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TOTALLY PURE SEQUENCES AND TOTALLY PURE PROJECTION

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Introduction

All groups in this paper are Abelian. The notation is similar to Fuchs (1970) and Maclane (1963).

A reduced group G is totally projective if $G/p^{\alpha}G$ is p^{α}-projective for all ordinals α. Nunke (1963) introduced this class of groups and derived a number of their properties. This class is of great interest since Hill (1967) has shown that Ulm's theorem holds for it. Nunke introduced the class via his study of p^{α}-purity, and his derivation of the various properties of totally projective groups depends heavily on the rather extensive theory described in Nunke (1963) and Nunke (1967). Crowley and Hales (1969) then introduced the concept of T-groups and proved Ulm's theorem for this class. Using Hill's theorem they showed that the class of T-groups coincides with the class of totally projective groups.

However, neither paper explicitly describes the elements of these groups. We shall investigate the relative homological algebra for which the totally projectives are projective. This will enable us to explicitly describe this class.

The α-Totally Pure Sequences

DEFINITION 1.1. Let $E: 0 \to A \to B \to C \to 0$ be exact. Then E is *α-totally pure* if and only if the induced sequences $p^{\beta}B[p] \to p^{\beta}C[p] \to 0$ are exact for each ordinal $\beta < \alpha$. If E is α-totally pure for each ordinal α, then E is totally pure.

Recall that the length of a p-group G is the least ordinal α such that $p^{\alpha}G = 0$ and is denoted by $\Lambda(G)$.

THEOREM 1.1. *Let $E: 0 \to A \to B \to C \to 0$ be exact. If α is a limit ordinal or $\Lambda(C) \leqq \alpha$ the following are equivalent.*

(i) *E is α-totally pure*

(ii) $0 \to p^\beta A[p] \to p^\beta B[p] \to p^\beta C[p] \to 0$ is *exact for* $\beta < \alpha$
(iii) $0 \to p^\beta A \to p^\beta B \to p^\beta C \to 0$ *is exact for* $\beta < \alpha$
(iv) $0 \to A/p^\beta A \to B/p^\beta B \to C/p^\beta C \to 0$ *is exact for* $\beta < \alpha$
(v) $0 \to p^\lambda A[p]/p^\beta A[p] \to p^\lambda B[p]/p^\beta B[p] \to p^\lambda C[p]/p^\beta C[p] \to 0$ *is exact for* $\lambda < \beta < \alpha$.

Proof. For any ordinal α, (i) is equivalent to (ii). If α is a limit ordinal, then (ii) implies (iii) easily. By using the 3×3 Lemma, (iii), (iv), (v) and (i) are equivalent.

If $\alpha = \lambda + n$ and $\Lambda(C) = \alpha$ with λ a limit ordinal and $n < \omega$ then $p^\lambda C$ is bounded. Thus the sequence $0 \to p^\lambda A \to p^\lambda B \to p^\lambda C \to 0$ is split and the five conditions are equivalent for each ordinal $\beta < \alpha$.

Corollary 1.1. *Let* $F_G(\alpha)$ *denote the* α*-th Ulm invariant of* G. *Let* $E: 0 \to A \to B \to C \to 0$ *be totally pure. Then* $F_B(\alpha) = F_A(\alpha) + F_C(\alpha)$ *for each ordinal* α.

Proof. By Condition (v) of Theorem 1.1, if E is totally pure then

$$0 \to p^\alpha A[p]/p^{\alpha+1}A[p] \to p^\alpha B[p]/p^{\alpha+1}B[p] \to p^\alpha C[p]/p^{\alpha+1}C[p] \to 0$$

is exact. Over the p element field this sequence splits. Hence $F_B(\alpha) = F_A(\alpha) + F_C(\alpha)$.

All split exact sequences are totally pure. There are totally pure sequences that are not split exact. Let G be a group with $p^\omega G = 0$ and G is not a direct sum cyclics. Then there is a totally pure sequence $E: 0 \to A \to B \to C \to 0$ with B a direct sum of cyclic groups. Hence E does not split.

