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A SUMMABILITY PROBLEM

BY
M. S. MACPHAIL

In a paper by Wilansky and the writer [4] there were five questions left open, four of which have been answered by Beekman and the writer, [1], [3]. We shall consider the fifth one, namely, "If $\Lambda_A^\perp = I_A$, must $\Lambda_D^\perp = I_D$ for every matrix D with $c_D = c_A$?" Here A is a conservative summability matrix with column limits $a_1, a_2, \ldots, c_A = \{x = \langle x_k \rangle : Ax \in c\}$, $I_A = \{X \in c_A : \sum a_k x_k \text{ converges}\}$, $\Lambda_A^\perp = \{x \in I_A : \lim_A x = \sum a_k x_k\}$.

A method A such that $\Lambda_D^\perp = I_D$ for every method D with $c_D = c_A$ will be said to have property E. There are simple examples of methods having the property, for instance, Bennett [2, Proposition 4] has shown that $\Lambda_A^\perp$ is invariant if I_A is, so if $\Lambda_A^\perp = I_A$ and I_A is invariant, then A has property E. We shall give an example of a method A which has $\Lambda_A^\perp = I_A$ but which does not have property E, so the broad answer to the question is negative. To show what is possible, however, we shall also give an example of a method where I_A is not invariant, so the Bennett proposition does not apply, nevertheless property E holds. As D varies (with $c_D = c_A$), I_D and $\Lambda_D^\perp$ vary while remaining equal to each other. So invariance of the equation $\Lambda_A^\perp = I_A$ is a property possessed by some matrices but not by all.

Before giving our first example, we recall a few facts about the method

$$J = \begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ t_1 & 1 & 0 & 0 & \cdots \\ t_1 & t_2 & 1 & 0 & \cdots \\ t_1 & t_2 & t_3 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

where $\langle t_n \rangle$ is any sequence in ℓ. We have $c_J = c$, and for any conservative method A, if $D = JA$ we have $c_D = c_A$. Moreover, for all $x \in c_A$, we have $\lim_D x = \lim_n y_n + \sum t_n y_n$, where $y_n = \sum_k a_{nk} x_k$. In particular, $d_k = \lim_D e^k = a_k + \sum_n t_n a_{nk}$, where $e^k = \langle 0, 0, \ldots, 0, 1, 0, \ldots \rangle$ (1 in the kth place).

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EXAMPLE 1. Let
$$A=\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -2 & 1 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Evidently $\Lambda_A^\perp = I_A = c_A$. Define J and D as above, with

$$\langle t_n\rangle = \langle 1, 0, \tfrac{1}{2}, 0, \tfrac{1}{4}, 0, \ldots\rangle.$$

Then $d_k = 0$ for each k. If we choose a sequence $\langle y_1, 0, y_3, 0, \ldots\rangle$ with $\lim_n y_n = 0$, $\sum t_n y_n \neq 0$, and determine x from the system of equations $y_n = \sum_k a_{nk}x_k$ $(n = 1, 2, \ldots)$, we have $x \in c_A = c_D$, $\sum d_k x_k = 0$, $\lim_D x \neq 0$, so E does not hold for A.

LEMMA. *Let the method A be such that $\lim_A x = 0$ for all $x \in c_A$. Then A has property E if and only if the following condition holds:*

(E') *For every $\langle t_n\rangle \in \ell$, $\langle x_k\rangle \in c_A$ such that $\sum_k \sum_n t_n a_{nk} x_k$ converges, we have*

$$\sum_k \sum_n t_n a_{nk} x_k = \sum_n \sum_k t_n a_{nk} x_k.$$

Proof. The general continuous linear functional on c_A under the FK topology is given by [5, equation (4)]

$$\begin{aligned} f(x) &= \mu \lim_A x + \sum_n t_n \sum_k a_{nk} x_k + \sum_k \alpha_k x_k \\ &= \mu \lim_A x + \sum_n t_n \sum_k a_{nk} x_k + \sum_k \left(f(e^k) - \mu a_k - \sum_n t_n a_{nk} \right) x_k \end{aligned}$$

where $\langle \alpha_k\rangle \in c_A^\beta$, $\langle t_n\rangle \in \ell$.

Under our hypothesis this reduces to

$$f(x) = \sum_n t_n \sum_k a_{nk} x_k + \sum_k \left(f(e^k) - \sum_n t_n a_{nk} \right) x_k,$$

or, with $f = \lim_D$,

$$\lim_D x = \sum_n t_n \sum_k a_{nk} x_k + \sum_k \left(d_k - \sum_n t_n a_{nk} \right) x_k.$$

It is now easily seen that $E' \Rightarrow E$; to obtain $E \Rightarrow E'$ we observe that every sequence $\langle t_n\rangle \in \ell$ is the sequence of coefficients in a representation of $\lim_D$ for a matrix D with $c_D = c_A$, namely, $D = JA$ where

$$J = \begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ t_1 & 1 & 0 & 0 & \cdots \\ t_1 & t_2 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

EXAMPLE 2. Let $A = \begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{matrix}$

Then

$$\sum_n \sum_k t_n a_{nk} x_k = t_1 x_1 + t_3(x_2 - x_1) + t_5(x_3 - x_2) + \cdots$$

and

$$\sum_k \sum_n t_n a_{nk} x_k = (t_1 - t_3)x_1 + (t_3 - t_5)x_2 + \cdots$$

$$= \lim_{p\to\infty} ((t_1 - t_3)x_1 + (t_3 - t_5)x_2 + \cdots + (t_{2p-1} - t_{2p+1})x_p)$$

$$= \lim_{p\to\infty} (t_1 x_1 + t_3(x_2 - x_1) + \cdots + t_{2p-1}(x_p - x_{p-1}) - t_{2p+1}x_p)$$

Since $t_1x_1 + t_3(x_2 - x_1) + \cdots + t_{2p-1}(x_p - x_{p-1})$ converges for $\langle t_n \rangle \in \ell$, $\langle x_k \rangle \in c_A$, the convergence of $\sum_k \sum_n t_n a_{nk} x_k$ for some $\langle x_k \rangle$ implies the existence of $L = \lim_p t_{2p+1} x_p$, as a finite number. But if $L \neq 0$ we get a contradiction of $\langle t_k \rangle \in \ell$, since $x_p = o(p)$. Hence $L = 0$, and by the lemma A has property E.

To show that I_A is not invariant for A, we note first that $I_A = c_A$, and again define D with $c_D = c_A$ by $D = JA$ where

$$J = \begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ t_1 & 1 & 0 & 0 & \cdots \\ t_1 & t_2 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Now $d_k = t_{2k-1} - t_{2k+1}$, and as in the foregoing work, for $\sum d_k x_k$ to converge we require that $\lim_p t_{2p+1} x_p$ exists finitely. We take $t_q = 2^{-k}$ when $q = 2 \cdot 9^k + 1$ $(k = 1, 2, \ldots)$, $t_q = 0$ otherwise, and we take $x_p = p^{1/2}$ $(p = 1, 2, \ldots)$. Then for $p = 9^k$ we have $t_{2p+1} x_p = 2^{-k} 3^k \to \infty$, so $\langle x_p \rangle \in c_D \setminus I_D$, and I_A is not invariant.

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DEPARTMENT OF MATHEMATICS,
CARLETON UNIVERSITY,
OTTAWA, ONTARIO, K1S 5B6