Canad. Math. Bull. Vol. **58** (1), 2015 pp. 196–206
http://dx.doi.org/10.4153/CMB-2014-041-3

Dihedral Groups of Order $2p$ of Automorphisms of Compact Riemann Surfaces of Genus $p-1$

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Abstract. In this paper we prove that there is only one conjugacy class of dihedral group of order $2p$ in the $2(p-1) \times 2(p-1)$ integral symplectic group that can be realized by an analytic automorphism group of compact connected Riemann surfaces of genus $p-1$. A pair of representative generators of the realizable class is also given.

1 Introduction

The problem we consider in this paper is the realizability of dihedral groups D_{2p} of order $2p$, where p is an odd prime, in $SP_{2(p-1)}(\mathbb{Z})$, the $2(p-1) \times 2(p-1)$ symplectic group over the ring of integers $\mathbb{Z}$, by analytic automorphisms of compact connected Riemann surfaces of genus $p-1$. This is a special case of a more general problem.

Let S be a connected compact Riemann surface of genus g ($g \geq 2$) without boundary and let G be a subgroup of $\mathrm{Aut}(S)$, the group of analytic automorphisms of S. Then G induces a faithful group action on $H_1(S) = H_1(S, \mathbb{Z})$, the first homology group of S,

$$G_* \colon H_1(S) \longrightarrow H_1(S).$$

Let $a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$ be a canonical basis of $H_1(S)$, that is, a basis for which the intersection matrix is

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the identity matrix of degree g. For any element σ_* in G_*, let X be the matrix of σ_* with respect to this basis. Since σ_* preserves the intersection numbers, $X'JX = J$, where X' is the transpose of X.

Definition 1.1 The set of $2n \times 2n$ unimodular matrices X in $M_{2n}(\mathbb{Z})$ such that

$$X'JX = J$$

is called the *symplectic group of genus n over* $\mathbb{Z}$ and is denoted by $SP_{2n}(\mathbb{Z})$. Two symplectic matrices X, Y of $SP_{2n}(\mathbb{Z})$ are said to be *conjugate* or *similar*, denoted by $X \sim Y$, if there is a matrix $Q \in SP_{2n}(\mathbb{Z})$ such that $Y = Q^{-1}XQ$. Two subgroups G, H of

Received by the editors May 20, 2013; revised July 9, 2014.
Published electronically November 13, 2014.
AMS subject classification: **20H25**, 57M60.
Keywords: dihedral group, automorphism group, Riemann surface, integral symplectic matrix, fundamental domain.

$SP_{2n}(\mathbb{Z})$ are said to be conjugate or similar, denoted by $G \sim H$, if there is a matrix $Q \in SP_{2n}(\mathbb{Z})$ such that $H = Q^{-1}GQ = \{Q^{-1}XQ \mid X \in G\}$.

If we fix a canonical basis of $H_1(S)$, there is a natural group monomorphism

$$\Psi\colon \operatorname{Aut}(S) \to SP_{2g}(\mathbb{Z});$$

see Farkas and Kra [3, p. 286]. Clearly, for any given subgroup G of $\operatorname{Aut}(S)$, the groups $\Psi(G)$ with respect to different canonical basis are conjugate in $SP_{2g}(\mathbb{Z})$.

Definition 1.2 A subgroup H of $SP_{2g}(\mathbb{Z})$ is said to be *realizable* if there is subgroup G of Aut (S) for some Riemann surface S of genus g such that $\Psi(G) = H$ with respect to some canonical basis of $H_1(S)$.

A question that naturally arises is "Which subgroups H of $SP_{2g}(\mathbb{Z})$ can be realized?"

Note that $\operatorname{Aut}(S)$ is finite, so we only need to consider finite subgroups of $SP_{2g}(\mathbb{Z})$. The case of cyclic groups of order p of $SP_{p-1}(\mathbb{Z})$, where $p \geq 5$ is an odd prime, was solved by Sjerve and Yang. They gave a complete list of realizable conjugacy classes of p-torsion in $SP_{p-1}(\mathbb{Z})$; see [11]. We have solved the problem for cyclic subgroups of $SP_4(\mathbb{Z})$, see [13]. In this paper we address the question of which classes of dihedral subgroups of order $2p$ of $SP_{2(p-1)}(\mathbb{Z})$ can be realized by a dihedral group action on some Riemann surface of genus $p-1$.

