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S-SUBGROUPS OF THE REAL HYPERBOLIC GROUPS

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1. Introduction. If H is a closed subgroup of a locally compact group G, with G/H having finite G-invariant measure, then, as observed by Atle Selberg [**8**], for any neighborhood U of the identity in G and any element g in G, there is an integer $n > 0$ such that g^n is in $U \cdot H \cdot U$. A subgroup satisfying this latter condition is said to be an *S-subgroup*, or satisfies *property* (S). If G is a solvable Lie group, then the converse of Selberg's result has been proved by S. P. Wang [**10**]: If H is a closed S-subgroup of G, then G/H is compact. Property (S) has been used by A. Borel in the important "density theorem" (see Section 2 or [**1**]). The main result of this paper is

THEOREM 1. *A discrete subgroup H of $G_n = SO(n + 1, 1)/(\pm I)$ is an S-subgroup if and only if it is a group of the first kind in the sense of Fuchsian and Kleinian groups* (*see Definition* 3.1 *below*).

A discrete subgroup for which G/H has finite measure is commonly called a *lattice*. G_1 and G_2 are respectively isomorphic to $PSL(2, \mathbf{R})$ and $PSL(2, \mathbf{C})$, and a discrete subgroup of G_1 (resp., G_2) is a *Fuchsian* (resp., *Kleinian*) group. So for discrete (finitely generated) subgroups of G_1, lattice and S-subgroup mean the same thing. But in G_2 this is false; there are finitely generated discrete subgroups of the first kind in G_2 having fundamental domain of infinite volume. Theorem 1 was first proved for G_1 and G_2 in [**7**].

In Section 2, property (S) is further described and related to Dirichlet's theorem in number theory. In Section 3, the groups G_n, and the notion "group of the first kind" are described. The "only if" part of Theorem 1 is proved in Section 4, and the "if" part in Section 5. In Section 6, the main result is reformulated for non-discrete subgroups.

2. Property (S) and Dirichlet's theorem. Because the study of Fuchsian groups often involves working with sequences, I will use the following version of property (S), which is equivalent if G is a connected Lie group, since the countability axioms are then satisfied.

Definition 2.1. H is said to be an *S-subgroup* of G if, given any g in G, there exists a sequence h_i in H and sequences u_i and v_i in G tending to the identity, e, as $i \to \infty$, and a sequence of positive integers n_i, such that $g^{n_i} = u_i \cdot h_i \cdot v_i$. Moreover, we can require that the n_i be strictly increasing, if G has a countable neighborhood base at e. When such equations hold, we say that *g satisfies the S-condition for H.*

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An important result depending on property (S) is

2.2. Density Theorem (Borel [**1**]). *Let G be a connected semisimple Lie group without compact factor, and H an S-subgroup of G. For any linear representation $f : G \to GL(n, \mathbf{R})$, the Zariski closure of $f(H)$ in $GL(n, \mathbf{R})$ contains $f(G)$.*

Remarks 1. As a corollary to (2.2), if G is a simple Lie group, then H is either discrete or dense in G. So in looking for closed S-subgroups of such G, we need only consider discrete subgroups.

2. Theorem 1 provides an easy counterexample to the converse of (2.2) in the case $G_1 \cong PSL(2, \mathbf{R})$ by letting H be the Fuchsian group generated by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$. H is a subgroup of infinite index in $\Gamma(3)$, the principal congruence subgroup of level 3 in $PSL(2, \mathbf{Z})$, and thus is not a lattice and so not of the first kind. By Theorem 1, H cannot be an S-subgroup. Nor can the subgroup H' in $SL(2, \mathbf{R})$ generated by the same two matrices be an S-subgroup, else property (S) would project down to G_1. But H' is Zariski dense in $SL(2, \mathbf{R})$ since the one-parameter subgroups $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right\}$ generate $SL(2, \mathbf{R})$.

If G is the additive group of real numbers with the usual topology, and H is the subgroup of integers, then Selberg's observation is simply Dirichlet's theorem on Diophantine approximation:

Theorem 2.3 (Hardy and Wright [**6**, p. 156]). *Given a real number x and any $\epsilon > 0$, there are integers m, n such that $|mx - n| \leqq \epsilon$. Moreover, if $x > 0$, m and n may both be taken as arbitrarily large positive integers.*

This will be a key tool in the proofs of Section 5, as will its generalization to higher dimensions:

Theorem 2.4 ([**6**, p. 170]). *Given real numbers $x_1, x_2, \ldots, x_k$ and any $\epsilon > 0$, there are integers $m, n_1, n_2, \ldots, n_k$ such that $|mx_j - n_j| < \epsilon$, $j = 1, \ldots, k$, and m and the n_j are arbitrarily large if the $x_j > 0$.*

