follows from [21, Theorem A]. (Note that the assumption in [21, Theorem A] that X is the complement in a proper rigid analytic k-curve of the disjoint union of finitely many k-rational open disks is satisfied after a finite extension of k [cf. [6, 2.6, Théorème 6 and Corollaire 1]].) Assertion (ii) follows from [23, Theorem A]. ∎

Let $S \overset{\mathrm{def}}{=} \{x_1,\ldots,x_n\} \subset X^{\mathrm{cl}}$ be a finite set of closed points and write $U \overset{\mathrm{def}}{=} X \setminus S$ viewed as an open subscheme of X (resp. $X = \mathrm{Spec}\,A$ in case X is an affinoid). Let $\pi_1(U,\eta)$ be the étale fundamental group of U with geometric point η (cf. [21, 2.1] for the definition of $\pi_1(U,\eta)$ in case X is affinoid) which sits in the exact sequence

$$1 \longrightarrow \pi_1(U,\eta)^{\mathrm{geo}} \longrightarrow \pi_1(U,\eta) \longrightarrow G_k \longrightarrow 1,$$

where $\pi_1(U,\eta)^{\mathrm{geo}} \overset{\mathrm{def}}{=} \mathrm{Ker}[\pi_1(U,\eta) \twoheadrightarrow G_k]$ (cf. [21, 2.1] in case X is affinoid). Let

$$\pi_1(U,\eta)^{(\mathrm{ab})} \overset{\mathrm{def}}{=} \pi_1(U,\eta)/\mathrm{Ker}[\pi_1(U,\eta)^{\mathrm{geo}} \twoheadrightarrow \pi_1(U,\eta)^{\mathrm{geo,ab}}]$$

be the *geometrically abelian* fundamental group of U (here, $\pi_1(U,\eta)^{\mathrm{geo,ab}}$ is the maximal abelian quotient of $\pi_1(U,\eta)^{\mathrm{geo}}$). We have an exact sequence

$$1 \to \widetilde{\Delta}_U \to \pi_1(U,\eta)^{(\mathrm{ab})} \to \pi_1(X,\eta)^{(\mathrm{ab})} \to 1, \tag{1.1}$$

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where $\widetilde{\Delta}_U \overset{\text{def}}{=} \text{Ker}[\pi_1(U,\eta)^{(\text{ab})} \twoheadrightarrow \pi_1(X,\eta)^{(\text{ab})}] = \text{Ker}[\pi_1(U,\eta)^{\text{geo,ab}} \twoheadrightarrow \pi_1(X,\eta)^{\text{geo,ab}}]$ and the (surjective) map $\pi_1(U,\eta)^{(\text{ab})} \twoheadrightarrow \pi_1(X,\eta)^{(\text{ab})}$ is induced by the natural projection $\pi_1(U,\eta) \twoheadrightarrow \pi_1(X,\eta)$. Note that $\widetilde{\Delta}_U$ has a natural structure of G_k-module.

Proposition 1.2. *We use the above notations. There exists a natural isomorphism*

$$\prod_{i=1}^{n} \text{Ind}_{k(x_i)}^{k} \hat{\mathbb{Z}}(1) \xrightarrow{\sim} \widetilde{\Delta}_U$$

of G_k-modules where the (1) *is a Tate twist.*

Proof. We have a natural surjective homomorphism $\prod_{i=1}^{n} \text{Ind}_{k(x_i)}^{k} \hat{\mathbb{Z}}(1) \twoheadrightarrow \widetilde{\Delta}_U$ of G_k-modules mapping $\text{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1)$ onto the inertia subgroup [of $\pi_1(U,\eta)^{(\text{ab})}$] at x_i, as follows from the structure of inertia groups of Galois extensions of Henselian discrete valuation rings of residue characteristic zero. We show this map is an isomorphism. To this end, we can, without loss of generality, assume that X admits a k-compactification Y (cf. Notations). Indeed, this holds for X affinoid (cf. [21, 2.1]), and holds after possibly replacing k by a finite field extension in case X is a formal p-adic germ (cf. Proposition 5.1) which does not alter the structure of $\widetilde{\Delta}_U$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X,\eta)^{\text{geo,ab}} & \longrightarrow & \pi_1(X,\eta)^{(\text{ab})} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(Y_{\bar{k}},\bar{\eta})^{\text{ab}} & \longrightarrow & \pi_1(Y,\eta)^{(\text{ab})} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array} \tag{1.2}$$

where $\pi_1(Y,\eta)^{(\text{ab})} \overset{\text{def}}{=} \pi_1(Y,\eta)/\text{Ker}[\pi_1(Y_{\bar{k}},\bar{\eta}) \twoheadrightarrow \pi_1(Y_{\bar{k}},\bar{\eta})^{\text{ab}}]$ and the middle vertical map is induced by the natural homomorphism $\pi_1(X,\eta) \to \pi_1(Y,\eta)$ (cf. Notations, diagram (0.1)).

Denote by x_i' the image of x_i in Y, $\forall 1 \le i \le n$ (note that $k(x_i) = k(x_i')$). Let $x_0' \in Y^{\text{cl}} \setminus \{x_1',\dots,x_n'\}$ be a closed point which is not in the image of X (cf. Notations). Write $S' \overset{\text{def}}{=} \{x_0',x_1',\dots,x_n'\} \subset Y^{\text{cl}}$ and $V \overset{\text{def}}{=} Y \setminus S'$ which is an affine k-curve. Let $\pi_1(V,\eta)$ be the étale fundamental group of V with geometric point η which sits in the exact sequence $1 \to \pi_1(V_{\bar{k}},\bar{\eta}) \to \pi_1(V,\eta) \to G_k \to 1$, where $\pi_1(V_{\bar{k}},\bar{\eta})$ is the étale fundamental group of $V_{\bar{k}} \overset{\text{def}}{=} V \times_k \bar{k}$ with geometric point $\bar{\eta}$ which is induced by η. Let $\pi_1(V,\eta)^{(\text{ab})} \overset{\text{def}}{=} \pi_1(V,\eta)/\text{Ker}[\pi_1(V_{\bar{k}},\bar{\eta}) \twoheadrightarrow \pi_1(V_{\bar{k}},\bar{\eta})^{\text{ab}}]$ be the geometrically abelian fundamental group of V. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{\Delta}_U & \longrightarrow & \pi_1(U,\eta)^{(\text{ab})} & \longrightarrow & \pi_1(X,\eta)^{(\text{ab})} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{\Delta}_V & \longrightarrow & \pi_1(V,\eta)^{(\text{ab})} & \longrightarrow & \pi_1(Y,\eta)^{(\text{ab})} & \longrightarrow & 1,
\end{array} \tag{1.3}$$

where $\widetilde{\Delta}_V \overset{\text{def}}{=} \text{Ker}[\pi_1(V,\eta)^{(\text{ab})} \twoheadrightarrow \pi_1(Y,\eta)^{(\text{ab})}]$. The middle vertical map in diagram (1.3) is induced by the natural homomorphism $\pi_1(U,\eta) \to \pi_1(V,\eta)$, which is induced by the scheme morphism $X \to Y$ in case X is a formal p-adic germ, and by the rigid analytic morphism $X \to Y^{\text{rig}}$ and the rigid GAGA functor in case X is affinoid (here, we use the fact that x_0' is not in the image of X). The right vertical map in diagram (1.3) is the middle vertical map in diagram (1.2).

One has an exact sequence of G_k-modules (as follows from the well-known structure of $\pi_1(V,\eta)^{(\mathrm{ab})}$; see, e.g., the discussion in [24, §0])

$$0 \to \hat{\mathbb{Z}}(1) \to \prod_{i=0}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) \to \widetilde{\Delta}_V \to 0.$$

Consider the composite homomorphism $\tau : \prod_{i=1}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) \to \widetilde{\Delta}_V$ of G_k-modules:

$$\prod_{i=1}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) \hookrightarrow \prod_{i=0}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) \twoheadrightarrow \widetilde{\Delta}_V,$$

where the first map is the natural embedding: $(\beta_1,\dots,\beta_n) \mapsto (0,\beta_1,\dots,\beta_n)$ and the second map is as in the above exact sequence. Thus, τ is injective (cf. above exact sequence). Consider the following commutative diagram:

$$\begin{array}{ccc} \prod_{i=1}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) & \longrightarrow & \widetilde{\Delta}_U \\ \downarrow & & \downarrow \\ \prod_{i=0}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) & \longrightarrow & \widetilde{\Delta}_V \end{array}$$

where the right vertical map is the one in diagram (1.3). The left vertical and lower horizontal maps are as explained above; hence, their composite is the map τ. The upper horizontal map is the natural projection $\prod_{i=1}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1) \twoheadrightarrow \widetilde{\Delta}_U$ mentioned at the start of the proof. This map is an isomorphism since it is onto and it is injective as τ is. ∎

REMARK 1.3. With the notations in Proposition 1.2 and the proof therein, assume that $x_0' \in Y(k)$ is a k-rational point. In this case $\tau(\prod_{i=1}^{n} \mathrm{Ind}_{k(x_i')}^{k} \hat{\mathbb{Z}}(1)) = \widetilde{\Delta}_V$, the map $\widetilde{\Delta}_U \to \widetilde{\Delta}_V$ is an isomorphism, and the right square in diagram (1.3) (cf. proof of Proposition 1.2) is cartesian.

Let $G_X \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L)$ (recall $L \overset{\mathrm{def}}{=} \mathrm{Fr}(A)$) which sits in the exact sequences

$$1 \to G_X^{\mathrm{geo}} \to G_X \to G_k \to 1,$$

where $G_X^{\mathrm{geo}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L\bar{k})$, and

$$1 \to \mathcal{H}_X \to G_X \to \pi_1(X,\eta) \to 1, \tag{1.4}$$

where $\mathcal{H}_X \overset{\mathrm{def}}{=} \mathrm{Ker}[G_X \twoheadrightarrow \pi_1(X,\eta)]$. Let

$$G_X^{(\mathrm{ab})} \overset{\mathrm{def}}{=} G_X/\mathrm{Ker}(G_X^{\mathrm{geo}} \twoheadrightarrow G_X^{\mathrm{geo,ab}})$$

which we shall refer to as the *geometrically abelian* Galois group of X (here, $G_X^{\mathrm{geo,ab}}$ is the maximal abelian quotient of G_X^{geo}). We have an exact sequence

$$1 \to \widetilde{\mathcal{H}}_X \to G_X^{(\mathrm{ab})} \to \pi_1(X,\eta)^{(\mathrm{ab})} \to 1, \tag{1.5}$$

where $\widetilde{\mathcal{H}}_X \overset{\mathrm{def}}{=} \mathrm{Ker}[G_X^{(\mathrm{ab})} \twoheadrightarrow \pi_1(X,\eta)^{(\mathrm{ab})}] = \mathrm{Ker}[G_X^{\mathrm{geo,ab}} \twoheadrightarrow \pi_1(X,\eta)^{\mathrm{geo,ab}}]$. Note that $\widetilde{\mathcal{H}}_X$ has a natural structure of G_k-module.

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Proposition 1.4. *We use the above notations. There exists a natural isomorphism of G_k-modules*

$$\prod_{x\in X^{\mathrm{cl}}} \mathrm{Ind}_{k(x)}^{k}\,\hat{\mathbb{Z}}(1) \xrightarrow{\sim} \widetilde{\mathcal{H}}_X,$$

where the product is over all closed points $x \in X^{\mathrm{cl}}$.

Proof. This follows from Proposition 1.2 and the fact that $\widetilde{\mathcal{H}}_X \xrightarrow{\sim} \varprojlim_U \widetilde{\Delta}_U$ where $U = X \setminus S$; S runs over all finite subsets of X^{cl}, and $\widetilde{\Delta}_U$ is as in the proof of Proposition 1.2. (Note that $G_X^{(\mathrm{ab})} \xrightarrow{\sim} \varprojlim_U \pi_1(U,\eta)^{(\mathrm{ab})}$ where the limit runs over all U as above.) ∎

§2. Cuspidally abelian arithmetic fundamental groups

In this section, we investigate the problem of *cuspidalization* of sections of the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$. This problem has been investigated in the case of proper and smooth hyperbolic p-adic curves in [19], [22]. We use the notations in §0 and §1.