Let D_{2p} be the dihedral group of $2p$ elements and let $T, R \in D_{2p}$ be two fixed generators of order p and 2. Let

$$C = \begin{pmatrix} 0 & & & & -1 \\ 1 & 0 & & & -1 \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -1 \\ & & & 1 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

be two $(p-1) \times (p-1)$ matrices, where C is the companion matrix for $\Phi_p(x)$, the cyclotomic polynomial of p-th unit roots, and U is an anti-diagonal matrix whose anti-diagonal entries are 1. Then $C^p = U^2 = (CU)^2 = I$. Let X and Y,

$$X = \begin{pmatrix} C & \\ & C'^{-1} \end{pmatrix}, \qquad Y = \begin{pmatrix} U & \\ & U \end{pmatrix},$$

be two $2(p-1) \times 2(p-1)$ matrices. It is easy to verify that X and Y are integral symplectic matrices and satisfy the relations $X^p = Y^2 = (XY)^2 = I$. Therefore, the group $\langle X, Y \rangle$ generated by X and Y is a dihedral subgroup of order $2p$ of $SP_{2(p-1)}(\mathbb{Z})$.

Main Theorem *Let dihedral group D_{2p} act on a Riemann surface S of genus $p-1$. There is a canonical basis $a_1, a_2, \dots, a_{p-1}, b_1, b_2, \dots, b_{p-1}$ of $H_1(S)$ such that $\Psi(D_{2p}) = \langle X, Y \rangle$ with $\Psi(T) = X$ and $\Psi(R) = Y$.*

Note that $p-1$ is the minimal genus larger than or equal to two for a Riemann surface that has D_{2p} as its group of automorphisms; see Breuer [1], Michael [5], or Yang [12].

2 Preliminaries

In this section we collect some preliminary material on Riemann surfaces. First we describe how all group actions on Riemann surfaces occur, and then we specialize to the case of the dihedral group of order $2p$.

If G is a finite group acting topologically on a surface S by orientation preserving homeomorphisms, then the positive solution of the Nielsen Realization Problem guarantees that there exists a complex analytic structure on S for which the action of G is by analytic automorphisms (see [2, 4, 8, 10]). Thus there is no loss of generality in assuming that the action of G is complex analytic to begin with, and we will tacitly do so.

The orbit space $\bar{S} = S/G$ of the action of G is also a Riemann surface, and the orbit map $p\colon S \to \bar{S}$ is a branched covering, with all ramifications occurring at fixed points of the action. If $x \in \bar{S}$ is a branch point, then each point in $p^{-1}(x)$ has a non-trivial stabilizer subgroup in G.

It is known that there is a one-to-one correspondence between analytic conjugacy classes of D_{2p} actions on compact connected Riemann surface S of genus $p-1$ and short exact sequences

$$1 \longrightarrow \Pi \longrightarrow \Gamma \stackrel{\theta}{\longrightarrow} D_{2p} \longrightarrow 1,$$

where Γ is a Fuchsian group of signature $(0; p, p, 2, 2)$ and the kernel Π is a torsion free subgroup of Γ. The short exact sequence corresponds to the induced action of D_{2p} on $S = \mathbb{H}/\Pi$, where $\mathbb{H}$ denotes the upper half plane.

As an abstract group Γ has the presentation:

$$\Gamma = \Gamma(0; p, p, 2, 2) = \langle A_1, A_2, B_1, B_2 \mid A_1^p = A_2^p = B_1^2 = B_2^2 = A_1A_2B_1B_2 = 1\rangle;$$

see Jones and Singerman[7, p. 262] or Harvey [6]. The epimorphism $\theta\colon \Gamma \to D_{2p}$ is determined by the images of the generators. The relations in Γ must be preserved and the kernel of θ must be torsion free. Let $T = \theta(A_1)$ and $R = \theta(B_2)$. Then

$$\theta\colon A_1 \mapsto T, \quad A_2 \mapsto T^{p-u}, \quad B_1 \mapsto T^{u-1}R, \quad B_2 \mapsto R,$$

where $u = 1, 2, \ldots, p-1$, is a fixed integer determined by θ.

The main tool we will use is the fundamental domain. We choose a particular embedding of Γ in $\mathrm{Aut}(\mathbb{H})$, namely, Γ is the subgroup generated by A_1, A_2, B_1, B_2, where A_1, A_2 are rotations by $2\pi/p$ and B_1, B_2 are rotations by π about the vertices v_1, v_2, v_3, v_4 of a quadrilateral P whose angles are $\pi/p, \pi/p, \pi/2, \pi/2$, respectively, ordered in the counterclockwise sense. A particular fundamental domain of Γ is given by $P \cup R(P)$, where $R(P)$ is a reflection of P in its side v_1v_4. Then fundamental domain of Π consists of $2p$ copies of the fundamental domain of Γ that can be chosen as the union

$$\bigcup_{i=0}^{p-1} A_1^i\big(P \cup R(P) \cup B_2\big(P \cup R(P)\big)\big),$$

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which is a hyperbolic polygon with $4p$ sides. Figure 1 illustrates a fundamental domain of Π for a particular embedding when $p = 5$. Label these sides e_1, f_1, g_1, $h_1, \dots, e_p$, f_p, g_p, h_p, and orient them as indicated in Figure 1. In the following, the subscripts will be in the set $\mathbb{Z}_p = \{1, 2, \dots, p\}$. For example, we write $e_j = e_i$ if $j - i = kp$ for some integer k.