While Dirichlet's theorem concerns approximating integers by multiples of a number x, Kronecker's theorem deals with approximating non-integral numbers by multiples of a number x. Stated in the higher-dimensional form needed for Section 5:

Theorem 2.5 ([**6**, p. 382]). *Given real numbers $a_1, a_2, \ldots, a_k$ and $x_1, x_2, \ldots, x_k$ such that $x_1, x_2, \ldots, x_k$, and 1 are linearly independent over the field of rational numbers (e.g., if $k = 1$, x_1 is irrational), then for any $\epsilon > 0$, there is a positive integer M, depending only on ϵ and the x_j, and integers $m, n_1, n_2, \ldots, n_k$ so that $|mx_j - n_j - a_j| < \epsilon$, $j = 1, 2, \ldots, k$, and $0 \leqq m \leqq M$.*

In other words, multiples of the x_j, modulo 1, taken as a k-tuple, are dense in the unit k-cube. If there are linear relations in the x_j, then the multiples are dense in the hyperplanes of the unit k-cube defined by those relations:

THEOREM 2.6 (Cassels [**2**, pp. 53–59]). *Given real numbers $a_1, \ldots, a_k$ and $x_1, \ldots, x_k$ such that for all sets of k integers $u_1, u_2, \ldots, u_k$,*

$$u_1a_1 + \ldots + u_ka_k = \textit{integer},$$

whenever

$$u_1x_1 + \ldots + u_kx_k = \textit{integer},$$

then for any $\epsilon > 0$, there is a positive integer M, depending only on ϵ and the x_j, and integers $m, n_1, \ldots, n_k$ so that $|mx_j - n_j - a_j| < \epsilon$, $j = 1, 2, \ldots, k$, and $0 \leqq m \leqq M$.

3. The hyperbolic groups and groups of the first kind. To describe the real hyperbolic groups [**3**, **4**, **5**], begin with Moebius n-space which is the one-point compactification of euclidean n-space $\mathbf{R}^n$, denoted by $\overline{\mathbf{R}^n} = \mathbf{R}^n \cup \infty$. The Moebius group, M_n, is the continuous group of transformations of $\overline{\mathbf{R}^n}$ generated by reflections in $(n-1)$-spheres and planes of $\mathbf{R}^n$. Embedding $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$, we see that the action of M_n extends to $\overline{\mathbf{R}^{n+1}}$ by reflections in n-spheres and planes meeting $\mathbf{R}^n$ orthogonally. This action leaves the two components of $\mathbf{R}^{n+1} - \mathbf{R}^n$ invariant. Denote one of these components by H^{n+1}, the "upper half-space." Letting $\overline{H^{n+1}} = H^{n+1} \cup \overline{\mathbf{R}^n}$, $B^n =$ the open unit ball in $\mathbf{R}^n$, $S^{n-1} =$ the boundary of B^n, and $\overline{B^n} = B^n \cup S^{n-1}$, it is well known that H^{n+1}, $\overline{\mathbf{R}^n}$, and $\overline{H^{n+1}}$ can be identified with B^{n+1}, S^n, and $\overline{B^{n+1}}$, respectively. As is well known, there is a Riemannian metric in H^{n+1} (or B^{n+1}) making H^{n+1} into $(n+1)$-dimensional hyperbolic space, with M_n its group of isometries.

Also, if $O(n+1, 1)$ denotes the orthogonal (or Lorentz) group of $(n+2)$ by $(n+2)$ real matrices leaving invariant the quadratic form

$$x_0^2 + x_1^2 + \ldots + x_n^2 - x_{n+1}^2,$$

then M_n is isomorphic to $O(n+1, 1)/(\pm I)$. Let G_n be the identity component of M_n. For all n, G_n is isomorphic to $SO^+(n+1, 1)$ (the sugbroup which preserves each half $x_{n+1} > 0$ and $x_{n+1} < 0$ of the cone $x_0^2 + x_1^2 + \ldots + x_n^2 - x_{n+1}^2 < 0$).

An element of G_n is generated by orientation preserving orthogonal maps, stretchings, and translations. Moreover, G_n sends k-spheres to k-spheres in $\overline{H^{n+1}}$.

G_1 is isomorphic to $PSL(2, \mathbf{R})$ and acts on $\overline{H^2}$ (identified with the extended complex upper half-plane) by linear fractional transformations, being the group of all directly conformal automorphisms.

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A discrete subgroup H of G_n acts discontinuously on H^{n+1}, but not in general on $\overline{\mathbf{R}^n}$. The *limit set*, $L(H)$, of H consists of those points x in $\overline{\mathbf{R}^n}$ (*limit points*) for which there is a z in $\overline{H^{n+1}}$ with $H(z)$ accumulating at x (actually, for each x, z may be any point in H^{n+1} with the possible exception of x and one other point).