Let $S \overset{\mathrm{def}}{=} \{x_1,\dots,x_n\} \subset X^{\mathrm{cl}}$ be a finite set of closed points, and let $U \overset{\mathrm{def}}{=} X \setminus S$ (cf. §1). Consider the exact sequence

$$1 \to \Delta_U \to \pi_1(U,\eta)^{\mathrm{geo}} \to \pi_1(X,\eta)^{\mathrm{geo}} \to 1,$$

where $\Delta_U \overset{\mathrm{def}}{=} \mathrm{Ker}[\pi_1(U,\eta)^{\mathrm{geo}} \twoheadrightarrow \pi_1(X,\eta)^{\mathrm{geo}}]$. The maximal abelian quotient Δ_U^{ab} of Δ_U is a $\pi_1(X,\eta)^{\mathrm{geo}}$-module. Let Δ_U^{cn} be the maximal quotient of Δ_U^{ab} on which $\pi_1(X,\eta)^{\mathrm{geo}}$ acts trivially. Define

$$\pi_1(U,\eta)^{\mathrm{geo,c-ab}} \overset{\mathrm{def}}{=} \pi_1(U,\eta)^{\mathrm{geo}}/\mathrm{Ker}(\Delta_U \twoheadrightarrow \Delta_U^{\mathrm{ab}})$$

and

$$\pi_1(U,\eta)^{\mathrm{geo,c-cn}} \overset{\mathrm{def}}{=} \pi_1(U,\eta)^{\mathrm{geo}}/\mathrm{Ker}(\Delta_U \twoheadrightarrow \Delta_U^{\mathrm{cn}}).$$

We shall refer to $\pi_1(U,\eta)^{\mathrm{geo,c-ab}}$ (resp. $\pi_1(U,\eta)^{\mathrm{geo,c-cn}}$) as the *cuspidally abelian* (resp. *cuspidally central*) quotient of $\pi_1(U,\eta)^{\mathrm{geo}}$. Further, define

$$\pi_1(U,\eta)^{(\mathrm{c-ab})} \overset{\mathrm{def}}{=} \pi_1(U,\eta)/\mathrm{Ker}(\Delta_U \twoheadrightarrow \Delta_U^{\mathrm{ab}})$$

and

$$\pi_1(U,\eta)^{(\mathrm{c-cn})} \overset{\mathrm{def}}{=} \pi_1(U,\eta)/\mathrm{Ker}(\Delta_U \twoheadrightarrow \Delta_U^{\mathrm{cn}}).$$

We shall refer to $\pi_1(U,\eta)^{(\mathrm{c-ab})}$ (resp. $\pi_1(U,\eta)^{(\mathrm{c-cn})}$) as the *(geometrically) cuspidally abelian* (resp. *[geometrically] cuspidally central*) quotient of $\pi_1(U,\eta)$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_U & \longrightarrow & \pi_1(U,\eta) & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_U^{\mathrm{ab}} & \longrightarrow & \pi_1(U,\eta)^{(\mathrm{c-ab})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_U^{\mathrm{cn}} & \longrightarrow & \pi_1(U,\eta)^{(\mathrm{c-cn})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{\Delta}_U & \longrightarrow & \pi_1(U,\eta)^{(\mathrm{ab})} & \longrightarrow & \pi_1(X,\eta)^{(\mathrm{ab})} & \longrightarrow & 1
\end{array} \tag{2.1}$$

where the middle vertical maps are surjective, and the middle vertical map in the lower diagram is induced by the natural surjective map $\pi_1(U,\eta)^{\mathrm{geo,c-ab}} \twoheadrightarrow \pi_1(U,\eta)^{\mathrm{geo,ab}}$. (Note that $\pi_1(X,\eta)^{\mathrm{geo}}$ acts trivially on the quotient $\widetilde{\Delta}_U$ of Δ_U^{ab}.)

Lemma 2.1. *We use the above notations. The homomorphism $\Delta_U^{\mathrm{cn}} \to \widetilde{\Delta}_U$ in diagram (2.1) is an isomorphism of G_k-modules. In particular, the lower right square in diagram (2.1) is Cartesian.*

Proof. The proof follows from Proposition 1.2 and the various definitions. More precisely, there exists a natural surjective homomorphism $\prod_{i=1}^n \mathrm{Ind}_{k(x_i)}^k \hat{\mathbb{Z}}(1) \twoheadrightarrow \Delta_U^{\mathrm{cn}}$ (mapping $\mathrm{Ind}_{k(x_i')}^k \hat{\mathbb{Z}}(1)$ onto the inertia subgroup of $\pi_1(U,\eta)^{(\mathrm{c-cn})}$ at x_i, as follows from the structure of inertia groups of Galois extensions of Henselian discrete valuation rings of residue characteristic zero) which composed with the projection $\Delta_U^{\mathrm{cn}} \twoheadrightarrow \widetilde{\Delta}_U$ is the isomorphism $\prod_{i=1}^n \mathrm{Ind}_{k(x_i)}^k \hat{\mathbb{Z}}(1) \xrightarrow{\sim} \widetilde{\Delta}_U$ in Proposition 1.2 hence our assertion. ∎

Let $s : G_k \to \pi_1(X,\eta)$ be a *section* of the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$.

Proposition 2.2 (Lifting of sections to cuspidally central arithmetic fundamental groups). *We use the above notations. There exists a section $s_U^{c-\mathrm{cn}} : G_k \to \pi_1(U,\eta)^{(c-\mathrm{cn})}$ of the projection $\pi_1(U,\eta)^{(c-\mathrm{cn})} \twoheadrightarrow G_k$ which lifts the section s, that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{s_U^{c-\mathrm{cn}}} & \pi_1(U,\eta)^{(c-\mathrm{cn})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X,\eta) \end{array}$$

where the right vertical map is the natural projection $\pi_1(U,\eta)^{(c-\mathrm{cn})} \twoheadrightarrow \pi_1(X,\eta)$. In particular, the set of sections of the projection $\pi_1(U,\eta)^{(c-\mathrm{cn})} \twoheadrightarrow G_k$ which lift the section s is non-empty, and is (up to conjugation by elements of Δ_U^{cn}) a torsor under $H^1(G_k, \Delta_U^{\mathrm{cn}})$.

Proof. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_U^{\mathrm{cn}} & \longrightarrow & E_U \overset{\mathrm{def}}{=} E_U[s] & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow s & & \\ 1 & \longrightarrow & \Delta_U^{\mathrm{cn}} & \longrightarrow & \pi_1(U,\eta)^{(\mathrm{c-cn})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1 \end{array}$$

where the right square is Cartesian. Thus, the group extension E_U is the pullback of the group extension $\pi_1(U,\eta)^{(\mathrm{c-cn})}$ by the section s. The set of (possible) splittings of the group extension E_U is in one-to-one correspondence with the set of sections of the projection $\pi_1(U,\eta)^{(\mathrm{c-cn})} \twoheadrightarrow G_k$ which lift the section s. We show that the group extension E_U splits.

To this end, we can replace k by a finite extension over which the points $\{x_i\}_{i=1}^n$ are rational, and we can also assume $n = 1$ (see the argument at the start of the proof of Lemma 2.3.1 in [19]). Further, we can replace X by a neighborhood X' of the section s: that is, an étale cover $X' \to X$ corresponding to an open subgroup $H = \pi_1(X',\eta) \subset \pi_1(X,\eta)$ containing the image $s(G_k)$ of s. Indeed, if $U' \overset{\mathrm{def}}{=} U \times_X X'$, there exists a commutative

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diagram of natural homomorphisms

$$\begin{array}{ccc} \pi_1(U',\eta)^{(\text{c}-\text{cn})} & \longrightarrow & \pi_1(U,\eta)^{(\text{c}-\text{cn})} \\ \downarrow & & \downarrow \\ \pi_1(X',\eta) & \longrightarrow & \pi_1(X,\eta) \end{array}$$

where the upper horizontal map is induced by the natural map $\pi_1(U',\eta) \to \pi_1(U,\eta)$ (note $\Delta_{U'} = \Delta_U$ and $\pi_1(X',\eta)^{\text{geo}}$ acts trivially on Δ_U^{cn}), and the various maps in this diagram commute with the projections onto G_k. The section s induces a section $\tilde{s} : G_k \to \pi_1(X',\eta)$ of the projection $\pi_1(X',\eta) \twoheadrightarrow G_k$, and a lifting $\tilde{s}_{U'}^{\text{c}-\text{cn}} : G_k \to \pi_1(U',\eta)^{(\text{c}-\text{cn})}$ of $\tilde{s}$ (as in the statement of Proposition 2.2) induces a lifting $s_U^{\text{c}-\text{cn}} : G_k \to \pi_1(U,\eta)^{(\text{c}-\text{cn})}$ of s as required (cf. above diagram). Now, it follows from [21, Theorem A] in case X is an affinoid, and Proposition 5.3 in this paper (cf. §5) in case X is a formal p-adic germ, that there exists (after possibly a finite extension of k) a neighborhood $X' \to X$ of s with X' hyperbolic (cf. Notations). We can thus assume, without loss of generality, that X possesses a k-compactification Y with Y hyperbolic and the set $S \overset{\text{def}}{=} \{x\} \subset X(k)$ consists of a single k-rational point, in which case $\Delta_U^{\text{cn}} \xrightarrow{\sim} \hat{\mathbb{Z}}(1)$ as a $\pi_1(X,\eta)$-module (cf. Lemma 2.1 and Proposition 1.2).

Consider the following maps (here, $X = \operatorname{Spec} A$ in case X is *affinoid*):

$$H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1)) \hookrightarrow H^2(X,\hat{\mathbb{Z}}(1)) \leftarrow \operatorname{Pic}(X),$$

where the map $H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1)) \hookrightarrow H^2(X,\hat{\mathbb{Z}}(1))$ arises from the Cartan–Leray spectral sequence and is injective (cf. [25, Proof of Proposition 1]), and the map $\operatorname{Pic}(X) \to H^2(X,\hat{\mathbb{Z}}(1))$ is the cycle class map arising from the Kummer exact sequence in étale topology. Let $[\pi_1(U,\eta)^{(\text{c}-\text{cn})}] \in H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1))$ be the class of the group extension $\pi_1(U,\eta)^{(\text{c}-\text{cn})}$. The image of $[\pi_1(U,\eta)^{(\text{c}-\text{cn})}]$ in $H^2(X,\hat{\mathbb{Z}}(1))$ coincides with the image of the line bundle $\mathcal{O}(x) \in \operatorname{Pic}(X)$ via the Kummer map $\operatorname{Pic}(X) \to H^2(X,\hat{\mathbb{Z}}(1))$. Indeed, this follows from the following commutative diagram:

$$\begin{array}{ccccc} H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1)) & \longrightarrow & H^2(X,\hat{\mathbb{Z}}(1)) & \longleftarrow & \operatorname{Pic}(X) \\ \uparrow & & \uparrow & & \uparrow \\ H^2(\pi_1(Y,\eta),\hat{\mathbb{Z}}(1)) & \longrightarrow & H^2(Y,\hat{\mathbb{Z}}(1)) & \longleftarrow & \operatorname{Pic}(Y) \end{array}$$

where the right and middle vertical maps are induced by the scheme morphism $X \to Y$ if X is a formal p-adic germ, and the rigid morphism $X \to Y^{\text{rig}}$ and the comparison theorems between étale cohomology and rigid analytic étale cohomology in case X is affinoid (cf. [11, Theorem 1.8 and Theorem 1.9]). The right horizontal maps are the cycle class maps arising from the Kummer exact sequence in étale topology, and the left lower horizontal map is an isomorphism arising from the Cartan–Leray spectral sequence (cf. [17, Proposition 1.1]). The pullback of the class $[\pi_1(V,\eta)^{(\text{c}-\text{cn})}] \in H^2(\pi_1(Y,\eta),\hat{\mathbb{Z}}(1))$ in $H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1))$, where V is the complement in Y of the image of $S = \{x\}$ (cf. [19, 2.1.1] for the definition of $\pi_1(V,\eta)^{(\text{c}-\text{cn})}$), coincides with the class $[\pi_1(U,\eta)^{(\text{c}-\text{cn})}]$ (this follows from Lemma 2.1 and the various definitions). The class $[\pi_1(V,\eta)^{(\text{c}-\text{cn})}] \in H^2(\pi_1(Y,\eta),\hat{\mathbb{Z}}(1)) \xrightarrow{\sim} H^2(Y,\hat{\mathbb{Z}}(1))$ coincides with the image of the Chern class of the line bundle $\mathcal{O}(y) \in \operatorname{Pic}(Y)$ where $y \in Y(k)$ is the image of x (cf. [19, Proof of Lemma 2.3.1]). Thus, the image of $[\pi_1(U,\eta)^{(\text{c}-\text{cn})}]$ in

$H^2(X,\hat{\mathbb{Z}}(1))$ coincides with the image of the line bundle $\mathcal{O}(x) \in \mathrm{Pic}(X)$ via the cycle class map $\mathrm{Pic}(X) \to H^2(X,\hat{\mathbb{Z}}(1))$ as claimed.