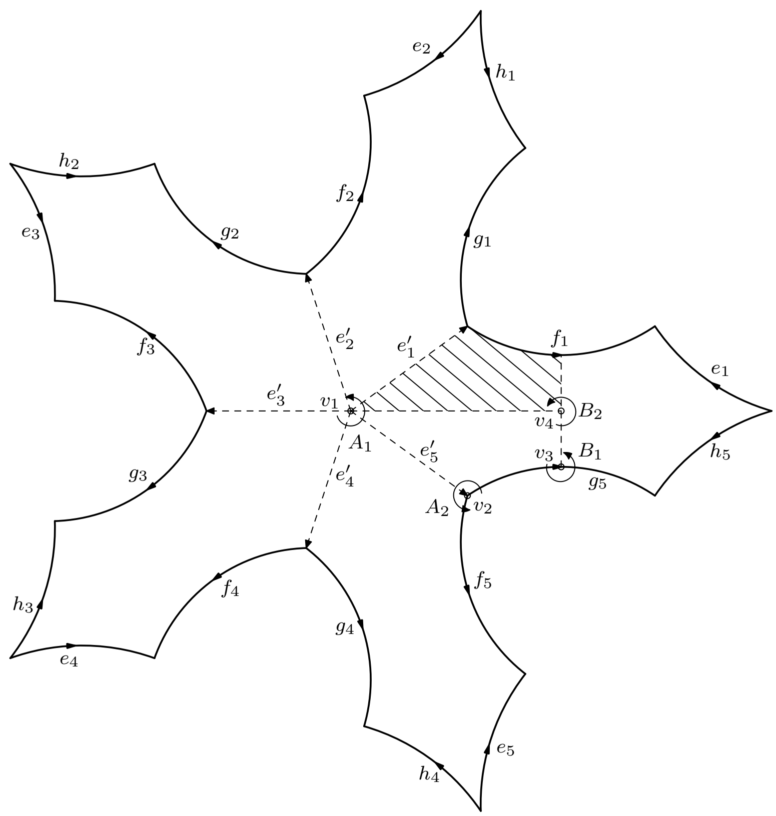

Figure 1: Fundamental Domain of Π.

The sides e_j and h_j, $j = 1, \dots, p$, are identified in the Riemann surface $S = \mathbb{H}/\Pi$. This can be seen from the fact that $A_1^j B_2 A_1 B_2 A_1^{1-j} \in \operatorname{Ker}\theta = \Pi$ and

$$e_j \xrightarrow{A_1^{1-j}} e_1 \xrightarrow{B_2} e'_p \xrightarrow{A_1} e'_1 \xrightarrow{B_2} h_p \xrightarrow{A_1^j} h_j.$$

Similarly, $A_1^{j-1} B_2 B_1 A_1^{u-j} \in \Pi$ and

$$g_{j-u} \xrightarrow{A_1^{u-j}} g_p \xrightarrow{B_1} -g_p \xrightarrow{B_2} f_1 \xrightarrow{A_1^{j-1}} f_j.$$

Therefore, the sides f_j and g_{j-u} are identified in S, for $j = 1, \dots, p$.

Let $\alpha_j = f_j - g_j$, $\beta_j = e'_j - e'_{j-u}$, $\gamma_j = e_j - e_{j+u}$, $j = 1, \dots, p$. Then $\alpha_1, \dots, \alpha_p$, $\beta_1, \dots, \beta_p, \gamma_i, \dots, \gamma_p$ are closed paths on S. It is clear that

$$\sum_{i=1}^{p}[\alpha_i] = 0, \qquad \sum_{i=1}^{p}[\beta_i] = 0, \qquad \sum_{i=1}^{p}[\gamma_i]. = 0,$$

Here we use the notation [ ] to denote homology classes in $H_1(S)$. We see that $[\beta_1], \dots, [\beta_{p-1}], [\gamma_1], \dots, [\gamma_{p-1}]$ forms a basis of $H_1(S)$; see Massey [9]. According to the Figure 1, and $[f_{j+u} - g_j] = 0$, we see that, for $j = 1, \dots, p$,

$$\begin{aligned}
&[e_j - e_{j+u}] + [f_{j+1} - g_{j+1}] + [f_{j+2} - g_{j+2}] + \cdots + [f_{j+u-1} - g_{j+u-1}] \\
&\quad = [e_j - e_{j+1}] + [f_{j+1} - g_{j+1}] + [e_{j+1} - e_{j+2}] + \cdots \\
&\qquad + [e_{j+u-2} - e_{j+u-1}] + [f_{j+u-1} - g_{j+u-1}] + [e_{j+u-1} - e_{j+u}] \\
&\quad = -[f_{j+u} - g_j] + [e'_j - e'_{j+u}] = [e'_j - e'_{j+u}].
\end{aligned}$$