Definition 3.1. A discrete subgroup H of G_n is said to be of the *first kind* if its limit set, $L(H)$, is all of $\overline{\mathbf{R}^n}$; otherwise, it is of the *second kind*.

The elements of G_n belong to three kinds of conjugacy classes according to their fixed points in $\overline{H^{n+1}}$ ([**3**], [**4**]). An element g in G_n having a fixed point in H^{n+1} is called *elliptic*, and its matrix form is conjugate to an element in the orthogonal group $SO(n + 1)$. If g has exactly two fixed points, x and y, both in $\overline{\mathbf{R}^n}$, g is called *loxodromic*, and can be written uniquely as $\tilde{g} \cdot \mathring{g} = \mathring{g} \cdot \tilde{g}$ where $\tilde{g}$ is a stretching (or *hyperbolic* element) of constant factor along circular arcs in $\overline{H^{n+1}}$ joining x and y, and $\mathring{g}$ is an elliptic element leaving pointwise fixed the geodesic (semicircle) A_g in H^{n+1} orthogonal to $\overline{\mathbf{R}^n}$ with x and y as its endpoints. A_g is called the *axis* of g. One of these two fixed points, say x, is called the *attracting fixed point* of g since, for any z in $\overline{H^{n+1}}$ except $z = y$, $g^n(z) \to x$, as $n \to +\infty$. And the other, y, is called the *repelling fixed point* of g since $g^{-n}(z) \to y$ as $n \to +\infty$, $z \neq x$. Transforming x to ∞ and y to the origin 0 in $\mathbf{R}^n$, g has the conjugate form $\tilde{g} \cdot \mathring{g}$, where $\tilde{g}(z) = r \cdot z$, $r > 1$, and $\mathring{g}$ is in $SO(n + 1)$. The number r (resp., r^{-1}) is called the *attraction*, or *stretching*, *factor* of g (resp., g^{-1}).

If g has exactly one fixed point, which is in $\overline{\mathbf{R}^n}$, g is called *parabolic*, and moving the fixed point to ∞, g is seen to be conjugate to a rigid euclidean motion of $\mathbf{R}^n$ which may be assumed to be in the form $\bar{g} \cdot \mathring{g} = \mathring{g} \cdot \bar{g}$, where $\bar{g}$ is a pure translation in the x_n-direction (in fact, $\bar{g} : z_n \to z_n + 1$), and $\mathring{g}$ is an orthogonal transformation.

We call the corresponding fixed points elliptic, loxodromic, or parabolic, also.

4. Property (S) **implies group of the first kind.** The "only if" part of the main result is easily proved:

PROPOSITION 4.1. *If a discrete subgroup H of G_n satisfies property* (S), *then H is of the first kind.*

Proof. Let x be a point in $\overline{\mathbf{R}^n}$. We must show that x is a limit point of H. Let g be any hyperbolic element (stretching) of G_n with fixed points x (attracting) and $y (\neq x)$. Select an n-sphere C in $\mathbf{R}^{n+1}$ centered at x (resp., at y, if $x = \infty$), so that y is exterior to C (resp., $x = \infty$ is exterior to C), and select a point z in both the interior (resp., exterior) of C and in H^{n+1}.

Positive integral powers of g move C and its interior (resp., exterior) arbitrarily close to x (e.g., within any predetermined $(n + 1)$-ball centered at x). Since H is an S-subgroup, by Definition (2.1) we have equations

$$g^{n_i} = u_i \cdot h_i \cdot v_i,$$

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where $u_i, v_i \to e$ in G_n, h_i is in H, and $n_i \to +\infty$. Then $v_i^{-1} \to e$, and for large i, $v_i^{-1}(z)$ is in the interior (resp., exterior) of C. Hence, $g^{n_i} \cdot v_i^{-1}(z) \to x$, as $i \to \infty$. But $g^{n_i} \cdot v_i^{-1} = u_i \cdot h_i$, and so $u_i \cdot h_i(z) \to x$. Since G_n acts continuously on $\overline{H^{n+1}}$, and since $u_i^{-1} \to e$, we have $u_i^{-1} \cdot u_i \cdot h_i(z) = h_i(z) \to x$. That is, x is a limit point of H, so H is a group of the first kind.