The Picard group $\mathrm{Pic}(X)$ is finite (cf. Proposition 4.1 in this paper in case X is affinoid and [22, Proposition 5.4] in case X is a formal p-adic germ). In particular, the image of $[\pi_1(U,\eta)^{(\mathrm{c-cn})}]$ in $H^2(X,\hat{\mathbb{Z}}(1))$ and hence the class $[\pi_1(U,\eta)^{(\mathrm{c-cn})}]$ is a torsion element of $H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1))$. The class $[E_U] \in H^2(G_k,\hat{\mathbb{Z}}(1))$ of the group extension E_U is the image of $[\pi_1(U,\eta)^{(\mathrm{c-cn})}]$ under the retraction map $H^2(\pi_1(X,\eta),\hat{\mathbb{Z}}(1)) \xrightarrow{s^\star} H^2(G_k,\hat{\mathbb{Z}}(1)) \xrightarrow{\sim} \hat{\mathbb{Z}}$ induced by s. Hence, the class $[E_U]$ is trivial since $\hat{\mathbb{Z}}$ is torsion-free, and the group extension E_U splits. ∎

THEOREM 2.3 (Lifting of sections to cuspidally abelian arithmetic fundamental groups). *We use the above notations. There exists a section* $s_U^{\mathrm{ab}} : G_k \to \pi_1(U,\eta)^{(c-\mathrm{ab})}$ *of the projection* $\pi_1(U,\eta)^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lifts the section* s*, that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{s_U^{\mathrm{c-ab}}} & \pi_1(U,\eta)^{(\mathrm{c-ab})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X,\eta) \end{array}$$

where the right vertical map is the natural projection $\pi_1(U,\eta)^{(c-\mathrm{ab})} \twoheadrightarrow \pi_1(X,\eta)$*. In particular, the set of sections of the projection* $\pi_1(U,\eta)^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lift the section* s *is non-empty, and is (up to conjugation by elements of* Δ_U^{ab}*) a torsor under* $H^1(G_k,\Delta_U^{\mathrm{ab}})$*.*

Proof. Let $\{H\}_{i\in I}$ be a projective system of open subgroups of $\pi_1(X,\eta)$ containing $s(G_k)$ such that $s(G_k) = \bigcap_{i\in I} H_i$. Thus, for $i \in I$, the open subgroup H_i corresponds to an étale finite cover $X_i \to X$ with X_i geometrically connected and H_i is identified with $\pi_1(X_i,\eta)$ which sits in the exact sequence $1 \to \pi_1(X_i,\eta)^{\mathrm{geo}} \to \pi_1(X_i,\eta) \to G_k \to 1$ (the geometric point; denote also η, of X_i is induced by the geometric point η of X). Further, the section s induces a section $s_i : G_k \to \pi_1(X_i,\eta)$ of the projection $\pi_1(X_i,\eta) \twoheadrightarrow G_k$. Let $U_i \overset{\mathrm{def}}{=} U \times_X X_i$, and let $\pi_1(U_i,\eta)^{(\mathrm{c-cn})}$ be the (geometrically) cuspidally central arithmetic fundamental group of U_i which sits in the exact sequence $1 \to \Delta_{U_i}^{\mathrm{cn}} \to \pi_1(U_i,\eta)^{(\mathrm{c-cn})} \to \pi_1(X_i,\eta) \to 1$.

Consider the following commutative diagrams:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_U^{\mathrm{ab}} & \longrightarrow & \mathcal{E}_U & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s} & & \\ 1 & \longrightarrow & \Delta_U^{\mathrm{ab}} & \longrightarrow & \pi_1(U,\eta)^{(\mathrm{c-ab})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1 \end{array}$$

and for $i \in I$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{U_i}^{\mathrm{cn}} & \longrightarrow & E_{U_i} & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s_i} & & \\ 1 & \longrightarrow & \Delta_{U_i}^{\mathrm{cn}} & \longrightarrow & \pi_1(U_i,\eta)^{(\mathrm{c-cn})} & \longrightarrow & \pi_1(X_i,\eta) & \longrightarrow & 1 \end{array}$$

where the right squares are Cartesian. Thus, $\mathcal{E}_U$ (resp. E_{U_i}) is the pullback of the group extension $\pi_1(U,\eta)^{(\mathrm{c-ab})}$ (resp. $\pi_1(U_i,\eta)^{(\mathrm{c-cn})}$) by the section s (resp. s_i). There is a natural isomorphism $\Delta_U^{\mathrm{ab}} = \varprojlim_{i\in I} \Delta_{U_i}^{\mathrm{cn}}$ as follows from the facts that $\Delta_U = \Delta_{U_i}$, $\forall i \in I$, and given a finite quotient $\Delta_U^{\mathrm{ab}} \twoheadrightarrow H$, there exists $i \in I$ such that $\pi_1(X_i,\eta)^{\mathrm{geo}}$ acts trivially on H. Further,

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there is a natural isomorphism $\mathcal{E}_U \xrightarrow{\sim} \varprojlim_{i\in I} E_{U_i}$ (the transition maps in the projective limit being surjective). The existence of a section $s_U^{\mathrm{c-ab}}: G_k \to \pi_1(U,\eta)^{(\mathrm{c-ab})}$ of the projection $\pi_1(U,\eta)^{(\mathrm{c-ab})} \twoheadrightarrow G_k$ which lifts the section s is equivalent to the splitting of the group extension $\mathcal{E}_U$, and the set of those (possible) liftings $s_U^{\mathrm{c-ab}}$ is in one-to-one correspondence with the set of sections of the projection $\mathcal{E}_U \twoheadrightarrow G_k$. The natural projection $E_{U_i} \twoheadrightarrow G_k$ splits for all $i \in I$ (see the proof of Proposition 2.2). We show that the group extension $\mathcal{E}_U$ splits.

Let $(P_j)_{j\in J}$ be a projective system of quotients $\mathcal{E}_U \twoheadrightarrow P_j$, where P_j sits in an exact sequence $1 \to F_j \to P_j \to G_k \to 1$ with F_j finite, and $\mathcal{E}_U = \varprojlim_{j\in J} P_j$. (More precisely, write $\mathcal{E}_U$ as a projective limit of finite groups $\{\tilde{P}_j\}_{j\in J}$ where $\tilde{P}_j$ sits in an exact sequence $1 \to F_j \to \tilde{P}_j \to G_j \to 1$ with G_j a quotient of G_k and F_j a quotient of $\mathrm{Ker}(\mathcal{E}_U \twoheadrightarrow G_k)$. Let $1 \to F_j \to P_j \to G_k \to 1$ be the pullback of the group extension $1 \to F_j \to \tilde{P}_j \to G_j \to 1$ by $G_k \twoheadrightarrow G_j$. Then $\mathcal{E}_U = \varprojlim_{j\in J} P_j$.) The set $\mathrm{Sect}(G_k, \mathcal{E}_U)$ of group-theoretic sections of the projection $\mathcal{E}_U \twoheadrightarrow G_k$ is naturally identified with the projective limit $\varprojlim_{j\in J} \mathrm{Sect}(G_k, P_j)$ of the sets $\mathrm{Sect}(G_k, P_j)$ of group-theoretic sections of the projections $P_j \twoheadrightarrow G_k$, $j \in J$. The set $\mathrm{Sect}(G_k, P_j)$ is non-empty, $\forall j \in J$. Indeed, P_j (being a quotient of $\mathcal{E}_U$) is a quotient of E_{U_i} for some $i \in I$, this quotient $E_{U_i} \twoheadrightarrow P_j$ commutes with the projections onto G_k, and we know the projection $E_{U_i} \twoheadrightarrow G_k$ splits, and hence the projection $P_j \twoheadrightarrow G_k$ splits. Moreover, the set $\mathrm{Sect}(G_k, P_j)$ is, up to conjugation by the elements of F_j, a torsor under the group $H^1(G_k, F_j)$ which is finite since k is a p-adic local field (cf. [18, (7.1.8) Theorem (iii)]). Thus, $\mathrm{Sect}(G_k, P_j)$ is a non-empty finite set. The set $\mathrm{Sect}(G_k, \mathcal{E}_U)$ is non-empty being the projective limit of non-empty finite sets. This finishes the proof of Theorem 2.3. ∎

Next, let

$$G_X^{(\mathrm{c-ab})} \stackrel{\mathrm{def}}{=} G_X / \mathrm{Ker}(\mathcal{H}_X \twoheadrightarrow \mathcal{H}_X^{\mathrm{ab}})$$

(cf. exact sequence (1.4) for the definition of $\mathcal{H}_X$). Thus, $G_X^{(\mathrm{c-ab})} = \varprojlim_U \pi_1(U,\eta)^{(\mathrm{c-ab})}$ where U runs over all subschemes of X as in Theorem 2.3.

Theorem 2.4 (Lifting of sections to cuspidally abelian Galois groups). *We use the above notations. There exists a section $s^{\mathrm{c-ab}}: G_k \to G_X^{(c-\mathrm{ab})}$ of the projection $G_X^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ which lifts the section s, that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{s^{c-\mathrm{ab}}} & G_X^{(c-\mathrm{ab})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X,\eta) \end{array}$$

where the right vertical map is the natural projection $G_X^{(c-\mathrm{ab})} \twoheadrightarrow \pi_1(X,\eta)$. In particular, the set of sections of the projection $G_X^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ which lift the section s is non-empty, and is (up to conjugation by elements of $\mathcal{H}_X^{\mathrm{ab}}$) a torsor under $H^1(G_k, \mathcal{H}_X^{\mathrm{ab}})$.

Proof. The proof follows, using the natural identification $G_X^{\mathrm{c-ab}} \xrightarrow{\sim} \varprojlim_U \pi_1(U,\eta)^{\mathrm{c-ab}}$ (where U runs over all subschemes of X as in Theorem 2.3), from Theorem 2.3 and a similar argument in our context to the one used in the proof of Theorem 2.3.5 in [19]. Alternatively, one can use Theorem 2.3 and a similar argument to the one used at the end of the proof of Theorem 2.3. ∎

The following is one of our main results in this section.

THEOREM 2.5. *Assume that X admits a k-compactification Y (cf. Notations). If the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$ splits, then* $\text{index}(Y) = 1$.