Thus we have

$$[\beta_{j+u}] + \sum_{i=1}^{u-1}[\alpha_{j+i}] = [\alpha_{j+1}] + \cdots + [\alpha_{j+u-1}] + [\beta_{j+u}] = -[\gamma_j] \tag{2.1}$$

or

$$[\beta_j] = -\sum_{i=1}^{u-1}[\alpha_{j-u+i}] - [\gamma_{j-u}] \tag{2.2}$$

where the sum $\sum_{i=1}^{u-1}$ is zero if $u = 1$. Hence $[\alpha_1], \dots, [\alpha_{p-1}], [\beta_1], \dots, [\beta_{p-1}]$ forms a basis of $H_1(S)$.

3 Matrices of T_* and R_*

Since

$$A_1 \colon e_j \mapsto e_{j+1}, \quad e'_j \mapsto e'_{j+1}, \quad f_j \mapsto f_{j+1}, \quad g_j \mapsto g_{j+1}$$

and $\theta(A_1) = T$, we get that T_* on $H_1(S)$ has the form

$$T_*[\alpha_j] = [\alpha_{j+1}], \qquad T_*[\beta_j] = [\beta_{j+1}], \qquad T_*[\gamma_j] = [\gamma_{j+1}].$$

Also, $\theta(A_1^k B_2 A_1^k) = R$. We see that the induced action R on S has the form

$$R\colon e_j \mapsto e'_{p+1-j}, \quad e'_j \mapsto e_{p+1-j}, \quad f_j \mapsto -g_{p+1-j}, \quad g_i \mapsto -f_{p+1-j}$$

and then induced action R_* on $H_1(S)$ is given by

$$R_*[\alpha_j] = [\alpha_{p+1-j}], \qquad R_*[\beta_j] = [\gamma_{p+1-j}], \qquad R_*[\gamma_j] = [\beta_{p+1-j}].$$

Let B_k denote the $(p-1) \times (p-1)$ matrix $C^k U$, where k is an integer. It is clear that $B_0 = U$, $B_k^2 = I$, and $B_k = B_{p+k}$.

Lemma 3.1 *Let T_u and R_u denote the matrices of T_* and R_* with respect to the basis*

$$\alpha_1, \alpha_2, \dots, \alpha_{p-1}, \beta_1, \beta_2, \dots, \beta_{p-1}$$

respectively. Then

$$T_u = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \qquad R_u = \begin{pmatrix} B_1 & -\sum_{i=1}^{u-1} B_{i+1} \\ 0 & -B_{u+1} \end{pmatrix}.$$

Proof Since $T_*[\alpha_i] = [\alpha_{i+1}]$ and $[\alpha_p] = -[\alpha_1] - \cdots - [\alpha_{p-1}]$, we have

$$\begin{aligned} T_*([\alpha_1], \dots, [\alpha_{p-2}], [\alpha_{p-1}]) &= ([\alpha_2], \dots, [\alpha_{p-1}], [\alpha_p]) \\ &= ([\alpha_2], \dots, [\alpha_{p-1}], -[\alpha_1] - \cdots - [\alpha_{p-1}]) \\ &= ([\alpha_1], \dots, [\alpha_{p-2}], [\alpha_{p-1}])C. \end{aligned}$$

Similarly, $T_*([\beta_1], \dots, [\beta_{p-2}], [\beta_{p-1}]) = ([\beta_1], \dots, [\beta_{p-2}], [\beta_{p-1}])C$. Therefore,

$$T_u = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

Note that $R_*[\alpha_i] = [\alpha_{p+1-i}]$, and

$$\begin{aligned} R_*([\alpha_1], \dots, [\alpha_{p-2}], [\alpha_{p-1}]) &= ([\alpha_p], [\alpha_{p-1}], \dots, [\alpha_3], [\alpha_2]) \\ &= ([\alpha_2], [\alpha_3], \dots, [\alpha_{p-1}], [\alpha_p])U \\ &= ([\alpha_1], \dots, [\alpha_{p-2}], [\alpha_{p-1}])CU \\ &= ([\alpha_1], \dots, [\alpha_{p-2}], [\alpha_{p-1}])B_1. \end{aligned}$$

Also note that $R_*[\beta_j] = [\gamma_{p+1-j}]$. From (2.1), we have

$$R_*[\beta_j] = -\sum_{i=1}^{u-1} [\alpha_{i+1-j}] - [\beta_{u+1-j}], \quad j = 1, 2, \dots, p-1.$$

