BULLETIN 13 (1), 1970

A NOTE ON ARTIN'S DIOPHANTINE CONJECTURE

BY
GEORGE MAXWELL

A well known theorem of Hasse [1] says that every quadratic form in at least 5 variables over the field Q_p of p-adic numbers has a nontrivial zero. This fact has led Artin to make the conjecture

(C): "Every form over Q_p of degree d in $n > d^2$ variables has a non-trivial zero." However, a counterexample has been provided by Terjanian [2] in the case $d=4$.

The case $d=2$ is distinguished by the fact that every quadratic form may be "diagonalized", i.e., assumed to be of the type $\sum a_i X_i^2$. One is therefore led to the weaker conjecture

(C′): "Every form $f=\sum a_i X_i^d$ over Q_p in $n>d^2$ variables has a nontrivial zero in Q_p,"

which still generalizes Hasse's theorem.

THEOREM. *Suppose* $(p, d)=1$. *Then* (C′) *is true.*

Proof. We may assume that every $a_i \neq 0$. By a suitable change of variable, f may be written as $f=f_0+pf_1+\cdots+p^{d-1}f_d$, where each f_i is of the same type as f but its coefficients are all units. At least one of the f_i will have more than d variables; if we can find a nontrivial zero of it then by setting the other variables equal to zero we shall have a nontrivial zero of f.

So consider a form $f=\sum a_i X_i^d$ in $n>d$ variables such that all the a_i are units. The reduction of f to Z/pZ has a nontrivial zero θ_1 by a theorem of Chevalley [3]. Suppose by induction that we have found nontrivial zeros θ_i of f reduced to Z/p^iZ for $1 \leqslant i \leqslant k$, such that the reduction of θ_i to Z/p^jZ is θ_j whenever $i>j$. Say $\theta_k=(x_1,\ldots,x_n)$; choose $y_1,\ldots,y_n \in Z/p^{k+1}Z$ such that $\bar{y}_i=x_i$. Let $\tilde{a}_i$ (resp. $\bar{a}_i$) be the reduction of a_i to $Z/p^{k+1}Z$ (resp. Z/p^kZ). Then $\bar{f}(y_1,\ldots,y_n)=\sum \bar{a}_i x_i^d=0$ so that $\tilde{f}(y_1,\ldots,y_n)=\sum \tilde{a}_i y_i^d$ is in $p^kZ/p^{k+1}Z$; say $f(y_1,\ldots,y_n)=p^kA$. Instead of the y_i we could have chosen $z_i=y_i+p^kt_i$ since $\bar{z}_i=x_i$ also. Now

$$\begin{aligned}\tilde{f}(z_1,\ldots,z_n) &= \sum \tilde{a}_i(y_i+p^kt_i)^d \\ &= \sum \tilde{a}_i y_i^d + dp^k \sum \tilde{a}_i y_i^{d-1} t_i.\end{aligned}$$

We are trying to make the R.H.S. zero by a suitable choice of t_i; i.e., solve

$$A^*+d^*\sum a_i^*(y_i^*)^{d-1}t_i^* = 0,$$

where * denotes reduction to Z/pZ.

Received by the editors June 26, 1969.

https://doi.org/10.4153/CMB-1970-024-3 Published online by Cambridge University Press

Since the a_i were units, $a_i^* \neq 0$; since $\theta_1 = (y_1^*, \ldots, y_n^*)$ is nontrivial, at least one of the $(y_i^*)^{d-1} \neq 0$; finally, $d^* \neq 0$ since $(p, \mathrm{d}) = 1$. Therefore a solution exists. We have thus found a zero θ_{k+1} of f reduced to $Z/p^{k+1}Z$ which is compatible with $\theta_1, \ldots, \theta_k$ in the above sense. The sequence $\theta_1, \theta_2, \ldots$ defines a nontrivial zero of f in $Z_p = \lim_{\leftarrow} Z/p^k Z$ and thus in Q_p.

It is easy to see that this proof may be generalized to yield the following

THEOREM. *Let K be a field with a discrete valuation v and residue class field $\bar{K}$ such that* (char $\bar{K}$, d) = 1. *If every form $f = \sum a_i X_i^d$ with coefficients in $\bar{K}$ has a nontrivial zero provided $n > d^k$, then every such form with coefficients in K has a nontrivial zero provided $n > d^{k+1}$.*

REFERENCES

1. H. Hasse, *Darstellbarkeit von Zahlen durch Quadratische Formen*, J. f. reine u. angew. Math. **153** (1923), 113–130.

2. G. Terjanian, *Un contre-exemple à une conjecture d'Artin*, Comptes Rendus de l'Acad. Sci. Paris **262** (1966), A612.

3. C. Chevalley, *Démonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Hamburg **11** (1935), 73–75.

QUEEN'S UNIVERSITY,
KINGSTON, ONTARIO