For any group G, let G^* be the completion of $G/p\omega G$ in the p-adiac topology.

Then $E: 0 \to A \to B \to C \to 0$ is $(\omega + 1)$-totally pure if and only if $0 \to A^* \to B^* \to C^* \to 0$ splits (Cook (to appear)). Thus if α is not a limit ordinal, Theorem 1.1 need not hold. If the conditions of Theorem 1.1 are fulfilled then A is an α-isotype subgroup of B, Irwin, Walker and Walker (1963). The converse need not hold. If $p^\omega G = 0$, then a basic subgroup is an $(\omega + 1)$-isotype subgroup, but not a $(\omega + 1)$-totally pure subgroup.

Definition 1.2. For each ordinal α, let $P\mathrm{ext}_\alpha(C, A)$ be the subgroup of $\mathrm{Ext}(C, A)$ consisting of equivalence classes of α-totally pure sequences.

Theorem 1.2. *If* α *is a limit ordinal, then* $A \subseteq p^\alpha B$ *if and only if the sequences*

$$E_D: 0 \to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, B/A) \to$$
$$\mathrm{Ext}(D, A) \to P\mathrm{ext}_\alpha(D, B) \to P\mathrm{ext}_\alpha(D, B/A)$$

are exact for all divisible groups D.

Proof. Suppose E_D is exact for all divisible groups D. Let $A \subset D$ with D divisible. Then under the map $\mathrm{Ext}(D/A, A) \to P\mathrm{ext}_\alpha(D/A, B)$ the element rep-

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resented by $0 \to A \to D \to D/A \to 0$ maps onto the element represented by $0 \to B \xrightarrow{f} D \oplus B/M \xrightarrow{g} D/A \to 0$ where $M = \{(a, -a) \mid a \in A\}$, $f(x) = (x,0) + M$ and $g((x,d)+M) = d + A$ for $x \in B$ and $d \in D$. Let $a \in A \subset D$. If $\beta < \alpha$, then $a \in p^\beta D$ implies

$$f(a) = (a,0) + M = (0,a) + M \in p^\beta(B \oplus D/M).$$

Since $g((0,a)+M) = 0$ and $0 \to p^\beta B \to p^\beta(B \oplus D/M) \to D/A \to 0$ is exact, there is $b \in p^\beta B$ such that $f(b) = (0,a) + M = f(a)$. Since f is a monomorphism $a = b$ and $a \in p^\beta B$ for $\beta < \alpha$. Hence, $a \in p^\beta B$ for each $\beta < \alpha$. Thus $A \subset p^\alpha B$.

If $A \subset p^\beta B$, then E_D is exact (Nunke (1963)).

Corollary 1.2. *Let α be an ordinal. If C is α-totally pure projective, then* $\mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, A/p^\alpha A)$ *for each group A.*

Proof. For $p^\alpha A$, the sequence $\mathrm{Hom}(C, A/p^\alpha A) \to \mathrm{Ext}(C, p^\alpha A) \to P\mathrm{ext}_\alpha(C, A)$, $= 0$ is exact. Since the sequence $\mathrm{Ext}(C, p^\alpha A) \to \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A/p^\alpha A) \to 0$ is exact, then $\mathrm{Ext}(C, A) \cong \mathrm{Ext}(C, A/p^\alpha A)$.

The α-Totally Pure Projectives

To describe the p-groups that are α-totally pure projectives, Nunke (1967) introduces a class of groups, one for each ordinal α. These groups are defined homologically and their basic properties are derived by the same means. Crowley and Hales (1969), define the concept of T-groups and using Hill's theorem show that these two classes coincide. In our development, we introduce similar classes of groups, but its definitions and basic properties will be in terms of the most elementary group theoretic principles. Via this class of groups, we will describe the α-totally pure projectives, derive some of their basic properties, and show that there are enough of them.