Then

$$\begin{aligned} &R_*([\beta_1], \dots, [\beta_{p-2}], [\beta_{p-1}]) \\ &= -\sum_{i=1}^{u-1} ([\alpha_i], [\alpha_{i-1}], \dots, [\alpha_{i-p+2}]) - ([\beta_u], [\beta_{u-1}], \dots, [\beta_{u-p+2}]) \\ &= -\sum_{i=1}^{u-1} ([\alpha_{i-p+2}], \dots, [\alpha_i])U - ([\beta_{u-p+2}], \dots, [\beta_u])U \\ &= -\sum_{i=1}^{u-1} ([\alpha_1], \dots, [\alpha_{p-1}])C^{i+1}U - ([\beta_1], \dots, [\beta_{p-1}])C^{u+1}U \\ &= -([\alpha_1], \dots, [\alpha_{p-1}]) \sum_{i=1}^{u-1} C^{i+1}U - ([\beta_1], \dots, [\beta_{p-1}])C^{u+1}U \\ &= -([\alpha_1], \dots, [\alpha_{p-1}]) \sum_{i=1}^{u-1} B_{i+1} - ([\beta_1], \dots, [\beta_{p-1}])B_{u+1}. \end{aligned}$$

Hence,

$$R_u = \begin{pmatrix} B_1 & -\sum_{i=1}^{u-1} B_{i+1} \\ 0 & -B_{u+1} \end{pmatrix}.$$

■

4 Intersection Matrix

It is easy to see that the intersection numbers of $[\alpha_i]$ and $[\alpha_j]$ are zero. But the other intersection numbers of $[\alpha_1], \dots, [\alpha_p], [\beta_1], \dots, [\beta_p]$ are somewhat complex.

Let l_{ij} be the intersection number $[\alpha_i]\cdot[\beta_j]$ of $[\alpha_i]$ and $[\beta_j]$. We have the following lemma.

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Lemma 4.1 *We have that* $l_{ij} = l_{i+1\,j+1}$, $l_{1j} = l_{p-j+2,1} = l_{p-1,j-2}$, *and*

$$l_{1j} = \begin{cases} 1, & j = 1, \\ -1, & j = u+1, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.1}$$

Proof T_* preserves the intersection number of closed chains. By Lemma 3.1,

$$l_{ij} = [\alpha_i] \cdot [\beta_j] = T_*\big([\alpha_i]\big) \cdot T_*\big([\beta_j]\big) = [\alpha_{i+1}] \cdot [\beta_{j+1}] = l_{i+1\,j+1}.$$

Iterating this formula, we see that $l_{1j} = l_{p-j+2,1} = l_{p-1,j-2}$.

If $j \neq 1$ and $j \neq u+1$, it is clear that α_1 and β_j do not intersect, so $l_{1j} = 0$. If $j = 1$ or $j = u+1$, the intersection of α_1 and β_j is one point. The verification of (4.1) is easy. ■

Let L_k, $k = 1, \dots, p-1$, denote the $(p-1) \times (p-1)$ matrix

$$L_k = \begin{pmatrix} 1 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ \vdots & & & \ddots & & & & -1 \\ 0 & & & & \ddots & & & 0 \\ -1 & & & & & \ddots & & \vdots \\ \vdots & \ddots & & & & & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

whose entries $x_{ij}^{(k)}$ are given by

$$x_{ij}^{(k)} = \begin{cases} 1, & i = j, \\ -1, & j - i = k \quad \text{or} \quad i - j = p - k, \\ 0, & \text{otherwise.} \end{cases}$$

According to Lemma 4.1, we have proved that the intersection matrix of

$$[\alpha_1], \dots, [\alpha_{p-1}] \quad \text{and} \quad [\beta_1], \dots, [\beta_{p-1}]$$

is L_u. That is

$$([\alpha_1], \dots, [\alpha_{p-1}])' \cdot ([\beta_1], \dots, [\beta_{p-1}]) = L_u.$$

Then

$$([\beta_1], \dots, [\beta_{p-1}])' \cdot ([\alpha_1], \dots, [\alpha_{p-1}]) = -L_u'.$$

It is clear that L_k is a persymmetric matrix, and $L_k U = U L_k' = U L_{p-k}$. Also, we have

Lemma 4.2 *For* $1 \le i, k \le p-1$,

$$(C')^{-i}L_k = L_k C^i = \begin{cases} L_{k-i} - L_{p-i}, & i < k, \\ -L_{p-k}, & i = k, \\ L_{p+k-i} - L_{p-i}, & i > k. \end{cases}$$