5. Group of the first kind implies property (S)**.** In order that a discrete subgroup H of the first kind in G_n be an S-subgroup, each element g in G_n must satisfy the S-condition for H. Observe that g may be assumed to be in any convenient conjugate form. For if $\bar{g} = T \cdot g \cdot T^{-1}$, with T in G_n, then $T \cdot H \cdot T^{-1}$ is a group of the first kind, because if x is a limit point of H in $\overline{\mathbf{R}^n}$, $T(x)$ is a limit point of $T \cdot H \cdot T^{-1}$, and $T(\overline{\mathbf{R}^n}) = \overline{\mathbf{R}^n}$, and, moreover, if we obtain equations $\bar{g}^{n_i} = u_i \cdot \bar{h}_i \cdot v_i$, with $\bar{h}_i = T \cdot h_i \cdot T^{-1}$, then

$$g^{n_i} = (T^{-1} \cdot u_i \cdot T) \cdot h_i \cdot (T^{-1} \cdot v_i \cdot T),$$

with the elements in parentheses approaching e, since $u_i, v_i \to e$.

The "if" part of the main result is proved below in Propositions (5.1), (5.3) and (5.4).

PROPOSITION 5.1. *If g is an elliptic element in G_n, then g satisfies the S-condition for any subgroup H of G_n.*

Proof. The elliptic element g has its matrix form conjugate to an element in $SO(n + 1)$, and so is further conjugate to a matrix in block-diagonal form with each of k blocks, A_j, being in $SO(2, \mathbf{R})$:

$$A_j = \begin{pmatrix} \cos 2\pi\theta_j & \sin 2\pi\theta_j \\ -\sin 2\pi\theta_j & \cos 2\pi\theta_j \end{pmatrix}, \quad 0 \leqq \theta_j < 1.$$

where $k = (n + 1)/2$ if n is odd, and $k = n/2$ (and $A_{(n+2)/2} = 1$) if n is even. Then the blocks of g^n are A_j^n for a positive integer n, and, calling the θ_j the *angular parts* of g (the numbers $\exp(\pm i2\pi\theta_j)$ are the eigenvalues of g), we know that the angular parts of g^n are the numbers $n\theta_j$, modulo 1. Applying Dirichlet's theorem (2.4) in k variables, we see that the $n\theta_j$ can be simultaneously made as close to integers as desired, making each block A_j^n as close to the two-by-two identity matrix as desired. Hence, there is a sequence of positive integers $n_i \to \infty$, such that $g^{n_i} \to e$, and so g trivially satisfies the S-condition for any H.

In order to obtain the S-condition for loxodromic and parabolic elements g, we need the known result that the loxodromic fixed points of a group of the first kind are "pairwise dense":

PROPOSITION 5.2 ([**4**, *Proposition* 12], [**7**, pp. 13, 14, 29], [**9**, *Proposition* 1.4]). *If H is a discrete subgroup of the first kind in G_n, and if x and y are* (*not necessarily distinct*) *points in $\overline{\mathbf{R}^n}$, then there is a sequence of loxodromic elements*

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h_i in H, so that h_i has attracting fixed point x_i and repelling fixed point y_i, with $x_i \to x$ and $y_i \to y$.

PROPOSITION 5.3. *If H is a discrete subgroup of the first kind and g is a loxodromic element in G_n, then g satisfies the S-condition for H.*

Proof. Take g to be in the conjugate form $g = \tilde{g} \cdot \mathring{g} = \mathring{g} \cdot \tilde{g}$ with fixed points ∞ (attracting) and 0, so that $\tilde{g}(z) = r \cdot z$, $r > 1$, and $\mathring{g}$ is orthogonal with k angular parts α_j, as in Proposition (5.1) above. By (5.2) there are loxodromic elements h_i in H with fixed points x_i (attracting) and y_i in $\overline{\mathbf{R}^n}$, so that $x_i \to \infty$ and $y_i \to 0$. The h_i are conjugate to elements $\bar{h}_i = u_i \cdot h_i \cdot u_i^{-1}$ having fixed points ∞ (attracting) and 0, with $u_i \to e$ in G_n. The u_i may be constructed by first letting d_i be a translation in $\overline{\mathbf{R}^n}$ fixing ∞ and moving y_i to 0, then letting f_i be a similar parabolic element fixing 0 and moving $d_i(x_i)$ to ∞, and finally setting $u_i = f_i \cdot d_i$.

The element $\bar{h}_i$ is a product of a hyperbolic element $\tilde{h}_i$ of the form $\tilde{h}_i(z) = r_i \cdot z$, $r_i > 1$, and an elliptic element $\mathring{h}_i$ fixing 0 and ∞, whose matrix representation has k angular parts θ_{ji}, $j = 1, \ldots, k$.