Proof. Assume that the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$ splits and let $s : G_k \to \pi_1(X,\eta)$ be a section of this projection. By Theorem 2.4, there exists a section $s^{\text{c}-\text{ab}} : G_k \to G_X^{(\text{c}-\text{ab})}$ of the projection $G_X^{(\text{c}-\text{ab})} \twoheadrightarrow G_k$ which lifts the section s. The section $s^{\text{c}-\text{ab}}$ induces naturally a section $\tilde{s} : G_k \to G_X^{(\text{ab})}$ of the projection $G_X^{(\text{ab})} \twoheadrightarrow G_k$ (see §1 for the definition of $G_X^{(\text{ab})}$ and note that $G_X^{(\text{ab})}$ is a quotient of $G_X^{(\text{c}-\text{ab})}$). Let $G_Y \overset{\text{def}}{=} \text{Gal}(\overline{K}/K)$ be the absolute Galois group of the function field K of Y, and let $G_Y^{(\text{ab})} \overset{\text{def}}{=} G_Y/\text{Ker}[\text{Gal}(\overline{K}/K\bar{k}) \twoheadrightarrow \text{Gal}(\overline{K}/K\bar{k})^{\text{ab}}]$ be its geometrically abelian quotient. We have a commutative diagram

$$\begin{array}{ccc} G_X^{(\text{ab})} & \longrightarrow & G_k \\ \downarrow & & \Vert \\ G_Y^{(\text{ab})} & \longrightarrow & G_k \end{array}$$

where the left vertical map is induced by the natural map $G_X \to G_Y$, which is induced by the scheme morphism $X \to Y$ in case X is a formal p-adic germ, and by the rigid analytic morphism $X \to Y^{\text{rig}}$ and the rigid GAGA functor in case X is affinoid. The section $\tilde{s} : G_k \to G_X^{(\text{ab})}$ induces a section $s^\dagger : G_k \to G_Y^{(\text{ab})}$ of the projection $G_Y^{(\text{ab})} \twoheadrightarrow G_k$ (cf. above diagram). The existence of the section $s^\dagger$ implies that $\text{index}(Y) = 1$ as was observed by Esnault and Wittenberg (see [5, Remark 2.3(ii)] and [24, Theorem A] for a more general result). ∎

§3. Geometric sections of arithmetic fundamental groups

We investigate *geometric* sections of the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$ (relative to a fixed compactification of X). We use the notations in §§0–2. We further assume that X possesses a *k-compactification* Y with Y *hyperbolic* (cf. Notations) which is fixed throughout §3.

Let

$$s : G_k \to \pi_1(X,\eta)$$

be a *section* of the projection $\pi_1(X,\eta) \twoheadrightarrow G_k$ fixed throughout §3, which induces a *(local) section*

$$s_Y : G_k \to \pi_1(Y,\eta)$$

of the projection $\pi_1(Y,\eta) \twoheadrightarrow G_k$ (cf. diagram (0.1) and §0).

We have an exact sequence

$$1 \to \mathcal{I}_Y \to G_Y \to \pi_1(Y,\eta) \to 1,$$

where $G_Y = \text{Gal}(\overline{K}/K)$ is the absolute Galois group of the function field K of Y and $\mathcal{I}_Y \overset{\text{def}}{=} \text{Ker}[G_Y \twoheadrightarrow \pi_1(Y,\eta)]$. Let

$$G_Y^{(\text{c}-\text{ab})} \overset{\text{def}}{=} G_Y/\text{Ker}(\mathcal{I}_Y \twoheadrightarrow \mathcal{I}_Y^{\text{ab}}).$$

Thus, $G_Y^{(\text{c}-\text{ab})} = \varprojlim_V \pi_1(V,\eta)^{(\text{c}-\text{ab})}$ where V runs over all open subschemes of Y (cf. [19, 2.1.1] for the definition of $\pi_1(V,\eta)^{(\text{c}-\text{ab})}$).

Theorem 3.1 (Lifting of sections to cuspidally abelian Galois groups). *We use the above notations. The followings hold.*

(i) *There exists a section* $s_Y^{\mathrm{c-ab}}: G_k \to G_Y^{(c-\mathrm{ab})}$ *of the projection* $G_Y^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lifts the section* $s_Y: G_k \to \pi_1(Y,\eta)$*, that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{s_Y^{c-\mathrm{ab}}} & G_Y^{(c-\mathrm{ab})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s_Y} & \pi_1(Y,\eta) \end{array}$$

where the right vertical map is the natural projection $G_Y^{(c-\mathrm{ab})} \twoheadrightarrow \pi_1(Y,\eta)$*. In particular, the set of sections of the projection* $G_Y^{(c-\mathrm{ab})} \twoheadrightarrow G_k$ *which lift the section* s_Y *is non-empty, and is (up to conjugation by elements of* $\mathcal{I}_Y^{\mathrm{ab}}$*) a torsor under* $H^1(G_k, \mathcal{I}_Y^{\mathrm{ab}})$*.*

(ii) *The (local) section* $s_Y: G_k \to \pi_1(Y,\eta)$ *is uniformly orthogonal to* Pic *in the sense of [19, Definition 1.4.1].*

Proof. Assertion (i) follows from Theorem 2.4 and the fact that there exists a natural homomorphism $G_X^{(\mathrm{c-ab})} \to G_Y^{(\mathrm{c-ab})}$, induced by the natural homomorphism $G_X \to G_Y$, which commutes with the projections to G_k. Assertion (ii) follows from assertion (i) and Theorem 2.3.5 in [19]. ∎

Consider the following push-out diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{H}_X & \longrightarrow & G_X & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathcal{H}_{X,1/p^2} & \longrightarrow & G_X^{(1/p^2-\mathrm{sol})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1 \end{array}$$

where $\mathcal{H}_{X,1/p^2}$ is the *maximal* $1/p^2$*-th solvable quotient* of $\mathcal{H}_X$ and $G_X^{(1/p^2-\mathrm{sol})} \overset{\mathrm{def}}{=} G_X/\mathrm{Ker}(\mathcal{H}_X \twoheadrightarrow \mathcal{H}_{X,1/p^2})$. Thus, $\mathcal{H}_{X,1/p^2}$ is the maximal quotient of $\mathcal{H}_X$ which is abelian and annihilated by p^2 (cf. [22, 1.2] for more details). We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{H}_{X,1/p^2} & \longrightarrow & G_X^{(1/p^2-\mathrm{sol})} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{I}_{Y,1/p^2} & \longrightarrow & G_Y^{(1/p^2-\mathrm{sol})} & \longrightarrow & \pi_1(Y,\eta) & \longrightarrow & 1 \end{array} \tag{3.1}$$

which is induced by the natural homomorphism $G_X \to G_Y$, where $G_Y^{(1/p^2-\mathrm{sol})}$ is defined in a similar way to $G_X^{(1/p^2-\mathrm{sol})}$. More precisely, $\mathcal{I}_{Y,1/p^2}$ is the maximal quotient of $\mathcal{I}_Y$ which is abelian and annihilated by p^2 and $G_Y^{(1/p^2-\mathrm{sol})} \overset{\mathrm{def}}{=} G_Y/\mathrm{Ker}(\mathcal{I}_Y \twoheadrightarrow \mathcal{I}_{Y,1/p^2})$ is the *geometrically cuspidally* $1/p^2$*-th step solvable quotient* of G_Y (cf. [22, 3.1]; recall the exact sequence $1 \to \mathcal{I}_Y \to G_Y \to \pi_1(Y,\eta) \to 1$).

The following Proposition 3.2, item (i), is weaker than (and follows from) Theorem 2.4, and we state it in connection with Theorem 3.5.2 in this section.

Proposition 3.2 (Lifting of sections to cuspidally $1/p^2$-th step solvable Galois groups). *We use the above notations. The followings hold.*

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(i) *There exists a section* $\tilde{s}: G_k \to G_X^{(1/p^2-\mathrm{sol})}$ *of the projection* $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ *which lifts the section* $s: G_k \to \pi_1(X,\eta)$, *that is, which inserts in the following commutative diagram:*

$$\begin{array}{ccc} G_k & \xrightarrow{\tilde{s}} & G_X^{(1/p^2-\mathrm{sol})} \\ \| & & \downarrow \\ G_k & \xrightarrow{s} & \pi_1(X,\eta) \end{array}$$

where the right vertical map is the natural projection $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow \pi_1(X,\eta)$. *In particular, the set of sections of the projection* $G_X^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ *which lift the section* s *is non-empty, and is (up to conjugation by elements of* $\mathcal{H}_{X,1/p^2}$*) a torsor under* $H^1(G_k, \mathcal{H}_{X,1/p^2})$.

(ii) *The section* $\tilde{s}: G_k \to G_X^{(1/p^2-\mathrm{sol})}$ *in (i) induces a section* $\tilde{s}_Y: G_k \to G_Y^{(1/p^2-\mathrm{sol})}$ *of the projection* $G_Y^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ *which lifts the section* $s_Y: G_k \to \pi_1(Y,\eta)$. *In particular, the (local) section* $s_Y: G_k \to \pi_1(Y,\eta)$ *is uniformly orthogonal to* Pic *mod-*p^2 *in the sense of [22, Definition 3.4.1].*

Proof. Assertion (i) follows from Theorem 2.4 and the fact that there exists a natural projection $G_X^{(\mathrm{c-ab})} \twoheadrightarrow G_X^{(1/p^2-\mathrm{sol})}$ which commutes with the projections onto G_k. Assertion (ii) follows from (i) and the fact that there exists a natural homomorphism $G_X^{(1/p^2-\mathrm{sol})} \to G_Y^{(1/p^2-\mathrm{sol})}$, induced by the homomorphism $G_X \to G_Y$, which commutes with the projections onto G_k (cf. diagram (3.1) and [22, Theorem 3.4.4]). ∎

3.3

Write

$$\Pi_Y[X] \stackrel{\mathrm{def}}{=} \varprojlim_{T \subset Y \setminus X} \pi_1(Y \setminus T, \eta)$$

and

$$\Pi_Y[X]^{\mathrm{geo}} \stackrel{\mathrm{def}}{=} \varprojlim_{T \subset Y \setminus X} \pi_1(Y \setminus T, \eta)^{\mathrm{geo}},$$

where the limits are over all subsets T consisting of finitely many closed points of Y *not in* X (cf. Notations), $Y \setminus T$ is the corresponding (affine if T is non-empty) curve, and $\pi_1(Y \setminus T, \eta)^{\mathrm{geo}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}[\pi_1(Y \setminus T, \eta) \twoheadrightarrow G_k]$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X,\eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi_Y[X]^{\mathrm{geo}} & \longrightarrow & \Pi_Y[X] & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \pi_1(Y_{\bar{k}},\bar{\eta}) & \longrightarrow & \pi_1(Y,\eta) & \longrightarrow & G_k & \longrightarrow & 1 \end{array} \tag{3.2}$$

where the middle upper map is induced by the rigid analytic morphism $X \to Y^{\mathrm{rig}}$ and the rigid GAGA functor in case X is *affinoid*, and the scheme morphism $X \to Y$ in case X is a *formal p-adic germ*. The left and middle lower vertical maps are the natural projections (they are surjective).

Proposition 3.3.1. *We use the above notations. The left and middle upper vertical maps in diagram (3.2) are injective in the case X is affinoid.*

Proof. The first assertion follows from Theorem A in [21] (see the comments in the proof of Proposition 1.1). The second assertion follows from the first and the commutativity of the upper part in diagram (3.2). ∎

The section $s: G_k \to \pi_1(X,\eta)$ induces a section (denoted also s)

$$s: G_k \to \Pi_Y[X]$$

of the projections $\Pi_Y[X] \twoheadrightarrow G_k$ (cf. diagram (3.2)).

Definition 3.3.2. We say that the section s is *geometric*, relative to Y, if the image $s(G_k)$ of the section $s: G_k \to \Pi_Y[X]$ is contained in a decomposition group $D_x \subset \Pi_Y[X]$ associated with a *rational* point $x \in Y(k)$.

Note that if s is geometric in the above sense, associated with $x \in Y(k)$, then the (local) section $s_Y: G_k \to \pi_1(Y,\eta)$ of the projection $\pi_1(Y,\eta) \twoheadrightarrow G_k$ induced by s is geometric and is associated with $x \in Y(k)$, that is, $s_Y(G_k)$ is contained in (hence equal to) a decomposition group $D_x \subset \pi_1(Y,\eta)$ associated to x.

3.4

In this subsection, we assume that $X = \operatorname{Spec}(A \otimes_{\mathcal{O}_k} k)$ is a *formal p-adic germ*.