Proof We only prove the case that $L_k C^i = L_{k-i} - L_{p-i}$, $1 \le i < k \le p-1$. From the definition of L_k, we have

$$L_k \varepsilon_j = \begin{cases} \varepsilon_j - \varepsilon_{p+j-k}, & 1 \le j < k, \\ \varepsilon_k, & j = k, \\ \varepsilon_j - \varepsilon_{j-k}, & k < j \le p-1, \end{cases}$$

where ε_j, $j = 1, \dots, p-1$, is the j-th basic unit vector whose j-th component is 1 and all other components are 0. Therefore, $L_k(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{p-1}) = \varepsilon_{p-k}$. Note that

$$C\varepsilon_j = \begin{cases} \varepsilon_{j+1}, & 1 \le j < p-1, \\ -(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{p-1}), & j = p-1. \end{cases}$$

By mathematical induction, we have

$$C^i\varepsilon_j = \begin{cases} \varepsilon_{i+j}, & 1 \le j < p-i, \\ -(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{p-1}), & j = p-i, \\ \varepsilon_{i+j-p}, & p-i < j \le p-1. \end{cases}$$

Hence

$$L_k C^i \varepsilon_j = \begin{cases} \varepsilon_{i+j} - \varepsilon_{p+i+j-k}, & 1 \le j < k-i, \\ \varepsilon_k, & j = k-i, \\ \varepsilon_{i+j} - \varepsilon_{i+j-k}, & k-i < j < p-i, \\ -\varepsilon_{p-k}, & j = p-i, \\ \varepsilon_{i+j-p} - \varepsilon_{i+j-k}, & p-i < j \le p-1. \end{cases}$$

When $1 \le j \le k-i-1$,

$$(L_k C^i + L_{p-i} - L_{k-i})\varepsilon_j = (\varepsilon_{i+j} - \varepsilon_{p+i+j-k}) + (\varepsilon_j - \varepsilon_{i+j}) - (\varepsilon_j - \varepsilon_{p+i+j-k}) = 0.$$

When $j = k-i$,

$$(L_k C^i + L_{p-i} - L_{k-i})\varepsilon_{k-i} = \varepsilon_k + (\varepsilon_{k-i} - \varepsilon_k) - \varepsilon_{k-i} = 0.$$

When $k-i+1 \le j \le p-i-1$,

$$(L_k C^i + L_{p-i} - L_{k-i})\varepsilon_j = (\varepsilon_{i+j} - \varepsilon_{i+j-k}) + (\varepsilon_j - \varepsilon_{i+j}) - (\varepsilon_j - \varepsilon_{i+j-k}) = 0.$$

When $j = p-i$,

$$(L_k C^i + L_{p-i} - L_{k-i})\varepsilon_{p-i} = \varepsilon_{p-k} + \varepsilon_{p-i} - (\varepsilon_{p-i} - \varepsilon_{p-k}) = 0.$$

When $p-i+1 \le j \le p-1$,

$$(L_k C^i + L_{p-i} - L_{k-i})\varepsilon_j = (\varepsilon_{i+j-p} - \varepsilon_{i+j-k}) + (\varepsilon_j - \varepsilon_{i+j-p}) - (\varepsilon_j - \varepsilon_{i+j-k}) = 0.$$

Thus $L_k C^i + L_{p-i} - L_{k-i} = 0$.

For other cases, the method is similar. ∎

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Now we compute the intersection matrix of $[\beta_1], \dots, [\beta_{p-1}]$. From (2.2), we have

$$\begin{aligned}\big([\beta_1], \dots, [\beta_{p-1}]\big) &= -\sum_{i=1}^{u-1}\big([\alpha_{1-u+i}], \dots, [\alpha_{p-1-u+i}]\big) - \big([\gamma_{1-u}], \dots, [\gamma_{p-1-u}]\big)\\ &= -\sum_{i=1}^{u-1}\big([\alpha_1], \dots, [\alpha_{p-1}]\big)C^{i-u} - \big([\gamma_{1-u}], \dots, [\gamma_{p-1-u}]\big).\end{aligned}$$

Since the paths β_j and γ_{k-u} have no intersection for any j and k, the intersection matrix of $[\beta_1], \dots, [\beta_{p-1}]$ is

$$\begin{aligned}&\big([\beta_1], \dots, [\beta_{p-1}])' \cdot ([\beta_1], \dots, [\beta_{p-1}]\big)\\ &= -\sum_{i=1}^{u-1}(C')^{i-u}([\alpha_1], \dots, [\alpha_{p-1}])' \cdot ([\beta_1], \dots, [\beta_{p-1}])\\ &= -\sum_{i=1}^{u-1}(C')^{i-u}L_u = -\sum_{i=1}^{u-1}(L_i - L_{p+i-u})\\ &= -\sum_{i=1}^{u-1}L_i + \sum_{i=1}^{u-1}L'_{u-i} = -\sum_{i=1}^{u-1}L_i + \sum_{i=1}^{u-1}L'_i\\ &= \sum_{i=1}^{u-1}(L'_i - L_i).\end{aligned}$$

Hence the intersection matrix of the basis $[\alpha_1], \dots, [\alpha_{p-1}], [\beta_1], \dots, [\beta_{p-1}]$ is

$$M_u = \begin{pmatrix} 0 & L_u \\ -L'_u & \sum_{i=1}^{u-1}(L'_i - L_i)\end{pmatrix}.$$

We see that $[\alpha_1], \dots, [\alpha_{p-1}], [\beta_1], \dots, [\beta_{p-1}]$ is not a canonical basis.