For each fixed i, by Dirichlet's theorem in $2k + 1$ variables, there exist integers m_i, $n_i > 0$ and p_{ji}, q_{ji}, $j = 1, \ldots, k$, such that (logarithms are to base r, the attraction factor of g):

$$\begin{aligned} &m_i \log (r_i) - n_i = a_i, \\ &m_i\theta_{ji} - p_{ji} = b_{ji}, \quad j = 1, \ldots, k, \quad \text{and} \\ &m_i \log (r_i) \cdot \alpha_j - q_{ji} = c_{ji}, \quad j = 1, \ldots, k, \end{aligned}$$

with a_i, b_{ji}, and $c_{ji} \to 0$, as $i \to \infty$.

Defining hyperbolic elements v_i in G_n by $v_i(z) = r^{-a_i} \cdot z$, we see that $\tilde{g}^{n_i} = v_i \cdot \tilde{h}_i^{m_i}$, with $v_i \to e$. Also, the angular parts of $\mathring{h}_i^{m_i}$ are $m_i\theta_{ji}$ (modulo 1) which get closer to 0 or 1, so $\mathring{h}_i^{m_i} \to e$. Similarly, the angular parts of $\mathring{g}^{n_i}$ are

$$n_i\alpha_j = m_i \log (r_i) \cdot \alpha_j - a_i\alpha_j = q_{ji} + c_{ji} - a_i\alpha_j \text{ (modulo 1)},$$

and so $\mathring{g}^{n_i} \to e$. Finally, we obtain

$$g^{n_i} = v_i \cdot \tilde{h}_i^{m_i} \cdot \mathring{g}^{n_i} = v_i \cdot \bar{h}_i^{m_i} \cdot (\mathring{h}_i^{-m_i} \cdot \mathring{g}^{n_i}),$$

with the element in parentheses approaching e, and so g satisfies the S-condition for H. This completes the proof of (5.3).

For parabolic g, except when dealing with elliptic parts, Dirichlet's theorem is not needed (and indeed is very awkward) since, for example, the powers of the translation $z_n \to z_n + 1$ are $z_n \to z_n + n_i$, which, for an appropriate integer n_i, can approximate the action of a nearby loxodromic element along the x_n-axis to within one unit.

PROPOSITION 5.4. *If H is a discrete subgroup of the first kind, and g is a parabolic element in G_n, then g satisfies the S-condition for H.*

Proof. Assume g is in the conjugate form $g = ts = st$, with t the translation, $z_n \to z_n + 1$, in the x_n-direction, and s, orthogonal. Then g has ∞ as its only fixed point. As Lemma (5.5) will show, there are loxodromic elements h_i in H which, after conjugation by elements u_i in G_n approaching e, have fixed points x_i (attracting) and y_i, both on the x_n-axis with $x_i \to +\infty$, and $y_i \to -\infty$, and with $y_i/x_i \to 0$ (identifying these points with their x_n-coordinates). The conjugated elements will be denoted again as h_i.

Case (i). First suppose $s = e$, so that $g = t$. Express the h_i as products of their hyperbolic and elliptic parts, $h_i = \tilde{h}_i \cdot \mathring{h}_i$. We can think of $\tilde{h}_i$ and t as being in $G_1 \cong PSL(2, \mathbf{R})$ which acts on the extended upper half-plane defined by the x_n-axis and the positive x_{n+1}-axis in H^{n+1}. If $\tilde{h}_i$ has an attraction factor r_i^2 ($r_i > 1$), then it is conjugate in G_1 to

$$\begin{pmatrix} r_i & 0 \\ 0 & r_i^{-1} \end{pmatrix},$$

and so has the form

$$\tilde{h}_i = \frac{1}{x_i - y_i}\begin{pmatrix} x_i r_i - y_i/r_i & -x_i y_i(r_i - 1/r_i) \\ r_i - 1/r_i & -y_i r_i + x_i/r_i \end{pmatrix}.$$

Replace each h_i by a sufficiently high power of itself in order to have $r_i \to \infty$ (the attraction factor of h_i^m is r_i^{2m}) and $\mathring{h}_i \to e$ (Dirichlet's theorem).

To have $t^{-n_i} \cdot \tilde{h}_i$ be close to the two-by-two identity matrix, examine

$$(1) \quad t^{-n_i} \cdot \tilde{h}_i = \begin{pmatrix} 1 & -n_i \\ 0 & 1 \end{pmatrix} \cdot \tilde{h}_i = \frac{1}{x_i - y_i}\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},$$

where

$$\begin{aligned} A_i &= x_i r_i - y_i/r_i - n_i(r_i - 1/r_i), \\ B_i &= -x_i y_i(r_i - 1/r_i) - n_i(-y_i r_i + x_i/r_i), \\ C_i &= r_i - 1/r_i, \quad \text{and} \\ D_i &= -y_i r_i + x_i/r_i. \end{aligned}$$