Let $\mathcal{Y} \to \operatorname{Spec}\mathcal{O}_k$ be a model of Y, let $y \in \mathcal{Y}^{\mathrm{cl}}$ be a closed point, and let $\hat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\sim} A$ be an isomorphism. Let $\mathcal{Y}_F \overset{\mathrm{def}}{=} \mathcal{Y} \times_{\operatorname{Spec}\mathcal{O}_k} \operatorname{Spec} F$ be the special fiber of $\mathcal{Y}$. Consider the following assumption **(*)**:

(*) *The gcd of the total multiplicities of the irreducible components of $\mathcal{Y}_F$ is* 1.

Let ξ be a geometric point of $\mathcal{Y}_F$ with values in the generic point of an irreducible component Y_{i_0} of $\mathcal{Y}_F$. Thus, ξ determines an algebraic closure $\overline{F}$ of F. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X,\eta)^{\mathrm{geo}} & \longrightarrow & \pi_1(X,\eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(Y_{\bar{k}},\bar{\eta}) & \longrightarrow & \pi_1(Y,\eta) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{Y}_{\overline{F}},\bar{\xi}) & \longrightarrow & \pi_1(\mathcal{Y}_F,\xi) & \longrightarrow & G_F & \longrightarrow & 1
\end{array} \tag{3.3}$$

where the middle upper map is induced by the scheme morphism $X \to Y$, the lower middle map (which is defined up to conjugation) is a specialization map, $\pi_1(\mathcal{Y}_F,\xi)$ (resp. $\pi_1(\mathcal{Y}_{\overline{F}},\bar{\xi})$) is the fundamental group of $\mathcal{Y}$ (resp. $\mathcal{Y}_{\overline{F}} \overset{\mathrm{def}}{=} \mathcal{Y} \times_{\operatorname{Spec}\mathcal{O}_k} \operatorname{Spec}\overline{F}$) with geometric point ξ (resp. $\bar{\xi}$ which is induced by ξ), $G_F \overset{\mathrm{def}}{=} \operatorname{Gal}(\overline{F}/F)$, and the lower right vertical map is the natural projection $G_k \twoheadrightarrow G_F$ (cf. [20, diagram 0.1] and the discussion thereafter). The left (hence also the middle) lower vertical map in diagram (3.3) is surjective under the assumption **(*)** (cf. [20, diagram 0.1] and the references therein).

The section $s : G_k \to \pi_1(X,\eta)$ induces the (local) section $s_Y : G_k \to \pi_1(Y,\eta)$ of the projection $\pi_1(Y,\eta) \twoheadrightarrow G_k$, as well as a homomorphism

$$\tilde{s} : G_k \to \pi_1(\mathcal{Y}_F,\xi)$$

obtained by composing the section $s_Y : G_k \to \pi_1(Y,\eta)$ with the specialization map $\pi_1(Y,\eta) \twoheadrightarrow \pi_1(\mathcal{Y}_F,\xi)$ in diagram (3.3).

LEMMA 3.4.1. *We use the above notations. The followings hold.*

(i) *The closed point $y \in \mathcal{Y}^{\text{cl}}$ is an F-rational point.*
(ii) *The section s_Y is unramified: the homomorphism $\tilde{s} : G_k \to \pi_1(\mathcal{Y}_F,\xi)$ factors through G_F and induces a section $\bar{s}_Y : G_F \to \pi_1(\mathcal{Y}_F,\xi)$ of the natural projection $\pi_1(\mathcal{Y}_F,\xi) \twoheadrightarrow G_F$.*
(iii) *The section $\bar{s}_Y : G_F \to \pi_1(\mathcal{Y}_F,\xi)$ in (ii) is geometric and arises from the rational point y, that is, arises from the scheme-theoretic morphism $y : \operatorname{Spec} F \to \mathcal{Y}_F$.*
(iv) *Assume that $\mathcal{Y}$ is regular. Then condition (*) holds.*

Proof. Assertion (i) is clear (recall $\hat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\sim} A$); it also follows from (ii). We prove (ii). We have a commutative diagram of scheme morphisms

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \mathcal{Y} \\ \uparrow & & \uparrow \\ \operatorname{Spec}(F) & \xrightarrow{y} & \mathcal{Y}_F \end{array} \tag{3.4}$$

where the lower horizontal morphism is induced by the closed point y of $\mathcal{Y}_F$, and the lower vertical morphisms are closed immersions. This diagram gives rise to a commutative diagram of homomorphisms between fundamental groups

$$\begin{array}{ccc} \pi_1(X,\eta) & \longrightarrow & \pi_1(Y,\eta) \\ \downarrow & & \downarrow \\ \pi_1(\operatorname{Spec} A,\eta) & \longrightarrow & \pi_1(\mathcal{Y},\eta) \\ {\scriptstyle\tau_y}\uparrow & & {\scriptstyle\sigma}\uparrow \\ G_F & \xrightarrow{s_y} & \pi_1(\mathcal{Y}_F,\xi) \end{array} \tag{3.5}$$

where the lower horizontal map is a section of the projection $\pi_1(\mathcal{Y}_F,\xi) \twoheadrightarrow G_F$ arising from the F-rational point $y \in \mathcal{Y}_F$, and is defined up to conjugation, the lower vertical maps are induced by the lower vertical maps in diagram (3.4) (they are defined up to conjugation) and are isomorphisms (cf. [8, Exposé X, Théorème 2.1] for the right vertical map σ being an isomorphism). Further, the composite $\psi : \pi_1(X,\eta) \to \pi_1(\operatorname{Spec} A,\eta) \xrightarrow{\tau_y^{-1}} G_F \xrightarrow{s_y} \pi_1(\mathcal{Y}_F,\xi)$ is the composite of the middle vertical maps in diagram (3.3) as follows from the definition of the specialization map $\pi_1(Y,\eta) \to \pi_1(\mathcal{Y}_F,\xi)$: this map is the composite of the maps $\pi_1(Y,\eta) \to \pi_1(\mathcal{Y},\eta) \xrightarrow{\sigma^{-1}} \pi_1(\mathcal{Y}_F,\xi)$. In particular, the homomorphism $\tilde{s} : G_k \to \pi_1(\mathcal{Y}_F,\xi)$ factors through G_F and induces a section $\bar{s}_Y : G_F \to \pi_1(\mathcal{Y}_F,\xi)$ of the natural projection

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$\pi_1(\mathcal{Y}_F,\xi) \twoheadrightarrow G_F$. This shows (ii). The section $\bar{s}_Y$ coincides (up to conjugation) with the section $G_F \xrightarrow{s_y} \pi_1(\mathcal{Y}_F,\xi)$ in diagram (3.5), hence is geometric and arises from the F-rational point y as claimed in (iii). The last assertion follows from Theorem 2.5 and the well-known fact that if $\mathcal{Y}$ is regular, then the gcd of the total multiplicities of the irreducible components of $\mathcal{Y}_F$ divides index(Y) (cf., e.g., [7, Theorem 8.2 and Remark 8.6]). ∎

REMARK 3.4.2. Assume that the morphism $\mathcal{Y} \to \operatorname{Spec}\mathcal{O}_k$ is *smooth*. If s is geometric, and arises from the rational point $x \in Y(k)$ (cf. Definition 3.3.2), it follows from Lemma 3.4.1(iii) and the fact that $\mathcal{Y}_F$ is hyperbolic that the point x specializes in y necessarily (cf. [27, Proposition (2.8)(i)]). In particular, the point x is the image of a (unique) k-rational point $\tilde{x} \in X(k)$ via the morphism $X \to Y$. The fact that $s_Y(G_k) = D_x \subset \pi_1(Y,\eta)$ does not imply a priori that the image $s(G_k)$ via the section $s : G_k \to \pi_1(X,\eta)$ is contained in a decomposition group $D_{\tilde{x}} \subset \pi_1(X,\eta)$ associated with $\tilde{x}$.

3.5

Let $H \subset \Pi_Y[X]$ be an open subgroup with $s(G_k) \subset H$ [recall $s : G_k \to \Pi_Y[X]$ is the section induced by $s : G_k \to \pi_1(X,\eta)$]. Thus, H corresponds to a (possibly ramified) finite cover $Y' \to Y$ with Y' geometrically connected. Let $H' \subset \pi_1(X,\eta)$ be the inverse image of H via the homomorphism $\pi_1(X,\eta) \to \Pi_Y[X]$ (cf. diagram (3.2)). Thus, H' is an open subgroup of $\pi_1(X,\eta)$ containing the image of the section $s : G_k \to \pi_1(X,\eta)$ and corresponds to an étale cover $X' \to X$ with X' geometrically connected. There is a natural morphism $X' \to (Y')^{\mathrm{rig}}$ of rigid analytic spaces in case X is *affinoid*, and a natural scheme morphism $X' \to Y'$ in case X is a *formal p-adic germ*. The generic point η induces naturally a generic point (denoted also η) of X' and Y'. Further, we have a natural identification $H' = \pi_1(X',\eta)$ and a natural homomorphism $\pi_1(X',\eta) \to \pi_1(Y',\eta)$ which commutes with the projections onto G_k.

The section $s : G_k \to \pi_1(X,\eta)$ induces naturally sections $s' : G_k \to \pi_1(X',\eta)$ and $s_{Y'} : G_k \to \pi_1(Y',\eta)$ of the natural projections $\pi_1(X',\eta) \twoheadrightarrow G_k$ and $\pi_1(Y',\eta) \twoheadrightarrow G_k$, respectively. The section $s' : G_k \to \pi_1(X',\eta)$ lifts to a section $\tilde{s}' : G_k \to G_{X'}^{(1/p^2\text{-sol})}$ of the projection $G_{X'}^{(1/p^2\text{-sol})} \twoheadrightarrow G_k$ and induces a section $\tilde{s}_{Y'} : G_k \to G_{Y'}^{(1/p^2\text{-sol})}$ of the projection $G_{Y'}^{(1/p^2\text{-sol})} \twoheadrightarrow G_k$ (cf. Proposition 3.2). Let $F \subset G_{Y'}^{(1/p^2\text{-sol})}$ be an open subgroup with $\tilde{s}_{Y'}(G_k) \subset F$. Thus, F corresponds to a (possibly ramified) finite cover $Y'' \to Y'$ with Y'' geometrically connected. The generic point η induces naturally a generic point (denoted also η) of Y''. Write $\pi_1(Y'',\eta)^{(1/p\text{-sol})}$ for the *geometrically* $1/p$*-th step solvable quotient* of $\pi_1(Y'',\eta)$ which sits in the following exact sequence:

$$1 \to \pi_1(Y''_{\bar{k}},\bar{\eta})_{1/p} \to \pi_1(Y'',\eta)^{(1/p\text{-sol})} \to G_k \to 1, \tag{3.6}$$

where $\pi_1(Y''_{\bar{k}},\bar{\eta})_{1/p}$ is the maximal $1/p$-th step solvable quotient of $\pi_1(Y''_{\bar{k}},\bar{\eta})$ (cf. [22, 1.2]) and the generic point $\bar{\eta}$ is induced by η. Thus, $\pi_1(Y''_{\bar{k}},\bar{\eta})_{1/p}$ is the maximal quotient of $\pi_1(Y''_{\bar{k}},\bar{\eta})$ which is abelian and annihilated by p (cf. [22, 1.2]).

DEFINITION 3.5.1. We use the above notations. We say that the section s is *admissible*, relative to Y, if for every open subgroup $H \subset \Pi_Y[X]$ with $s(G_k) \subset H$, corresponding to (a possibly ramified) cover $Y' \to Y$, the following holds. There exists a section $\tilde{s}_{Y'} : G_k \to G_{Y'}^{(1/p^2\text{-sol})}$ of the projection $G_{Y'}^{(1/p^2\text{-sol})} \twoheadrightarrow G_k$ (such a section exists unconditionally [see above discussion]) satisfying the following property: *for each open sub-*

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group $F \subset G_{Y'}^{(1/p^2-\mathrm{sol})}$ with $\tilde{s}_{Y'}(G_k) \subset F$, corresponding to a (possibly ramified) cover $Y'' \to Y'$ with Y'' geometrically connected, the natural projection $\pi_1(Y'',\eta)^{(1/p-\mathrm{sol})} \twoheadrightarrow G_k$ splits (cf. above discussion). Note that this latter condition is equivalent to (cf. [22, Lemma 3.4.8]): *the class of* $\mathrm{Pic}^1_{Y''}$ *in* $H^1(G_k, \mathrm{Pic}^0_{Y''})$ *is divisible by* p.