Definition 2.1. (Crowley and Hales) Let X be a set and $V \subset X^2$, let u and v be maps of X and V into the nonnegative integers respectively. Let $G(X, V, u, v)$ be the Abelian group generated by X subject only to the relations

$$p^{u(x)} = 0 \text{ for all } x \in X$$

$$p^{v(x,y)} = y \text{ for all } (x, y) \in V.$$

Then an Abelian p-group G is a T-group if $G \simeq G(X, V, u, v)$ for some quadruple (X, V, u, v).

Definition 2.2. Let α be an ordinal, $X_\alpha = \{\alpha_1\alpha_2 \cdots \alpha_n \mid \alpha_i \text{ an ordinal}, \alpha > \alpha_1 > \alpha_2 \cdots \alpha_n\}$; F_α the free group on X_α; K_α the subgroup of F_α generated by the elements $\{p\alpha_1\alpha_2 \cdots \alpha_{n+1} - \alpha_1\alpha_2 \cdots \alpha_n, p\beta\}$; and $P_\alpha = F_\alpha/K_\alpha$.

The element $\alpha_1\alpha_2\cdots\alpha_n + K_\alpha$ in P_α will be simply denoted by $\alpha_1\alpha_2\cdots\alpha_n$. It is clear that P_α is a T-group. Moreover $p^n\alpha_1\alpha_2\cdots\alpha_n = 0$.

To motivate this definition suppose that $E: 0 \to A \to B \to C \to 0$ is totally pure. If $c \in p^\alpha C$ then $= pc_{\alpha_1}$ with $c_{\alpha_1} \in p^{\alpha_1}C$ and $\alpha > \alpha_1$. Now $c_{\alpha_1} = pc_{\alpha_2}$ with $c_{\alpha_2} \in p^{\alpha_2}C$ and $\alpha > \alpha_1 > \alpha_2$. Since every decreasing sequence of ordinals is finite we obtain a set $c, c_{\alpha_1}, c_{\alpha_2}\cdots c_{\alpha_n}$ subject to certain relations. Since E is totally pure we can lift each c_{α_i} to $b_{\alpha_i} \in p^{\alpha_i}B$ and the set $b_\alpha, b_{\alpha_i}\cdots b_{\alpha_n}$ is also subject to these relations. What we have done is to construct a class with only these relations. This class should be, and indeed is, the class of totally pure projectives. We first define some important subgroups of the P_α's.

Definition 2.3. Let $H_\alpha = < \{\alpha\alpha_1\cdots\alpha_n \in P_{\alpha+1} \mid \alpha > \alpha_1\cdots\alpha_n\} >$

The next three propositions may be proven by transfinite induction, or they follow directly from statements (3.3), (3.4) and (3.5) in Crowley and Hales (1969).

Proposition 2.1. *Let α be an ordinal. Then $P_\alpha = \sum_{\beta<\alpha} H_\beta$.*

Proposition 2.2. *Each element $y \in P_\alpha$ can be written uniquely in the form $y = \sum_{x \in X_\alpha} s_x s$, $0 \leqq s_x < p$.*

Proposition 2.3. *Let γ be an ordinal and $X_\alpha^\gamma = \{\alpha_1\alpha_2\cdots\alpha_n \mid \alpha_n \geqq \gamma\}$. Then $p^\gamma P_\alpha$ is the subgroup generated by X_α^γ.*

In view of Proposition 2.3, it can be noted that the height of the ordinal β in P_α is precisely β.

The next theorem starts to connect the H_α's with the α-totally pure sequences.

Theorem 2.1. *For $c \in C$, $c \in p^\alpha C[p]$ if and only if there is $f \in \operatorname{Hom}(H_\alpha, C)$ such that $f(\alpha) = c$.*

Proof. If $f \in \operatorname{Hom}(H_\alpha, C)$, then $f(\alpha) \in p^\alpha C[p]$. Suppose $c \in p^\alpha C[p]$. We will define a set map

$$f : H_\alpha \cap X_\alpha \to C \text{ such that}$$

(i) $f(\alpha) = c$

(ii) $f(\alpha\alpha_1\cdots\alpha_n) \in p^{\alpha_n}C[p^{n+1}]$

(iii) $pf(\alpha\alpha_1\cdots\alpha_n) = f(p\alpha\alpha_1\cdots\alpha_n)$.