For the lower right entry of (1) to be close to 1, we should have $x_i \doteq -r_i y_i$. And if we had $x_i = -r_i y_i a_i$, with $a_i \to 1$, the upper right entry of (1) would be

$$\frac{1}{-r_i a_i + 1}\left(y_i a_i(r_i^2 - 1) + n_i(r_i + a_i)\right).$$

For this to be close to zero, we should have

$$(2) \quad n_i \doteq -y_i a_i(r_i^2 - 1)/(r_i + a_i) = \frac{a_i(r_i + 1)}{r_i + a_i} \cdot (-y_i(r_i - 1)).$$

As Lemma (5.6) will show, if we select $a_i \to 1$ so that (2) is an integer (call it n_i), then there are elements $w_i \to e$ in G_1 leaving y_i fixed and moving x_i to $-r_i y_i a_i$ by which $h_i' = w_i \cdot \tilde{h}_i \cdot w_i^{-1}$ is as desired. That is, replacing $\tilde{h}_i$

by h_i' in (1), the lower right entry becomes $(r_i + a_i)/(r_i a_i + 1) \to 1$. The upper right entry becomes 0, with $n_i = (2)$. Hence, the upper left entry is $(r_i a_i + 1)/(r_i + a_i) \to 1$, and the lower left entry is $(r_i - 1/r_i)/(-r_i y_i a_i - y_i) \to 0$.

Thus, we have $t^{-n_i} \cdot h_i' \to e$. So in G_n, for some $v_i \to e$,

$$t^{n_i} = h_i' \cdot v_i = w_i \cdot \tilde{h}_i \cdot w_i^{-1} \cdot v_i = w_i \cdot h_i \cdot (\mathring{h}_i^{-1} \cdot w_i^{-1} \cdot v_i),$$

where the elements in parentheses approach e. And t satisfies the S-condition for H.

Case (ii). Now suppose that s, the orthogonal part of g, is not the identity, e. Then $g^{n_i} = t^{n_i} \cdot s^{n_i}$, and we will force $s^{n_i} \to e$, thus obtaining the S-condition for g. By changing a_i slightly in (2) above, we will show that n_i can be replaced by a nearby integer $n_i + m_i = n_i(1 + m_i/n_i)$ where $m_i/n_i \to 0$, so that t still satisfies the S-condition, and $s^{n_i+m_i} \to e$.

Suppose s has k angular parts θ_j. The angular parts of s^{-n_i} (n_i is as obtained from case (i) ignoring s) are the numbers $-n_i\theta_j$ (modulo 1), $j = 1, \ldots, k$. The numbers θ_j and 1 are linearly independent over the field of rational numbers if and only if the numbers $-n_i\theta_j$ and 1 are (for fixed i). So applying Kronecker's theorem, as in (2.5) or (2.6), we see that, given $\epsilon_i = 1/i$, there is some positive integer M_i (depending only on M_i and the θ_j, and not on n_j) and integers m_i, $p_{1i}, p_{2i}, \ldots, p_{ki}$, such that

$$|m_i\theta_j - p_{ji} + n_i\theta_j| < \epsilon_i, j = 1, \ldots, k,$$

with $0 \leqq m_i \leqq M_i$. In other words, the angular parts of $s^{n_i+m_i}$ are within $1/i$ of 0 or 1. Therefore, $s^{n_i+m_i} \to e$.

In order to have $m_i/n_i \to 0$, first note that by expression (2) above, $n_i \doteq -y_i(r_i - 1)$. Then, when deriving the S-condition for the translation part of g, namely t, using case (i), replace h_i by a sufficiently high power of itself in order to have $-y_i(r_i - 1) > i \cdot M_i$, in addition to having $\mathring{h}_i \to e$. With this, let n_i be chosen as in case (i) so that $n_i \geqq -y_i(r_i - 1)$. Then in (2), modify a_i (and consequently w_i of Lemma 5.6) so that n_i is replaced by an appropriate $n_i + m_i$, $0 \leqq m_i \leqq M_i$, with $s^{n_i+m_i} \to e$. Note that

$$m_i/n_i \leqq M_i/i \cdot M_i \to 0,$$

so we still have $a_i \to 1$.

Finally, we have

$$g^{n_i} = t^{n_i} \cdot s^{n_i} = w_i \cdot h_i \cdot z_i \cdot s^{n_i},$$

with w_i, z_i, and $s^{n_i} \to e$, and this completes the proof of (5.4).

LEMMA 5.5. *In the proof of Proposition* (5.4) *above*, h_i, x_i, y_i, *and* u_i *can be chosen so that* $u_i \to e$, $u_i \cdot h_i \cdot u_i^{-1}$ *has fixed points* x_i (*attracting*) *and* y_i *both on the* x_n-*axis with* $x_i \to +\infty$, *and* $y_i \to -\infty$, *and* $y_i/x_i \to 0$.