Our main result in this section is the following.

THEOREM 3.5.2. *We use the above notations. The section $s: G_k \to \pi_1(X,\eta)$ is geometric relative to Y (cf. Definition 3.3.2) if and only if s is admissible relative to Y (cf. Definition 3.5.1).*

Proof. Assume first that the section $s: G_k \to \pi_1(X,\eta)$ is admissible (relative to Y). We prove that s is geometric (relative to Y). Using a well-known limit argument due to Tamagawa (cf. [27, Proposition 2.8(iv)]), it suffices to show the following. For every open subgroup $H \subset \Pi_Y[X]$ with $s(G_k) \subset H$, corresponding to (a possibly ramified) cover $Y' \to Y$ with Y' hyperbolic, $Y'(k) \neq \emptyset$ holds. By assumption, there exists a section $\tilde{s}_{Y'}: G_k \to G_{Y'}^{(1/p^2-\mathrm{sol})}$ of the projection $G_{Y'}^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$ satisfying the condition in Definition 3.5.1. In [22, 3.3], we defined a certain quotient $G_{Y'} \twoheadrightarrow G_{Y'}^{(p,2)} \twoheadrightarrow G_{Y'}^{(1/p^2-\mathrm{sol})}$ of $G_{Y'}$ (we refer to [22, 3.3] for more details on the definition of $G_{Y'}^{(p,2)}$). Let $F \subset G_{Y'}^{(1/p^2-\mathrm{sol})}$ be an open subgroup with $\tilde{s}_{Y'}(G_k) \subset F$ corresponding to a (possibly ramified) cover $Y'' \to Y'$ with Y'' geometrically connected. By assumption, the natural projection $\pi_1(Y'',\eta)^{(1/p-\mathrm{sol})} \twoheadrightarrow G_k$ splits (cf. Definition 3.5.1). This latter condition (for every F as above) implies that (in fact is equivalent to) the section $\tilde{s}_{Y'}: G_k \to G_{Y'}^{(1/p^2-\mathrm{sol})}$ lifts to a section $s_{Y'}^{\dagger}: G_k \to G_{Y'}^{(p,2)}$ of the projection $G_{Y'}^{(p,2)} \twoheadrightarrow G_k$ (cf. [22, Theorem 3.4.10 and Lemma 3.4.8]). Further, the existence of the section $s_{Y'}^{\dagger}: G_k \to G_{Y'}^{(p,2)}$ as above implies that $Y'(k) \neq \emptyset$ by [22, Proposition 4.6], as required.

Next, we assume that s is geometric (relative to Y) and prove that s is admissible (relative to Y). By assumption $s(G_k)$ is contained in $D_x \subset \Pi_Y[X]$ where D_x is a decomposition group associated with a rational point $x \in Y(k)$. Let $H \subset \Pi_Y[X]$ be an open subgroup with $s(G_k) \subset H$ corresponding to (a possibly ramified) cover $Y' \to Y$. Then $Y'(k) \neq \emptyset$. A rational point $x' \in Y'(k)$ gives rise to a section $\tilde{s}_{Y'}: G_k \to G_{Y'}^{(1/p^2-\mathrm{sol})}$ of the projection $G_{Y'}^{(1/p^2-\mathrm{sol})} \twoheadrightarrow G_k$. Let $F \subset G_{Y'}^{(1/p^2-\mathrm{sol})}$ be an open subgroup with $\tilde{s}_{Y'}(G_k) \subset F$ corresponding to a (possibly ramified) cover $Y'' \to Y'$ with Y'' geometrically connected. Then $Y''(k) \neq \emptyset$ holds since the section $\tilde{s}_{Y'}: G_k \to G_{Y'}^{(1/p^2-\mathrm{sol})}$ arises from the rational point x' and $\tilde{s}_{Y'}(G_k) \subset F$. In particular, the natural projection $\pi_1(Y'',\eta) \twoheadrightarrow G_k$, and a fortiori the projection $\pi_1(Y'',\eta)^{(1/p-\mathrm{sol})} \twoheadrightarrow G_k$, splits. Thus, s is admissible as required. ∎

§4. Picard groups of affinoid p-adic curves

The following is our main result in this section; it may be of interest independently of the topics discussed in §§1–3.

PROPOSITION 4.1. *Let $X = \mathrm{Sp}(A)$ be a* smooth *and geometrically connected k-*affinoid *curve. Then the Picard group* $\mathrm{Pic}(X)$ *is* finite.

The rest of this section is devoted to the proof of Proposition 4.1.

Let $\mathcal{X} = \mathrm{Spf}\, B$ be an excellent normal $\mathcal{O}_k$-formal scheme of finite type with generic fiber X, that is, $A = B \otimes_R k$. Write $\mathcal{X}^{\mathrm{reg}}$ for the set of regular points of $\mathcal{X}$. Thus, $\mathcal{X} \setminus \mathcal{X}^{\mathrm{reg}} =$

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$\{z_1, \dots, z_t\}$ consists of finitely many closed points of $\mathcal{X}$. By Lipman's theorem of resolution of singularities for excellent two-dimensional schemes, there exists a birational and proper morphism $\lambda : \mathcal{S} \to \mathcal{X}$ with $\mathcal{S}$ regular and $\lambda^{-1}(\mathcal{X}^{\text{reg}}) \to \mathcal{X}^{\text{reg}}$ an isomorphism (cf. [15]; here, we view $\mathcal{X}$ as the ordinary affine scheme $\operatorname{Spec} B$). For $n \geq 1$, write $B_n \overset{\text{def}}{=} B/(\pi^n)$, $\mathcal{X}_n \overset{\text{def}}{=} \operatorname{Spec} B_n$, and $\mathcal{S}_n \overset{\text{def}}{=} \mathcal{S} \times_{\mathcal{X}} \mathcal{X}_n$. Further, denote $\mathcal{X}_0 \overset{\text{def}}{=} \mathcal{X}_n^{\text{red}}$ and $\mathcal{S}_0 \overset{\text{def}}{=} \mathcal{S}_n^{\text{red}}$. Thus, $\mathcal{X}_0$ and $\mathcal{S}_0$ are one-dimensional reduced schemes over F. Further, there exists a morphism $\lambda : \mathcal{S} \to \mathcal{X}$ as above with S_0 a divisor with strict normal crossings (cf. [3, Corollary 0.4]), which we assume from now on.

We have a surjective homomorphism $\operatorname{Pic}(\mathcal{X}^{\text{reg}}) \twoheadrightarrow \operatorname{Pic}(X)$. To prove $\operatorname{Pic}(X)$ is finite, it suffices to prove that $\operatorname{Pic}(\mathcal{X}^{\text{reg}})$ is finite. For each singular point z_i of $\mathcal{X}$, let $E_i \overset{\text{def}}{=} \lambda^{-1}(z_i)^{\text{red}}$ and let $\{D_{i,j}\}_{1 \leq j \leq n_i}$ be the set of irreducible components of E_i, $1 \leq i \leq t$. Thus, E_i is a reduced proper curve over the residue field $k(z_i)$ at z_i which is a finite field. We have an exact sequence

$$M \overset{\text{def}}{=} \oplus_{i=1}^{t} (\oplus_{j=1}^{n_i} \mathbb{Z}) \xrightarrow{\ \beta\ } \operatorname{Pic}(\mathcal{S}) \to \operatorname{Pic}(\mathcal{X}^{\text{reg}}) \to 0,$$

where β maps the copy of $\mathbb{Z}$ indexed by the pair (i, j) to the class of the divisor $D_{i,j}$. Further, we have an isomorphism

$$\operatorname{Pic}(\mathcal{S}) \xrightarrow{\sim} \varprojlim_{n \geq 1} \operatorname{Pic}(\mathcal{S}_n)$$

(cf. [9, première partie, Corollaire 5.1.6]).

Lemma 4.2. *We use notations as above. To prove that* $\operatorname{Pic}(\mathcal{X}^{\text{reg}})$ *is finite, it suffices to prove the following two assertions:*

(A) *The cokernel of the composite map*

$$\phi_n : M \overset{\text{def}}{=} \oplus_{i=1}^{t} (\oplus_{j=1}^{n_i} \mathbb{Z}) \xrightarrow{\ \beta\ } \operatorname{Pic}(\mathcal{S}) \to \operatorname{Pic}(\mathcal{S}_n)$$

is finite for $n \geq 1$.

(B) *There exists* $n_0 > 0$ *such that the map*

$$\operatorname{Pic}(\mathcal{S}_{n+1}) \to \operatorname{Pic}(\mathcal{S}_n)$$

is an isomorphism for $n > n_0$.

Proof of Lemma 4.2. Follows from the above discussion and the fact that we have an exact sequence

$$M \to \varprojlim_{n \geq 1} \operatorname{Pic}(\mathcal{S}_n) \to \varprojlim_{n \geq 1} \operatorname{coker}(\phi_n) \to 0,$$

where the first map is induced by the maps $\phi_n : M \to \operatorname{Pic}(\mathcal{S}_n)$, $n \geq 1$, and $\varprojlim_{n \geq 1} \operatorname{coker}(\phi_n)$ is finite if assertions **(A)** and **(B)** are satisfied.

This finishes the proof of Lemma 4.2. □

The rest of this section is devoted to the proofs of assertions **(A)** and **(B)**.

*Proof of assertion **(A)**.* Let $\{\eta_r\}_{r=1}^{s}$ be the generic points of $\mathcal{X}_0$, let $\rho : \mathcal{S}_0^{\text{nor}} \to \mathcal{S}_0$ be the morphism of normalization, let $\widetilde{E}_i \overset{\text{def}}{=} \rho^{-1}(E_i)$, $1 \leq i \leq t$, and let $H_r = \overline{\{\eta_r\}}$ be the closure in $\mathcal{S}_0^{\text{nor}}$ of the (inverse image in $\mathcal{S}_0$ of the) generic point η_r of $\mathcal{X}_0$, $1 \leq r \leq s$. Thus, H_r is a

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connected affine normal one-dimensional scheme over F. Let

$$d : \mathrm{Pic}(\mathcal{S}_0) \xrightarrow{\rho^*} \mathrm{Pic}(\mathcal{S}_0^{\mathrm{nor}}) \xrightarrow{\deg} M = \oplus_{i=1}^{t}(\oplus_{j=1}^{n_i}\mathbb{Z})$$

be the composite map where the first map is the pullback of line bundles via the normalization morphism $\rho : \mathcal{S}_0^{\mathrm{nor}} \to \mathcal{S}_0$, and the map deg is obtained by taking the degree of a line bundle on each irreducible component $D_{i,j}$ of E_i.

