

# 3

## The Lagrangian formulation of mechanics

In most introductory texts on quantum mechanics you will find ‘Hamiltonian’ in the index (see our equation (3.8)) but you are less likely to find ‘Lagrangian’. However, quantum field theories are most conveniently described in a Lagrangian formalism, to which this chapter is an introduction.

### 3.1 Hamilton’s principle

The classical dynamics of a mechanical (non-dissipative) system is most elegantly derived from *Hamilton’s principle*. A closed mechanical system is completely characterised by its Lagrangian  $L(q, \dot{q})$ ; the variables  $q(t)$ , which are functions of time, are a set of coordinates  $q_1(t), q_2(t), \dots, q_s(t)$  which determine the configuration of the system at time  $t$ . In particular, the  $q_i$  might be the Cartesian coordinates of a set of interacting particles. We restrict our discussion to the case where all the  $q_i(t)$  are independent. In non-relativistic mechanics we take  $L = T - V$ , where  $T(q, \dot{q})$  is the kinetic energy of the system and  $V(q)$  is its potential energy.

Given  $L$ , the action  $S$  is defined by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt. \tag{3.1}$$

The value of  $S$  depends on the path of integration in  $q$ -space. The end-points of the path are fixed at times  $t_1$  and  $t_2$ , but the path is otherwise unrestricted.  $S$  is said to be a *functional* of  $q(t)$ . Hamilton’s principle states that  $S$  is stationary for that particular path in  $q$ -space determined by the equations of motion, so that if we consider a variation to an arbitrary neighbouring path (Fig. 3.1),  $\delta S = 0$ , where

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q}) dt \\ &= \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt. \end{aligned}$$

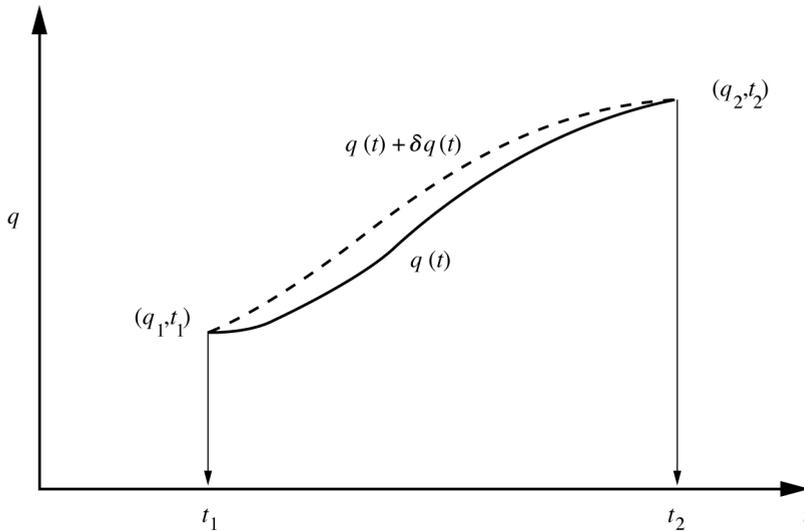


Figure 3.1 A schematic representation of the path in  $q$ -space determined by the equations of motion (full line) and a neighbouring path (dashed line).

Since  $\delta\dot{q} = d(\delta q)/dt$ , we can integrate the second term in this integral by parts, to give

$$\delta S = \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt. \quad (3.2)$$

The ‘end-point’ contributions from the integration by parts are zero, since  $\delta q(t_1) = \delta q(t_2) = 0$ .

The variations  $\delta q_i(t)$  are arbitrary. It follows from (3.2) that the condition  $\delta S = 0$  requires

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, s. \quad (3.3)$$

These are the *Euler–Lagrange equations of motion*. In classical non-relativistic mechanics they are equivalent to Newton’s equations of motion. As a simple example, consider a particle of mass  $m$  moving in one dimension in a potential  $V(x)$ . Then  $L = T - V = (m\dot{x}^2/2) - V(x)$ . From (3.3) we have immediately  $m\ddot{x} = -\partial V/\partial x$ , which is Newton’s equation of motion for the particle.