Proof. Let I_i and J_i be closed balls in $\mathbf{R}^n$ of radii $1/i$ (euclidean distance), and centered at the numbers i^2 and $-i$, respectively, along the x_n-axis. By

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Proposition (5.2) above, there are loxodromic elements h_i in H with attracting fixed point x_i' in I_i and repelling fixed point y_i' in J_i.

Let a_i be a planar rotation (elliptic element) in G_n leaving x_i' and ∞ fixed and taking y_i' to $a_i(y_i')$ in $J_i \cap$ (x_n-axis). If necessary, we may assume that y_i' is sufficiently close to the center of J_i so that such a rotation from x_i' does result in $a_i(y_i')$ being in J_i. If z is any point in $\overline{H^{n+1}}$ not in the extended plane defined by x_i', y_i', and $a_i(y_i')$ (which can be assumed non-collinear), then a_i moves z by its orthogonal projection in this plane.

Transforming x_i' to the origin 0 in $\overline{\mathbf{R}^n}$, a_i becomes a rotation, f_i, about the origin of angular measure θ_i, and so a_i has the form

$$\begin{aligned} a_i(z) &= f_i(z - x_i') + x_i' \\ &= f_i(z) + x_i' - f_i(x_i'). \end{aligned}$$

Along a circle centered at the origin with radius $||x_i'||$, f_i moves x_i' an arc length of $|\theta_i| \cdot ||x_i'|| \doteq ||x_i' - f_i(x_i')||$. Now, by the construction of a_i, $|\theta_i|$ is approximately $(1/i)/(i + i^2)$, or less. Hence,

$$||x_i' - f_i(x_i')|| \doteq |\theta_i| \cdot ||x_i'|| \doteq 1/i,$$

or less, and so $x_i' - f_i(x_i') \to 0$, the origin. And since $\theta_i \to 0$, $f_i(z) \to z$, and so $a_i \to e$ in G_n.

Similarly, let b_i be a rotation leaving $a_i(y_i')$ and ∞ fixed and taking x_i' to $b_i(x_i')$ in $I_i \cap$ (x_n-axis). As with a_i, it is clear that $b_i \to e$. Letting $u_i = b_i \cdot a_i$, we see that $u_i \cdot h_i \cdot u_i^{-1}$ has fixed points $x_i = b_i(x_i')$ (attracting) and $y_i = a_i(y_i')$ both on the x_n-axis with $x_i \doteq i^2$, and $y_i \doteq -i$, and the Lemma follows.

In the proof of Proposition (5.4) the following Lemma is used with $z_i = -y_i r_i a_i$.

LEMMA 5.6. *Given points x_i, y_i, and z_i in* $\mathbf{R}$, *all approaching* $\pm\infty$ *with y_i/x_i and $y_i/z_i \to 0$, there are elements w_i in $PSL(2, \mathbf{R})$ (acting as linear fractional transformations) fixing 0 and y_i and moving x_i to z_i such that $w_i \to e$, the identity.*

Proof. Since $w_i(0) = 0$, w_i has the form

$$w = \begin{pmatrix} a_i & 0 \\ c_i & a_i^{-1} \end{pmatrix}.$$

Since

$$\begin{aligned} w_i(y_i) &= a_i^2 y_i/(a_i c_i y_i + 1) = y_i, \quad \text{and} \\ w_i(x_i) &= a_i^2 x_i/(a_i c_i x_i + 1) = z_i, \end{aligned}$$

solving simultaneously, one obtains

$$\begin{aligned} a_i^2 &= (1 - y_i/x_i)/(1 - y_i/z_i) \to 1, \quad \text{and} \\ c_i &= (a_i^2 - 1)/a_i y_i \to 0. \end{aligned}$$

Hence, $w_i \to e$, and the Lemma follows.

6. Reformulation for non-discrete subgroups. If H is a non-discrete subgroup of G_n, the notion of "group of the first kind" is not particularly relevant and will be replaced by the notion of "pairwise density" of loxodromic fixed points as in Proposition (5.2).

Definition 6.1. A subgroup H of G_n is said to be *pairwise of the first kind* if the set

$$P(H) = \{(x(h), y(h)) : h \text{ is a loxodromic element in } H \text{ with } x(h) \text{ and } y(h) \text{ being the attracting and repelling fixed points of } h\}$$

is dense in $\overline{\mathbf{R}^n} \times \overline{\mathbf{R}^n}$ with the usual product topology.