Let $f(\alpha) = c \in p^\alpha C[p]$. Suppose for all $k \leqq n-1$, $f(\alpha\alpha_1\cdots\alpha_k)$ is defined su[illegible]h that $f(\alpha\alpha_1\cdots\alpha_k) \in p^{\alpha_k}C[p^{k+1}]$. Given $\alpha\alpha_1\cdots\alpha_n$, then $f(\alpha\alpha_1\cdots\alpha_{n+1}) \in p^{\alpha_n}C[p^n]$. Since $\alpha_{n-1} > \alpha_n$, then there is a $c_{\alpha_n} \in p^{\alpha_n}[p^{n+1}]$ such that $pc_{\alpha_n} = f(\alpha\alpha_1\cdots$ [illegible]$_{-1})$. Let $f(\alpha\alpha_1\cdots\alpha_n) = c_{\alpha n}$. Then $pf(\alpha\alpha_1\cdots\alpha_n) = pc_{\alpha_n} = f(p\alpha\alpha_1\cdots\alpha_n)$. Thus b[illegible] [illegible]nite induction, f is defined on all the generating elements of H_α. Since the rel[illegible]on[illegible]

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among these elements are preserved and f is a height increasing, order decreasing set map, then f extends to a group homomorphism from H_α to C.

The next theorem characterizes the α-totally pure sequences in terms of the H_β's.

THEOREM 2.2. *The sequence* $E: 0 \to A \to B \xrightarrow{\sigma} C \to 0$ *is α-totally pure if and only if H_β is projective for all $\beta < \alpha$.*

PROOF. Suppose H_β is projective for all $\beta < \alpha$ and $c \in p^\beta C[p]$. Then by Theorem 2.1 there is a homomorphism $f: H_\beta \to C$ such that $f(\beta) = c$. Since H_β is projective there is a homomorphism $g: H_\beta \to B$ with $\sigma g = f$. Thus $\sigma_\beta g(\beta) = c$ and $p^\beta B[p] \to 0$ is exact.

Suppose E is α-totally projective. If P_β is projective for $\beta \leqq \alpha$, then H_β is projective for $\beta < \alpha$. Thus suppose for all $\gamma < \beta$, P_γ is projective. If γ is a limit ordinal, then $P_\gamma = \sum_{\beta < \gamma} H_\beta$ and P_γ is projective. If $\gamma = \beta + 1$, then $P_\gamma = H_\beta \oplus P_\beta$. Hence it suffices to show H_β is projective. Suppose $f: H_\beta \to C$. Then $f(\beta) \in p^\beta C[p]$ and there is $b \in p^\beta B[p]$ with $\sigma(b) = f(\beta)$. Thus $b = \rho(\beta)$ for some $\rho: H_\beta \to B$. Since $(f - \sigma\rho)(\beta) = 0$ and $H_\beta / < \beta > = P_\beta$, then $f - \sigma\rho = \lambda\pi$ where $\pi: H_\beta \to P_\beta$. Since P_β is projective $\lambda = \sigma\delta$ so that $f = \sigma(\rho + \delta\pi)$. Thus P_γ is projective. Hence for $\beta < \alpha$, H_β is projective.

The reader is refered. to Maclane (1963) for a proof of the following corollary.

COROLLARY 2.1. *The α-totally pure sequences are a proper class.*

COROLLARY 2.2. *The sequence* $0 \to A \to B \to C \to 0$ *is α-totally pure if and only if the sequence of p-primary subgroups,* $0 \to A_p \to B_p \to C_p \to 0$, *is α-totally pure.*

PROOF. This follows readily since $\mathrm{Hom}(H_\beta, G_p) = \mathrm{Hom}(H_\beta, G)$ for each H_β.

THEOREM 2.3. *Let α be an ordinal and G a group. Then there is a sequence* $E: 0 \to K \to P \to G \to 0$ *with P α-totally projective and E α-totally pure.*

PROOF. Let $P = F \oplus T$ where F is free and $F \to G \to 0$ is exact, and $T = \sum_{\beta < \alpha} \sum_{f \in Hom(H_\beta, G)} H_\beta$.