An external, and possibly time-dependent, field can be included in the Lagrangian formalism through a time-dependent potential. In our one-dimensional example above,  $V(x)$  may be replaced by  $V(x, t)$ . Making the Lagrangian  $L$  depend explicitly on  $t$  does not affect the derivation of the field equations.

It is important to note that the Lagrangian of a given system is *not unique*: we can add to  $L$  any function of the form  $df(q,t)/dt$  where  $f(q,t)$  is an arbitrary function of  $q$  and  $t$ . Such a term gives a contribution  $[f(q_2, t_2) - f(q_1, t_1)]$  to  $S$ , independent of the path, and hence leaves the equations of motion unchanged.

### 3.2 Conservation of energy

In the case of a closed system of particles, interacting only among themselves, the equations of motion of the system do not depend explicitly on the time  $t$ , since the physics of a closed system does not depend on our choice of the origin of time. There is no reason to doubt that the laws of physics at the time of Archimedes, or the time of Newton, were the same as they are for us. Hence for a closed system we must be able to construct a Lagrangian  $L(q, \dot{q})$  that does not depend explicitly on  $t$ . For such a Lagrangian,

$$\frac{dL}{dt} = \sum_i \left[ \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right].$$

Taking the  $q_i(t)$  to obey the equations of motion and substituting for  $\partial L/\partial \dot{q}_i$  from (3.3) we obtain

$$\frac{dL}{dt} = \sum_i \left[ \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right] = \sum_i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

or

$$\frac{d}{dt} \left[ \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right] = 0. \quad (3.4)$$

Thus

$$E = \left[ \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right] \quad (3.5)$$

remains constant during the motion, and is called the *energy* of the system. This result exemplifies Noether's theorem (Section 1.2): we have here a conservation law stemming from the symmetry of the Lagrangian under a translation in time.

For a closed system of non-relativistic particles, with a potential function  $V(q_i)$ ,  $\partial L/\partial \dot{q}_i = \partial T/\partial \dot{q}_i$ . Since the kinetic energy  $T$  is a quadratic function of the  $\dot{q}_i$  (Problem 3.1),  $(\partial T/\partial \dot{q}_i) \dot{q}_i = 2T$ . Hence

$$E = 2T - (T - V) = T + V.$$

We recover the result of elementary mechanics.

The *generalised momenta*,  $p_i$ , are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (3.6)$$

The *Hamiltonian* of a system is defined by

$$H(p, q) = \sum_i p_i \dot{q}_i - L. \quad (3.7)$$

In terms of  $p$  and  $q$ , the energy equation (3.5) for a closed system becomes

$$H(p, q) = E. \quad (3.8)$$

This equation, which is a consequence of the homogeneity of time, is a foundation stone for making the transition from classical to quantum mechanics.

### 3.3 Continuous systems

To see how Hamilton's principle may be extended to continuous systems, we consider a flexible string, of mass  $\rho$  per unit length, stretched under tension  $F$  between two fixed points at  $x = 0$  and  $x = l$ , say, but subject to small transverse displacements in a plane. Gravity is neglected. If  $\phi(x, t)$  is the transverse displacement from equilibrium of an element  $dx$  of the string at  $x$ , at time  $t$ , then the length of the string is

$$\int_0^l (dx^2 + d\phi^2)^{1/2} = \int_0^l [1 + (\partial\phi/\partial x)^2]^{1/2} dx.$$

To leading order in  $\partial\phi/\partial x$ , which we take to be small for small displacements, the extension of the string is  $\int_0^l \frac{1}{2}(\partial\phi/\partial x)^2 dx$ , and the potential energy of stretching under the tension  $F$  is  $\int_0^l \frac{1}{2}F(\partial\phi/\partial x)^2 dx$ . The kinetic energy of the string is  $\int_0^l \frac{1}{2}\rho(\partial\phi/\partial t)^2 dx$ . Hence

$$L = T - V = \int_0^l \mathcal{L} dx, \quad (3.9)$$

where

$$\mathcal{L} = \frac{1}{2}\rho \left( \frac{\partial\phi}{\partial t} \right)^2 - \frac{1}{2}F \left( \frac{\partial\phi}{\partial x} \right)^2 \quad (3.10)$$

is called the *Lagrangian density*.