The main result can now be stated as

THEOREM 6.2. *If H is any subgroup of G_n, then H is an S-subgroup if and only if it is pairwise of the first kind.*

Proof. If H is pairwise of the first kind, then H must be an S-subgroup as proved in Section 5, since the proofs there make direct use of the parwise density of loxodromic fixed points and not of discreteness.

Now suppose H is an S-subgroup. Let x and y be points in $\overline{\mathbf{R}^n}$. Consider two cases according as x and y are distinct, or not:

Case (i). Suppose x and y are distinct (for convenience, assume $x, y \neq \infty$, although the proof readily extends). Let g be a hyperbolic element in G_n with fixed points x (attracting) and y. Then g satisfies the S-condition for H by which there are equations $g^{n_i} = u_i \cdot h_i \cdot v_i$, with $u_i, v_i \to e$, and we may assume the n_i's to be strictly increasing. Let I, J, and K be closed n-balls in $\mathbf{R}^n$ centered at x (euclidean distance for radii) with $K \subset$ interior (I), and $I \subset$ interior (J), and so that y is not in J

As is known, convergence in G_n implies uniform convergence in $\mathbf{R}^{n+1}$, and so for large i, $v_i^{-1}(I) \subset J$, and $u_i^{-1}(K) \subset I$. Also, $g^{n_i}(J) \subset K$, since x is the attracting fixed point of g. That is, for large i,

$$h_i(I) = u_i^{-1} \cdot g^{n_i} \cdot v_i^{-1}(I) \subset I,$$

and so by the Brouwer fixed point theorem, h_i has a fixed point in I. We can do the same thing at y, which is the attracting fixed point of g^{-1}, using the same S-condition equations. Taking a sequence of such I, J, and K with radii decreasing to zero, we see that, at least for a subsequence of the h_i, each h_i has fixed points in $\mathbf{R}^n$, $x_i \to x$, and $y_i \to y$.

Moreover, the h_i must be loxodromic for all but finitely many i. Otherwise, there would be infinitely many elliptic h_i, each with fixed points x_i and y_i and pointwise fixed geodesic axis joining x_i and y_i. But then, taking a point z_i on each such axis exterior to some fixed n-ball, J, centered at x and not containing y, we see that for large i the points $v_i^{-1}(z_i)$ are exterior to some n-ball,

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I, centered at x and contained in interior (J), as are the points

$$u_i(z_i) = u_i \cdot h_t \cdot v_i \cdot v_i^{-1}(z_i) = g^{n_i} \cdot v_i^{-1}(z_i).$$

This contradicts the fact that g^{n_i} moves the points $v_i^{-1}(z_i)$ arbitrarily close to x (provided the z_i were chosen so as not to accumulate at y).

Finally, replacing h_i by h_i^{-1} if necessary, we have that x_i is the attracting fixed point of h_i, and so H is pairwise of the first kind.

Case (ii). If $x = y$, take any sequence of points $x_j \neq x$ in $\mathbf{R}^n$ converging to x. For each pair (x, x_j), apply case (i) above to obtain a sequence of loxodromic elements h_{ij} with fixed points $x_{ij} \to x_j$ and $y_{ij} \to x$, as $i \to \infty$. Then, for some diagonal sequence, $h_{ii'}$, we have $x_{ii'}, y_{ii'} \to x$, as required. This completes the proof of (6.2).

References

1. A. Borel, *Density properties for certain subgroups of semisimple groups without compact components*, Ann. of Math. (2) *72* (1960), 179–188.

2. J. W. S. Cassels, *An introduction to diophantine approximation* (Cambridge Univ. Press, Cambridge, 1965).

3. S. S. Chen and L. Greenberg, *Hyperbolic spaces*, in *Contributions to analysis* (Academic Press, New York, 1974).

4. L. Greenberg, *Discrete subgroups of the Lorentz group*, Math. Scand. *10* (1962), 85–107.

5. ——— *Commensurable groups of Moebius transformations*, in *Discontinuous groups and Riemann surfaces*, Annals of Math. Studies, *79* (Princeton Univ. Press, Princeton, 1974), 227–237.

6. G. H. Hardy and E. M. Wright, *An Introduction to the theory of numbers* (Oxford Univ. Press, London, 1960).

7. T. J. O'Malley, *The geometry of S-subgroups of the hyperbolic groups*, Ph.D. dissertation, Cornell University, 1974.

8. A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, in *Contributions to function theory*, Tata Institute, Bombay, 1960, 147–164.

9. P. Tukia, *On discrete groups of the unit disk and their isomorphisms*, Ann. Acad. Sci. Fenn., Series A.I., No. *504*, 1972.

10. S. P. Wang, *On S-subgroups of solvable Lie groups*, Amer. J. Math. *92* (1970), 389–397.

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