5 Proof of Main Theorem

We need some more properties of L_k and B_k to prove the Main Theorem.

Lemma 5.1 *For any* $1 \le k \le p-1$,

(i) $C^{-i}B_kC^i = B_{k-2i}$,
(ii) $L_kB_kL_k^{-1} = -U$,
(iii) $(C')^iL_kC^i = L_k$,
(iv) $B'_iL_kB_i = L_{p-i}$.

Proof (i) By definition,

$$C^{-i}B_kC^i = C^{-i}C^kUC^i = C^{-i}C^kC^{-i}U = C^{k-2i}U = B_{k-2i}.$$

(ii) From Lemma 4.2, $L_kB_kL_k^{-1} = (L_kC^k)(UL_k^{-1}) = -L_{p-k}(L'_k)^{-1}U = -U$.
(iii) This is the special case of Lemma 4.2.
(iv) $B'_iL_kB_i = (C^iU)'L'_k(C^iU) = UC'^iL_kC^iU = UL_kU = L_{p-k}$. ■

Now we can prove the Main Theorem by showing that there is an invertible integral matrix Q such that $Q^{-1}T_uQ = X$, $Q^{-1}R_uQ = Y$, and $Q'M_uQ = J$.

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Proof of Main Theorem By Lemma 5.1, we see that $C^{-(p+1)/2}B_kC^{(p+1)/2} = B_{k-1}$. Let

$$Q_1 = \begin{pmatrix} C^{(p+1)/2} & \\ & C^{(p+1)/2} \end{pmatrix}.$$

Then by Lemma 5.1, we have

$$Q_1^{-1}T_uQ_1 = T_u, \quad Q_1^{-1}R_uQ_1 = \begin{pmatrix} U & -\sum_{i=1}^{u-1} B_i \\ 0 & -B_u \end{pmatrix}, \quad Q_1'M_uQ_1 = M_u.$$

Note that $L_kB_kL_k^{-1} = -U$. Let $Q_2 = \left(\begin{smallmatrix} I & \\ & L_u^{-1} \end{smallmatrix}\right)$. Then

$$Q_2^{-1}T_uQ_2 = X, \quad Q_2^{-1}\begin{pmatrix} U & -\sum_{i=1}^{u-1} B_i \\ 0 & -B_u \end{pmatrix} Q_2 = \begin{pmatrix} U & -(\sum_{i=1}^{u-1} B_i)L_u^{-1} \\ 0 & U \end{pmatrix},$$

and

$$Q_2'M_uQ_2 = Q_2'\begin{pmatrix} 0 & L_u \\ -L_u' & \sum_{i=1}^{u-1}(L_i' - L_i) \end{pmatrix} Q_2 = \begin{pmatrix} 0 & I \\ -I & A' - A \end{pmatrix},$$

where $A = L_u'^{-1}(\sum_{i=1}^{u-1} L_i)L_u^{-1}$. Let $Q_3 = \left(\begin{smallmatrix} I & -A \\ 0 & I \end{smallmatrix}\right)$. Then $Q_3^{-1} = \left(\begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix}\right)$ and

$$Q_3^{-1}\begin{pmatrix} C & \\ & C'^{-1} \end{pmatrix} Q_3 = \begin{pmatrix} C & AC'^{-1} - CA \\ 0 & C'^{-1} \end{pmatrix},$$

$$Q_3^{-1}\begin{pmatrix} U & -(\sum_{i=1}^{u-1} B_i)L_u^{-1} \\ 0 & U \end{pmatrix} Q_3 = \begin{pmatrix} U & AU - UA - (\sum_{i=1}^{u-1} B_i)L_u^{-1} \\ 0 & U \end{pmatrix},$$

$$Q_3'\begin{pmatrix} 0 & I \\ -I & A' - A \end{pmatrix} Q_3 = J.$$

Using Lemma 5.1 again, we have

$$\begin{aligned} AC'^{-1} - CA &= L_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1}C'^{-1} - CL_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1} \\ &= L_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) CL_u^{-1} - L_u'^{-1}C'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1} \\ &= L_u'^{-1}\Big(\sum_{i=1}^{u-1}(L_iC - C'^{-1}L_i)\Big) L_u^{-1} = 0 \end{aligned}$$