The corresponding action is

$$S = \int_0^1 dx \int_{t_1}^{t_2} dt \mathcal{L}(\dot{\phi}, \phi'),$$

writing  $\partial\phi/\partial t = \dot{\phi}$  and  $\partial\phi/\partial x = \phi'$ .

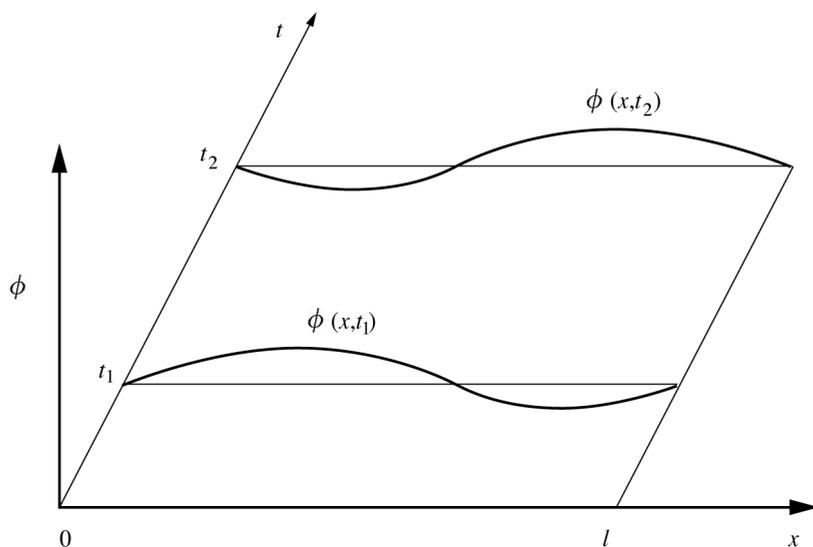


Figure 3.2 The actual motion of the string between an initial displacement  $\phi(x, t_1)$  and a final displacement  $\phi(x, t_2)$  generates a surface in space-time.

Hamilton's principle states that the action is stationary for that surface that describes the actual motion of the string between its initial displacement  $\phi(x, t_1)$  and its final displacement  $\phi(x, t_2)$  (Fig. 3.2). We have

$$\delta S = \int_0^l dx \int_{t_1}^{t_2} dt \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta(\phi) + \frac{\partial \mathcal{L}}{\partial \phi'} \delta(\phi') \right].$$

Using  $\delta(\dot{\phi}) = \partial(\delta\phi)/\partial t$  and  $\delta(\phi') = \partial(\delta\phi)/\partial x$  we integrate each term by parts. Again, the boundary contributions are zero since

$$\begin{aligned} \delta\phi(x, t_1) = \delta\phi(x, t_2) &= 0 \quad \text{for all } x, \\ \delta\phi(0, t) = \delta\phi(l, t) &= 0 \quad \text{for all } t. \end{aligned}$$

We are left with

$$\delta S = - \int_0^l dx \int_{t_1}^{t_2} dt \left[ \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \phi'} \right) \right] \delta\phi. \quad (3.11)$$

Since  $\delta\phi(x, t)$  is arbitrary, the condition  $\delta S = 0$  gives

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \phi'} \right) = 0. \quad (3.12)$$

Inserting the Lagrangian density (3.10), we obtain the familiar wave equation for small amplitude waves on a string:

$$\rho \frac{\partial^2 \phi}{\partial t^2} - F \frac{\partial^2 \phi}{\partial x^2} = 0.$$

Thus continuous systems can be described in a Lagrangian formalism by a suitable choice of Lagrangian density, and clearly the method can be extended to waves in any number of dimensions. By analogy with (3.6) and (3.7), we can define the momentum density

$$\Pi(\dot{\phi}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

and the Hamiltonian density

$$\mathcal{H} = \Pi \dot{\phi} - \mathcal{L}. \quad (3.13)$$

Since the Lagrangian density (3.10) does not depend explicitly on  $t$ , it follows that

$$E = \int \mathcal{H} dx = \int \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right) dx \quad (3.14)$$

remains constant during the motion (Problem 3.2). This result is the analogue of (3.5).