Claim 1. $\ker(d)$ is *finite*. ∎

Proof of Claim 1. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_1 & \longrightarrow & \ker(d) & \longrightarrow & \ker(\deg) = \oplus_{r=1}^{s}\mathrm{Pic}(H_r) \oplus (\oplus_{i=1}^{t}\mathrm{Pic}^0(\widetilde{E}_i)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_2 & \longrightarrow & \mathrm{Pic}(\mathcal{S}_0) & \longrightarrow & \mathrm{Pic}(S_0^{\mathrm{nor}}) = \oplus_{r=1}^{s}\mathrm{Pic}(H_r) \oplus (\oplus_{i=1}^{t}\mathrm{Pic}(\widetilde{E}_i)) \\
 & & & & d\downarrow & & \deg\downarrow \\
 & & & & M & = & M
\end{array}$$

where A_1 and A_2 are defined so that the above sequences are exact, and A_2 is finite as follows from the facts that the sheaf $\rho_*(\mathcal{O}^\times_{S_0^{\mathrm{nor}}})/\mathcal{O}^\times_{S_0}$ is a skyscraper sheaf and the residue fields at closed points of $\mathcal{S}_0$ are finite fields. The kernel $\ker(\deg) = \oplus_{r=1}^{s}\mathrm{Pic}(H_r) \oplus (\oplus_{i=1}^{t}\mathrm{Pic}^0(\widetilde{E}_i))$ of the right lower vertical map is finite: $\mathrm{Pic}^0(\widetilde{E}_i)$ is finite since $\widetilde{E}_i$ is a proper and non-singular curve over a finite field, and for $1 \le r \le s$ it holds $\mathrm{Pic}(H_r)$ is finite since H_r is an affine and normal one-dimensional scheme of finite type over the finite field F. Indeed, assume for simplicity that H_r is geometrically connected over F. Let ℓ/F be a finite extension such that $U_r \overset{\mathrm{def}}{=} H_r \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \ell$ admits a smooth and connected compactification C_r with $(C_r \setminus U_r)(\ell) \neq \emptyset$. Let $U_r \to H_r$ be the canonical morphism, and let $\mathrm{Pic}(H_r) \to \mathrm{Pic}(U_r)$ be the induced map of pullback of line bundles. Then $\mathrm{Ker}[\mathrm{Pic}(H_r) \to \mathrm{Pic}(U_r)]$ is finite (cf. [10, Theorem 1.8]). Further, the map $\mathrm{Pic}^0(C_r) \to \mathrm{Pic}(U_r)$ obtained by restricting a degree 0 line bundle on C_r to U_r is surjective (if $x \in (C_r \setminus U_r)(\ell)$ and $D \in \mathrm{Pic}(U_r)$ has degree m then $D - mx \in \mathrm{Pic}^0(C_r)$ restricts to D on U_r); hence, $\mathrm{Pic}(U_r)$ is finite since $\mathrm{Pic}^0(C_r)$ is finite. From the above, it follows that $\mathrm{Pic}(H_r)$ is finite.

This finishes the proof of Claim 1. ∎

Consider the composite map

$$\psi_n : \mathrm{Pic}(\mathcal{S}_n) \to \mathrm{Pic}(\mathcal{S}_0) \xrightarrow{d} M = \oplus_{i=1}^{t}(\oplus_{j=1}^{n_i}\mathbb{Z}).$$

Claim 2. $\ker(\psi_n)$ is *finite*.

Proof of Claim 2. First, we prove that the kernel of the map $\mathrm{Pic}(\mathcal{S}_n) \to \mathrm{Pic}(\mathcal{S}_{n-1})$ is finite for $n \ge 2$. Write $\mathcal{I}_n$ for the sheaf of ideals of $\mathcal{O}_{\mathcal{S}}$ defining $\mathcal{S}_n$. We have an exact sequence of sheaves on $\mathcal{S}_n$:

$$1 \to 1 + (\mathcal{I}_{n-1}/\mathcal{I}_n) \to \mathcal{O}^\times_{S_n} \to \mathcal{O}^\times_{S_{n-1}} \to 1$$

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which induces an exact sequence in cohomology

$$H^1(\mathcal{S}_n, 1+(\mathcal{I}_{n-1}/\mathcal{I}_n)) \to \operatorname{Pic}(\mathcal{S}_n) \to \operatorname{Pic}(\mathcal{S}_{n-1}) \to H^2(\mathcal{S}_n, 1+(\mathcal{I}_{n-1}/\mathcal{I}_n)).$$

Further, the truncated exponential map $\alpha \mapsto 1+\alpha$ induces an isomorphism of sheaves $\mathcal{I}_{n-1}/\mathcal{I}_n \xrightarrow{\sim} 1+(\mathcal{I}_{n-1}/\mathcal{I}_n)$ $[(\mathcal{I}_{n-1}/\mathcal{I}_n)^2 = 0]$; hence, $H^2(\mathcal{S}_n, 1+\mathcal{I}_{n-1}/\mathcal{I}_n) = 0$ and the map $\operatorname{Pic}(\mathcal{S}_n) \to \operatorname{Pic}(\mathcal{S}_{n-1})$ is surjective. Moreover, $H^1(\mathcal{S}_n, \mathcal{I}_{n-1}/\mathcal{I}_n)$ is finite. Indeed, $H^1(\mathcal{S}_n, \mathcal{I}_{n-1}/\mathcal{I}_n)$ is a finitely generated B_n-module with finite support since the morphism $\lambda_n^{-1}(\mathcal{Z}_n \setminus \{z_1,\dots,z_t\}) \to \mathcal{Z}_n \setminus \{z_1,\dots,z_t\}$ is affine and $R^1(\pi_n)_*(\mathcal{I}_{n-1}/\mathcal{I}_n)$ is the sheaf associated with the B_n-module $H^1(\mathcal{S}_n, \mathcal{I}_{n-1}/\mathcal{I}_n)$; here, $\lambda_n : \mathcal{S}_n \to \mathcal{Z}_n$ is the proper morphism induced by λ. This shows that the kernel of the map $\operatorname{Pic}(\mathcal{S}_n) \to \operatorname{Pic}(\mathcal{S}_{n-1})$ is finite for all $n \geq 2$. A similar argument shows that the kernel of the map $\operatorname{Pic}(\mathcal{S}_1) \to \operatorname{Pic}(\mathcal{S}_0)$ is finite. Hence, using Claim 1, $\ker(\psi_n)$ is finite.

This finishes the proof of Claim 2. ∎

In light of Claim 2, and in order to prove assertion **(A)**, it suffices to prove that the cokernel of the composite map

$$M \overset{\text{def}}{=} \oplus_{i=1}^t (\oplus_{j=1}^{n_i} \mathbb{Z}) \xrightarrow{\ \beta\ } \operatorname{Pic}(\mathcal{S}) \to \operatorname{Pic}(\mathcal{S}_n) \to \operatorname{Pic}(\mathcal{S}_0) \xrightarrow{\ d\ } M = \oplus_{i=1}^t (\oplus_{j=1}^{n_i} \mathbb{Z})$$

is finite. The latter follows from the nondegeneracy of the intersection pairing $(\oplus_{j=1}^{n_i}\mathbb{Z}) \times (\oplus_{j=1}^{n_i}\mathbb{Z}) \to \mathbb{Z}$ on each fiber E_i (cf. [26, Lemma on page 69 and the discussion on page 71 after this lemma]), $1 \leq i \leq t$.

This finishes the proof of assertion **(A)**.

*Proof of assertion **(B)**.* Let $\mathcal{J}$ be an ample invertible $\mathcal{O}_S$-ideal such that $\operatorname{Supp}(\mathcal{O}_S/\mathcal{J}) = \mathcal{S}_0$. The existence of such $\mathcal{J}$ follows from the facts that H_r is affine (cf. Proof of Assertion A), $1 \leq r \leq s$, the intersection pairing $(\oplus_{j=1}^{n_i}\mathbb{Z}) \times (\oplus_{j=1}^{n_i}\mathbb{Z}) \to \mathbb{Z}$ on each fiber E_i is negative definite (cf. [26, Lemma on page 69 and the discussion on page 71 after this lemma]), and the numerical criterion of ampleness on curves. More precisely, $\forall\ 1 \leq i \leq t$, one can find a divisor $D = \sum_{j=1}^{n_i} m_{ij} D_{i,j}$ with $m_{i,j} < 0$ and $D.D_{i,j} > 0$ for all $1 \leq j \leq n_j$.

For $m \geq 1$, let $\mathcal{S}'_m$ be the closed subscheme of $\mathcal{S}$ defined by the sheaf of ideals $\mathcal{J}^m$. To prove Assertion B, it suffices to prove that there exists $m_0 > 0$ such that the map

$$\operatorname{Pic}(\mathcal{S}'_{m+1}) \to \operatorname{Pic}(\mathcal{S}'_m)$$

is an isomorphism for any $m > m_0$. We have an exact sequence of shaves on $\mathcal{S}'_{m+1}$:

$$1 \to \mathcal{J}^m/\mathcal{J}^{m+1} \to \mathcal{O}^{\times}_{S'_{m+1}} \to \mathcal{O}^{\times}_{S'_m} \to 1,$$

where the map $\mathcal{J}^m/\mathcal{J}^{m+1} \to \mathcal{O}^{\times}_{S'_{m+1}}$ maps a local section α to $1+\alpha$, which induces an exact sequence in cohomology

$$H^1(\mathcal{S}'_{m+1}, \mathcal{J}^m/\mathcal{J}^{m+1}) \to \operatorname{Pic}(\mathcal{S}'_{m+1}) \to \operatorname{Pic}(\mathcal{S}'_m) \to 0.$$

Now, there exists $m_0 > 0$ such that $H^1(\mathcal{S}'_{m+1}, \mathcal{J}^m/\mathcal{J}^{m+1}) = 0$ if $m \geq m_0$ by [9, première partie, Proposition 2.2.1].

This finishes the proof of assertion **(B)**.

This finishes the proof of Proposition 4.1. ∎

∎

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§5. Compactification of formal germs of p-adic curves

In this section, we use the following notations: K is a complete discrete valuation field with valuation ring R, uniformizing parameter π, and with perfect residue field $\ell \overset{\text{def}}{=} R/\pi R$. Further, A is a *two-dimensional normal complete local ring* containing R with maximal ideal $\mathfrak{m}_A$ containing π and residue field $\ell = A/\mathfrak{m}_A$. We assume that $X \overset{\text{def}}{=} \operatorname{Spec}(A \otimes_R K)$ is geometrically connected. Given a finite extension L/K, we write $\mathcal{O}_L$ for the valuation ring of L, $A_L \overset{\text{def}}{=} A \otimes_{\mathcal{O}_L} L$, $A_{\mathcal{O}_L} \overset{\text{def}}{=} A \otimes_R \mathcal{O}_L$, and $A^{\text{nor}}_{\mathcal{O}_L}$ the normalization of $A_{\mathcal{O}_L}$ in its total ring of fractions.

PROPOSITION 5.1 (Compactification of formal germs of p-adic curves). *We use the above notations. There exists a finite extension L/K, a flat, proper, connected, and normal $\mathcal{O}_L$-relative curve $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_L$, a closed point $y \in \mathcal{Y}$, and an isomorphism $\hat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\sim} A^{\text{nor}}_{\mathcal{O}_L}$ where $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is the completion of the local ring $\mathcal{O}_{\mathcal{Y},y}$ of $\mathcal{Y}$ at y.*

Proof. By the main result in [4, Introduction], there exists a finite extension L/K with uniformizing parameter π_L such that $A^{\text{nor}}_{\mathcal{O}_L}/\pi_L A^{\text{nor}}_{\mathcal{O}_L}$ is reduced. Note that $A^{\text{nor}}_{\mathcal{O}_L}$ is a normal two-dimensional complete local ring with perfect residue field (cf. [2, Chap. IX, §4, Lemma 1] and our assumption that X is geometrically connected). Without loss of generality, we will assume that $A/\pi A$ is reduced. We show that there exist a proper, flat, connected, and normal relative R-curve $\mathcal{Y} \to \operatorname{Spec} R$, a closed point $y \in \mathcal{Y}$, and an isomorphism $\hat{\mathcal{O}}_{\mathcal{Y},y} \xrightarrow{\sim} A$.