If G is a reduced p-group and $\Lambda(G) = \beta$ and $\alpha \geqq \beta$, then any α-totally pure sequence, $E: 0 \to K \to P \to G \to 0$ is totally pure. If P is α-totally pure projective, ʻh . P is totally pure projective. Hence if G is a reduced p-group, then there is a ʻoʻally pure projective P and epimorphism $f: P \to G$ with $\mathrm{Ker} f$ totally pure i⁻ P. The next theorem shows that if G is a divisible p-group, then G is totally . re p.ojective.

-OREM ˊ 4. *The group $Z(p^\infty)$ is totally pure projective.*

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PROOF. Suppose $E: 0 \to A \to B \to C \to 0$ is totally pure. Then there is an ordinal α so that $p^\alpha A$, $p^\alpha B$, and $p^\alpha C$ are divisible. Since the sequence $0 \to p^\alpha A \to p^\alpha B \twoheadrightarrow p^\alpha C \to 0$ splits, $Z(p^\infty)$ is totally pure projective.

Combining Theorems 2.3 and 2.4 we hàve the following theorem.

THEOREM 2.5. *Let G be a group. Then there is a totally pure sequence $E: 0 \to K \to P \to G \to 0$ with P totally pure projective.*

THEOREM 2.6. *Let G be a reduced p-group. If G is α-totally pure projective, then G is a direct summand of a direct sum of H_β's for $\beta < \alpha$. Hence $\Lambda(G) \leqq \alpha$.*

PROOF. Let $E: 0 \to K \to P \to G \to 0$ be α-totally pure with P as in Theorem 2.3. Then E splits and G is a summand of P. Since G is a p-group and summands of free groups are free, then G is a summand of H_β for $\beta < \alpha$.

Recall that the set $X_\beta^\gamma = \{\beta\alpha_1 \cdots \alpha_n \in H_\beta | \alpha_n \geqq \gamma\}$. Then $H_\beta / p^\gamma H_\beta = \sum_{\Gamma \in X_\beta^\gamma} (P_\gamma)_\Gamma$. If G is a α-totally pure projective, then there is a group H such that $G \oplus H = \sum H_\beta$ for β's less than α. Then $p^\gamma G \oplus p^\gamma H = \sum p^\gamma H_\beta$ and $p^\gamma G$ is α-totally projective. Since $G \approx H/p^\beta G \oplus p^\beta H \approx G/p^\beta G \oplus H/p^\beta H \approx \sum H_\gamma / p^\beta H_\gamma \approx \sum(\sum P_\beta)$, $G/p_\gamma G$ is also α-totally projective. Using the results of this section, one can show directly that if $p^\gamma G$ and $G/p^\gamma G$ are α-totally projective, then so is G.

Nunke (1967) defines totally projective groups. These are the reduced totally pure projective groups.

The following theorem and proof is due to Professor Fred Richman.

THEOREM 2.7. *A p-group G is totally injective if $G = T \oplus D$ with T torsion complete and D divisible.*

PROOF. Choose B, a direct sum of cyclics, such that $|G|^{|B|} < 2^{|B|}$. Then the sequence $\mathrm{Hom}(B, G) \to P\mathrm{ext}_\omega(\bar{B}/B, G) \to P\mathrm{ext}_\omega(\bar{B}, G)$ is exact. Any pure extension by a group without elements of infinite height is totally pure, so $P\mathrm{ext}_\omega(\bar{B}, G) = 0$. $|\mathrm{Hom}(B,G)| \leqq |G|^{|B|} < 2^{|B|}$. So $P\mathrm{ext}_\omega(\bar{B}/B, G) = \Pi\, P\mathrm{ext}_\omega(Z_{p\infty}, G)$ is smaller than $2^{|B|}$. Thus $P\ \mathrm{ext}_\omega(z_{p\infty}, G) = 0$ and so G is torsion ${}^{|B|}$ complete $\oplus$ divisible.

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