### 3.4 A Lorentz covariant field theory

In three spatial dimensions, the action is of the form

$$S = \int \mathcal{L} dx dy dz dt = \int \mathcal{L} dx^0 dx^1 dx^2 dx^3. \quad (3.15)$$

The ‘volume element’  $dx^0 dx^1 dx^2 dx^3 = d^4x$  is a Lorentz invariant (Section 2.4). Hence  $S$  is a Lorentz invariant if the Lagrangian density  $\mathcal{L}$  transforms like a scalar field. The covariance of the field equations is then assured. Other symmetries required of a theory may be built into  $\mathcal{L}$ .

Consider a Lorentz invariant Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi), \quad (3.16)$$

where  $\phi(x) = \phi(x^0, \mathbf{x})$  is a scalar field. At any point  $x$  in space-time, such a Lagrangian density depends only on the field and its first derivatives at that point. The field theory is said to be *local*: there is no ‘action at a distance’. This will be an important feature of the Standard Model. The field equation is easily derived from the condition  $\delta S = 0$ , together with the condition that the field vanishes at large

distances, and we find

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0. \quad (3.17)$$

### 3.5 The Klein–Gordon equation

The Lorentz invariant Lagrangian density

$$\mathcal{L} = \frac{1}{2} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2] = \frac{1}{2} [\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2], \quad (3.18)$$

where  $\phi(x)$  is a real scalar field, is a particular case of (3.16). The field equation (3.17) becomes

$$-\partial_\mu \partial^\mu \phi - m^2 \phi = 0,$$

or

$$\left( -\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 \right) \phi = 0. \quad (3.19)$$

This equation is known as the *Klein–Gordon equation*.

The equation has wave-like solutions

$$\phi(\mathbf{r}, t) = a \cos(\mathbf{k} \cdot \mathbf{r} - \omega_{\mathbf{k}} t + \theta_{\mathbf{k}})$$

where the frequency  $\omega_{\mathbf{k}}$  is related to the wave vector  $\mathbf{k}$  by the dispersion relation

$$\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + m^2, \quad (3.20)$$

and  $\theta_{\mathbf{k}}$  is an arbitrary phase angle.

For mathematical simplicity we shall take the solutions  $\phi(\mathbf{r}, t)$  to lie in a large cube of side  $l$ , volume  $V = l^3$ , and apply periodic boundary conditions, so that  $\mathbf{k} = (2\pi n_1/l, 2\pi n_2/l, 2\pi n_3/l)$  where  $n_1, n_2, n_3$  are any integers  $0, \pm 1, \pm 2, \dots$

The general solution of (3.19) is a superposition of such plane waves:

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left( \frac{a_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_{\mathbf{k}} t)} + \frac{a_{\mathbf{k}}^*}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_{\mathbf{k}} t)} \right). \quad (3.21)$$

The factors  $\sqrt{2\omega_{\mathbf{k}}}$  are introduced for later convenience, and the phase factors have been absorbed into the complex wave amplitudes  $a_{\mathbf{k}}$ . The sum is over all allowed values of  $\mathbf{k}$ .

With the de Broglie identifications of  $E = \omega_{\mathbf{k}}$ ,  $\mathbf{p} = \mathbf{k}$  (recall  $\hbar = 1$ ,  $c = 1$ ) the dispersion relation for  $\omega_{\mathbf{k}}$  is equivalent to the Einstein equation for a free particle,

$$E^2 = \mathbf{p}^2 + m^2.$$

We may conjecture that the Klein–Gordon equation for  $\phi$  describes a scalar particle of mass  $m$ . There is no vector associated with a one-component scalar field, and the intrinsic angular momentum associated with such a particle is zero.