First, $A/\pi A$ is a (reduced) one-dimensional complete local ring with residue field ℓ, hence is isomorphic to a quotient $\ell[[x_1,\dots,x_t]]/\mathfrak{a}$ of a formal power series ring $\ell[[x_1,\dots,x_t]]$ over ℓ (cf. [2, chapitre IX, §3]). It then follows from [1, Theorem 3.8] and basic facts on the theory of algebraic curves, that there exist a proper and reduced connected (but not necessarily irreducible) ℓ-curve Z, a closed point $y \in Z$, and an isomorphism $\hat{\mathcal{O}}_{Z,y} \xrightarrow{\sim} A/\pi A$ where $\hat{\mathcal{O}}_{Z,y}$ is the completion of the local ring $\mathcal{O}_{Z,y}$ of Z at y. Moreover, Z is non-singular outside y. There exists a rational function f on Z which defines a finite generically separable morphism $f: Z \to \mathbb{P}^1_\ell$ such that $y = f^{-1}(\infty)$ (cf. [12, Proof of Theorem 3]). Thus, by considering the completion of the morphism f above ∞, we obtain a finite generically separable morphism $\bar{g}: \operatorname{Spec}(A/\pi A) \to \operatorname{Spec}(\ell[[t]])$ where t is a local parameter at ∞. This morphism lifts to a finite morphism $g: \operatorname{Spf} A \to \operatorname{Spf}(R[[T]])$ of formal schemes (cf. [12, Lemma 2]). Let $\widetilde{Z} \to Z$ be the morphism of normalization, and let $\{x_1,\dots,x_m\} \subset \widetilde{Z}$ be the pre-image of y. There is a one-to-one correspondence between the set $\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\} \subset \operatorname{Spec} A$ of prime ideals of height 1 containing π and the set $\{x_1,\dots,x_m\}$, $\mathfrak{p}_i$ corresponds to x_i, $1 \le i \le m$. The composite morphism $\widetilde{Z} \to Z \to \mathbb{P}^1_\ell$ induces, by completion above ∞, finite separable morphisms $\bar{g}_i: \operatorname{Spec} \operatorname{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i}) \to \operatorname{Spec} \ell((t))$ where $\operatorname{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i})$ is the fraction field of the completion $\hat{\mathcal{O}}_{\widetilde{Z},x_i}$ of the local ring $\mathcal{O}_{\widetilde{Z},x_i}$ of $\widetilde{Z}$ at x_i, $1 \le i \le m$ (with the above notations $t = T \mod \pi$).

Consider the formal closed unit disk $D = \operatorname{Spf} R < \frac{1}{T} >$ with parameter $\frac{1}{T}$ and its special fiber $D_\ell = \operatorname{Spec} \ell[\frac{1}{t}]$ ($D_\ell \xrightarrow{\sim} \mathbb{A}^1_\ell$). By a result of Gabber and Katz (cf. [14, Main Theorem 1.4.1]), there exists, for $1 \le i \le m$, a finite cover $\bar{h}_i: C_i \to D_\ell$ with C_i connected, which only (tamely) ramifies above the point $\frac{1}{t} = 0$ and such that the completion of $\bar{h}_i$ above $t = 0$ is generically isomorphic to the cover $\bar{g}_i: \operatorname{Spec} \operatorname{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i}) \to \operatorname{Spec} \ell((t))$. Using formal patching techniques (cf. [23, 1.2]), one can lift the covers $\bar{h}_i$ to finite covers $h_i: Y_i \to D$ which only ramify above the point $\frac{1}{T} = 0$, $1 \le i \le m$. (Outside $\frac{1}{T} = 0$, the existence of such a lifting follows from the theorems of lifting of étale covers [cf. [8, Exposé I, Corollaire 8.4]]. In a formal neighborhood of $\frac{1}{T} = 0$, such a lifting is possible under the tameness condition: étale locally near $\frac{1}{t}$ the cover $\bar{h}_i$ is defined by an equation $y^s = \frac{1}{t^e}$, where $s \geq 1$ is an integer prime to the characteristic of ℓ, and one lifts to the cover defined by $Y^s = \frac{1}{T^e}$.) For $1 \leq i \leq m$, let $\hat{A}_{\mathfrak{p}_i}$ be the completion of the localization $A_{\mathfrak{p}_i}$ of A at $\mathfrak{p}_i$. Thus, $\hat{A}_{\mathfrak{p}_i}$ is a complete discrete valuation ring with uniformizing parameter π (recall $A/\pi A$ is reduced) and residue field $\mathrm{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i})$. Let B be the completion of the localization of $R[[T]]$ at π. Thus, B is a complete discrete valuation ring with residue field $\ell((t))$. The finite cover $g : \mathrm{Spf}\, A \to \mathrm{Spf}(R[[T]])$ induces, by pullback to $\mathrm{Spf}\, B$, finite covers $g_i : \mathrm{Spf}\, \hat{A}_{\mathfrak{p}_i} \to \mathrm{Spf}\, B$ which (by construction) lift the covers $\bar{g}_i : \mathrm{Spec}\,\mathrm{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i}) \to \mathrm{Spec}\,\ell((t))$, $1 \leq i \leq m$. Further, the cover $h_i : Y_i \to D$ induces, by pullback to $\mathrm{Spf}\, B$, a finite cover $\tilde{h}_i : \mathrm{Spf}\, B_i \to \mathrm{Spf}\, B$ which by construction lifts the cover $\bar{g}_i : \mathrm{Spec}\,\mathrm{Fr}(\hat{\mathcal{O}}_{\widetilde{Z},x_i}) \to \mathrm{Spec}\,\ell((t))$. Thus, the covers $\tilde{h}_i : \mathrm{Spf}\, B_i \to \mathrm{Spf}\, B$ and $g_i : \mathrm{Spf}\, \hat{A}_{\mathfrak{p}_i} \to \mathrm{Spf}\, B$ are isomorphic since $\bar{g}_i$ is generically separable. Using formal patching techniques (cf. [8, Exposé I, Corollaire 8.4]), one can patch the covers $g : \mathrm{Spf}\, A \to \mathrm{Spf}(R[[T]])$ and $h_i : Y_i \to D$, $1 \leq i \leq m$, to construct a finite cover $\mathcal{Y} \to \mathbb{P}^1_R$ in the category of formal schemes with $\mathcal{Y}$ normal, connected, proper, and flat over $\mathrm{Spf}\, R$. The special fiber $\mathcal{Y}_\ell \overset{\mathrm{def}}{=} \mathcal{Y} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \ell$ of $\mathcal{Y}$ consists of m irreducible components which intersect at the point y and is (by construction) non-singular outside y. The formal curve $\mathcal{Y}$ is algebraic by formal GAGA and (by construction) $\hat{\mathcal{O}}_{\mathcal{Y},y} \overset{\sim}{\to} A$ as required. ∎

Remark 5.2. Proposition 5.1 asserts the existence, after possibly a finite extension of K, of a proper R-curve $\mathcal{Y}$ and a closed point $y \in \mathcal{Y}^{\mathrm{cl}}$ such that $\hat{\mathcal{O}}_{\mathcal{Y},y} \overset{\sim}{\to} A$. The special fiber $\mathcal{Y}_\ell \overset{\mathrm{def}}{=} \mathcal{Y} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \ell$ of $\mathcal{Y}$ consists of $m_y \overset{\mathrm{def}}{=} m$ (cf. the proof of Proposition 5.1 for the definition of m) irreducible components $\{C_1, \dots, C_m\}$ which intersect at y, $\mathcal{Y}_\ell$ is non-singular outside y, and the normalization morphism $C_i^{\mathrm{nor}} \to C_i$ is a homeomorphism, $1 \leq i \leq m$. In fact, one can, assuming the existence of a compactification of $\mathrm{Spec}\, A$ as in Proposition 5.1, construct such a compactification $\mathcal{Y}$ of $\mathrm{Spec}\, A$ with the additional property that $C_i^{\mathrm{nor}} \overset{\sim}{\to} \mathbb{P}^1_\ell$, $\forall 1 \leq i \leq m$ (cf. [23, Remark 3.1]).

Proposition 5.3. *We use the above notations. There exist a finite extension L/K and a finite morphism* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A^{\mathrm{nor}}_{\mathcal{O}_L}$ *with B local, normal, hyperbolic (cf. Notations), and the morphism* $\mathrm{Spec}\, B_L \to \mathrm{Spec}\, A_L$ *is geometric and étale.*

Proof. This follows easily from Proposition 5.1, Remark 5.2, and Theorem 3 in [23]. ∎

Acknowledgments. I would like to thank Akio Tamagawa for several discussions we had on the topic of this paper. I would like to thank the referee for his/her careful reading of the paper and comments.

References

[1] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 23–58.
[2] N. Bourbaki, *Algèbre Commutative: Chapitre 9*, Masson, Paris, 1983.
[3] V. Cossart, U. Jannsen and S. Saito, *Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes*, preprint, arXiv:0905.2191, 2013.
[4] P. H. Epp, *Eliminating wild ramification*, Invent. Math. **19** (1973), 235–249.
[5] H. Esnault and O. Wittenberg, *On abelian birational sections*, J. Amer. Math. Soc. **23** (2010), no. 3, 713–724.
[6] J. Fresnel and M. Matignon, *Sur les espaces analytiques quasi-compacts de dimension 1 Sur un corps valué, complet, ultramétrique*, Ann. Mat. Pura. Appl. (IV). **CXLV** (1986), 159–210.

[7] O. Gabber, Q. Liu and D. Lorenzini, *The index of an algebraic variety*, Invent. Math. **192** (2013), no. 3, 567–626.
[8] A. Grothendieck, *Revêtements étales et Groupe Fondamental*, Lecture Notes in Mathematics, Vol. 224, Springer, Heidelberg, 1971.
[9] A. Grothendieck and J. Dieudonné, *Étude cohomologique des faisceaux cohérents*, Publ. Math. Inst. Hautes Études Sci. **11** (1961), **17** (1963).
[10] R. Guralnick, D. B. Jaffe and W. Raskind, *On the Picard group: Torsion and the kernel induced by a faithfully flat map*, J. Algebra. **183** (1996), 420–455.
[11] D. Hansen, *Vanishing and comparison theorems in rigid analytic geometry*, Compos. Math. **156** (2020), no. 2, 299–324.
[12] D. Harbater and K. Stevenson, *Patching and thickening problems*, J. Algebra. **212** (1999), 272–304.
[13] Y. Hoshi, *Existence of nongeometric pro-p Galois sections of hyperbolic curves*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 4, 829–848.
[14] N. Katz, *Local-to-global extensions of representations of fundamental groups*, Ann. Inst. Fourier (Grenoble). **36** (1986), no. 4, 69–106.
[15] J. Lipman, *Desingularization of two dimensional schemes*, Ann. of Math. **107** (1978), 151–207.
[16] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, Vol. 6, Oxford University Press, Oxford, 2002.
[17] S. Mochizuki, *Absolute anabelian cuspidalizations of proper hyperbolic curves*, J. Math. Kyoto Univ. **47** (2007), no. 3, 451–539.
[18] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, first edition, Grundlehren der mathematischen Wissenschaften, Vol. 323, Springer, Berlin–Heidelberg, 2000.
[19] M. Saïdi, *The cuspidalisation of sections of arithmetic fundamental groups*, Adv. Math. **230** (2012), 1931–1954.
[20] M. Saïdi, *On the existence of non-geometric sections of arithmetic fundamental groups*, Math. Z. **277** (2014), nos. 1–2, 361–372.
[21] M. Saïdi, *Étale fundamental groups of affinoid p-adic curves*, J. Algebraic Geom. **27** (2018), 727–749.
[22] M. Saïdi, *The cuspidalisation of sections of arithmetic fundamental groups II*, Adv. Math. **354** (2019), 106737, DOI 10.1016/j.aim.2019.106737.
[23] M. Saïdi, *On étale fundamental groups of formal germs of p-adic curves*, Tohoku Math. J. **72** (2020), no. 1, 63–76.
[24] M. Saïdi, *Arithmetic of p-adic curves and sections of geometrically abelian fundamental groups*, Math. Z. **297** (2021), nos. 3–4, 1191–1203.
[25] J.-P. Serre, *Construction de revêtements de la droite affine en caractéristique* $p>0$, C. R. Acad. Sci. Paris. **311** (1990), 341–346.
[26] I. R. Shafarevich, *Lectures on Minimal Models and Birational Transformations of Two Dimensional Schemes*, Lectures on Mathematics and Physics: Mathematics, Vol. 37, Tata Institute of Fundamental Research, Bombay, 1966.
[27] A. Tamagawa, *The Grothendieck conjecture for affine curves*, Compos. Math. **109** (1997), no. 2, 135–194.

Mohamed Saïdi
College of Engineering, Mathematics, and Physical Sciences
University of Exeter
Harrison Building
North Park Road
Exeter EX4 4QF
United Kingdom
m.saidi@exeter.ac.uk