and

$$\begin{aligned} &AU - UA - \Big(\sum_{i=1}^{u-1} B_i\Big) L_u^{-1} \\ &= L_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1}U - UL_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1} - \Big(\sum_{i=1}^{u-1} B_i\Big) L_u^{-1} \\ &= L_u'^{-1}\Big[\Big(\sum_{i=1}^{u-1} L_i\Big) L_u^{-1}UL_u - L_u'UL_u'^{-1}\Big(\sum_{i=1}^{u-1} L_i\Big) - L_u'\Big(\sum_{i=1}^{u-1} B_i\Big)\Big] L_u^{-1} \\ &= L_u'^{-1}\Big[-\big(\sum_{i=1}^{u-1} L_i\big) B_u + B_u'\Big(\sum_{i=1}^{u-1} L_i\Big) - L_{p-u}\Big(\sum_{i=1}^{u-1} B_i\Big)\Big] L_u^{-1} \end{aligned}$$

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$$= L_u'^{-1}\Big[-\Big(\sum_{i=1}^{u-1} L_i\Big) + B_u'\Big(\sum_{i=1}^{u-1} L_i\Big) B_u - \Big(\sum_{i=1}^{u-1} L_{p-u}B_i\Big) B_u\Big] B_u L_u^{-1}$$

$$= L_u'^{-1}\Big[-\sum_{i=1}^{u-1} L_i + \sum_{i=1}^{u-1} L_{p-i} - \sum_{i=1}^{u-1} L_{p-u}C^{p+i-u}\Big] B_u L_u^{-1}$$

$$= L_u'^{-1}\Big[-\sum_{i=1}^{u-1} L_i + \sum_{i=1}^{u-1} L_{p-i} - \sum_{i=1}^{u-1} (L_{p-i} - L_{u-i})\Big] B_u L_u^{-1} = 0.$$

Here we also use Lemma 4.2 for $L_{p-u}C^{p+i-u}$. If we let $Q = Q_1Q_2Q_3$, then

$$Q^{-1}T_uQ = X, \qquad Q^{-1}R_uQ = Y, \qquad Q'M_uQ = J.$$

This concludes the proof. ∎

Acknowledgment We would like to thank the referee for his help on the paper.

References

[1] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*. London Mathematical Society Lecture Note Series, 280, Cambridge University Press, Cambridge, 2000.

[2] B. Eckmann and H. Müller, *Plane motion groups and virtual Poincaré duality of dimension two*. Invent. Math. **69**(1982), no. 2, 293–310. http://dx.doi.org/10.1007/BF01399508

[3] H. M. Farkas and I. Kra, *Riemann surfaces*. Second ed., Graduate Texts in Mathematics, 71, Springer-Verlag, New York, 1992.

[4] D. Gabai, *Convergence groups are Fuchsian groups*. Ann. of Math. **136**(1992), no. 3, 447–510. http://dx.doi.org/10.2307/2946597

[5] A. A. George Michael, *Metacyclic groups of automorphisms of compact Riemann surfaces*. Hiroshima Math. J. **31**(2001), no. 1, 117–132.

[6] W. J. Harvey, *Spaces of discrete groups*. In: Discrete groups and automorphic functions (Proc. Conf. Cambridge, 1975), Academic Press, London, 1977, pp. 295–348.

[7] G. Jones and D. Singerman, *Complex functions. An algebraic and geometric viewpoint.* Cambridge University Press, Cambridge, 1987.

[8] S. P. Kerckhoff, *The Nielsen realization problem*. Ann. of Math. **117**(1983), no. 2, 235–265. http://dx.doi.org/10.2307/2007076

[9] W. S. Massey, *Singular homology theory*. Graduate Texts in Mathematics, 70, Spring-Verlag, New York-Berlin, 1980.

[10] J. Nielsen, *Abbildungklassen Endliche Ordung*. Acta Math. **75**(1943), 23–115. http://dx.doi.org/10.1007/BF02404101

[11] D. Sjerve and Q. Yang, *Conjugacy classes of p-torsion in* $SP_{p-1}(\mathbb{Z})$. J. Algebra **195**(1997), no. 2, 580–603. http://dx.doi.org/10.1006/jabr.1997.7084

[12] Q. Yang, *Dihedral groups of automorphisms of compact Riemann surfaces*. Canad. Math. Bull. **41**(1998), no. 2, 252–256. http://dx.doi.org/10.4153/CMB-1998-036-7

[13] ———, *Matrix classes realized by automorphisms of compact Riemann surfaces of genus two*. Comm. Algebra **40**(2012), no. 7, 2343–2357. http://dx.doi.org/10.1080/00927872.2011.565491

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