We shall see a Lagrangian density of the form (3.18) arising in the Standard Model to describe the Higgs particle. At a less fundamental level, the overall motion of the  $\pi^0$  meson, which is an uncharged composite particle, is described by a similar Lagrangian density.

### 3.6 The energy–momentum tensor

The equations expressing both conservation of energy and conservation of linear momentum are obtained by considering the change in  $\mathcal{L}$  corresponding to a uniform infinitesimal space-time displacement

$$x^\mu \rightarrow x^\mu + \delta a^\mu, \quad (3.22)$$

where  $\delta a^\mu$  does not depend on  $x$ . The corresponding change in  $\phi$  is

$$\delta\phi = (\partial_\nu\phi)\delta a^\nu. \quad (3.23)$$

Since  $\mathcal{L}$  does not depend explicitly on the  $x^\mu$ ,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi).$$

Using the field equation (3.17) for  $\partial\mathcal{L}/\partial\phi$ , and the fact that  $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$ , we can rewrite this as

$$\delta\mathcal{L} = \partial_\mu \left[ \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) \delta\phi \right],$$

and then, from (3.23),

$$\delta\mathcal{L} = \partial_\mu \left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu\phi \right] \delta a^\nu.$$

We have also

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu} \delta a^\mu = \delta^\mu_\nu \frac{\partial\mathcal{L}}{\partial x^\mu} \delta a^\nu,$$

where, as in (2.14),

$$\delta^\mu_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu. \end{cases}$$

Since the  $\delta a^{\nu}$  are arbitrary, it follows on comparing these expressions for  $\delta\mathcal{L}$  that

$$\partial_{\mu} \left[ \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta_{\nu}^{\mu} \mathcal{L} \right] = 0, \quad (3.24)$$

or

$$\partial_{\mu} T_{\nu}^{\mu} = 0, \quad \text{where } T_{\nu}^{\mu} = \left[ \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta_{\nu}^{\mu} \mathcal{L} \right]. \quad (3.25)$$

$T_{\nu}^{\mu}$  is the *energy–momentum tensor*. The component

$$T_0^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \dot{\phi} - \mathcal{L}$$

corresponds to the Hamiltonian density defined in equation (3.13), and is interpreted as the energy density of the field; in a relativistic theory, the energy density transforms like a component of a tensor. The  $\nu = 0$  component of (3.25) may be written

$$\frac{\partial}{\partial t} (T_0^0) + \nabla \cdot \mathbf{T}_0 = 0, \quad (3.26)$$

and expresses local conservation of energy, with  $\mathbf{T}_0 = (T_0^1, T_0^2, T_0^3)$  interpreted as the energy flux. Integrating (3.26) over all space and using the divergence theorem yields

$$\frac{\partial}{\partial t} \int T_0^0 d^3\mathbf{x} = 0, \quad (3.27)$$

provided the field vanishes at large distances. This equation expresses the overall conservation of energy.

Similarly the  $\nu = 1, 2, 3$  components of (3.24) correspond to local conservation of momentum, with the overall total momentum of the field given by

$$P_i = \int T_i^0 d^3\mathbf{x}. \quad (3.28)$$

As with the energy, the total momentum of the field is conserved if the field vanishes at large distances.

In the case of the Klein–Gordon Lagrangian density (3.19),

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi},$$

and the energy density of the field is

$$T_0^0 = \frac{1}{2} [\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]. \quad (3.29)$$

Expressing  $\phi$  in terms of the field amplitudes  $a_{\mathbf{k}}$  and  $a_{\mathbf{k}}^*$ , and integrating over all space, gives the total field energy

$$H = \int T_0^0 d^3\mathbf{x} = \sum_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} \omega_{\mathbf{k}}. \quad (3.30)$$

In obtaining this expression we have used the orthogonality of the plane waves

$$\frac{1}{V} \int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} d^3\mathbf{x} = \delta_{\mathbf{k}\mathbf{k}'}$$

Similarly from (3.28) the total momentum of the field can be shown to be

$$\mathbf{P} = \sum_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} \mathbf{k}. \quad (3.31)$$

### 3.7 Complex scalar fields

It is instructive to consider also complex scalar fields  $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$  satisfying the Klein–Gordon equation. We shall see in Section 7.6 that if the field  $\Phi$  carries charge  $q$ , then the field  $\Phi^*$  carries charge  $-q$ . The Klein–Gordon equation for a complex field  $\Phi$  is obtained from the (real) Lagrangian density

$$\mathcal{L} = \partial_{\mu}\Phi^* \partial^{\mu}\Phi - m^2\Phi^*\Phi. \quad (3.32)$$

We introduce here a device that we shall often find useful. Instead of varying the real and imaginary parts of  $\Phi$  to obtain the field equations, we may vary  $\Phi$  and its complex conjugate  $\Phi^*$  independently. These procedures are equivalent. Varying  $\Phi^*$  in the action constructed from (3.32) yields, easily,

$$-\partial_{\mu}\partial^{\mu}\Phi - m^2\Phi = 0. \quad (3.33)$$

(Varying  $\Phi$  gives the complex conjugate of this equation.)

Note that the Lagrangian density (3.32) is the sum of contributions from the scalar fields  $\phi_1$  and  $\phi_2$ :

$$\begin{aligned} \mathcal{L} = \partial_{\mu}\Phi^* \partial^{\mu}\Phi - m^2\Phi^*\Phi &= \frac{1}{2}[\partial_{\mu}\phi_1 \partial^{\mu}\phi_1 - m^2\phi_1^2] \\ &+ \frac{1}{2}[\partial_{\mu}\phi_2 \partial^{\mu}\phi_2 - m^2\phi_2^2]. \end{aligned} \quad (3.34)$$

The general solution of (3.33) is a superposition of plane waves of the form

$$\Phi = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left( \frac{a_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \frac{b_{\mathbf{k}}^*}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right) \quad (3.35)$$

where  $a_{\mathbf{k}}$  and  $b_{\mathbf{k}}$  are now independent complex numbers. The field energy becomes

$$H = \sum_{\mathbf{k}} (a_{\mathbf{k}}^* a_{\mathbf{k}} + b_{\mathbf{k}}^* b_{\mathbf{k}}) \omega_{\mathbf{k}}. \quad (3.36)$$

We shall see that we can interpret this expression as being made up of the distinct contributions of positively and negatively charged fields. (The  $\pi^+$  and  $\pi^-$  mesons are composite particles whose overall motion is described by complex scalar fields.)

### Problems

- 3.1** Show that the kinetic energy of a system of particles, whose positions are determined by  $q(t)$ , is a quadratic function of the  $\dot{q}_i$ .
- 3.2** Show that  $dE/dt = 0$ , where  $E$  is given by equation (3.14).
- 3.3** For the stretched string of Section 3.3, show that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\rho \left( \frac{\partial\phi}{\partial t} \right)^2 + \frac{1}{2}F \left( \frac{\partial\phi}{\partial x} \right)^2.$$

The  $n$ th normal mode of oscillation, with wave amplitude  $A_n$ , is given by

$$\phi_n(x, t) = A_n \sin(k_n x) \sin(\omega_n t)$$

where  $k_n = n\pi/l$ ,  $\omega_n = (F/\rho)^{1/2}k_n$ . Show that the total energy is  $A_n^2\omega_n^2\rho l/4$  and oscillates harmonically between potential energy and kinetic energy.

- 3.4** Verify the expressions (3.30) and (3.31) for the energy and momentum of the scalar field given by equation (3.21).
- 3.5** Show that the Schrödinger equation for the wave function  $\psi(\mathbf{r}, t)$  of a particle of mass  $m$  moving in a potential  $V(\mathbf{r})$  may be obtained from the Lagrangian density

$$\mathcal{L} = -(1/2i) \left( \psi^* \frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t} \psi \right) - (1/2m) \nabla\psi^* \cdot \nabla\psi - \psi^* V \psi.$$

(Note that  $\mathcal{L}$  is real, but not Lorentz invariant.)