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PREDICTION OF FRACTIONAL BROWNIAN MOTION WITH HURST INDEX LESS THAN 1/2

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We give a proof based on an integral equation for an explicit prediction formula for fractional Brownian motion with Hurst index less than 1/2.

1. INTRODUCTION

A fractional Brownian motion with Hurst index $H \in (0,1)$ is a real centred Gaussian process $(B_H(t) : t \in \mathbf{R})$ with autocovariance

$$E\big[B_H(t)B_H(s)\big] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big) \qquad (t, s \in \mathbf{R}).$$

The case $H = 1/2$ corresponds to the ordinary Brownian motion. Starting from zero, fractional Brownian motion has stationary increments satisfying $E\Big[\big(B_H(t) - B_H(s)\big)^2\Big] = |t-s|^{2H}$. For $H \in (0,1) \setminus \{1/2\}$, fractional Brownian motion has the following asymptotic behaviour:

$$E\Big[\big\{B_H(t+1) - B_H(t)\big\}\big\{B_H(s+1) - B_H(s)\big\}\Big] \sim H(2H-1)(t-s)^{2H-2} \qquad (t-s \to \infty).$$

Fractional Brownian motion was discovered by Kolmogorov [**4**] but much of recent works on fractional Brownian motion originate from the seminal paper [**6**] by Mandelbrot and Van Ness. We refer to Samorodnitsky and Taqqu [**9**, Sections 7.2 and 14.7] for this background. Fractional Brownian motion has been widely used to model various phenomena in hydrology, network traffic, finance et cetera, which exhibit *long-range dependence*.

Let t_0, t_1, and T be real constants such that

$$-\infty < -t_0 \leqslant 0 \leqslant t_1 < T < \infty, \qquad t_0 < t_1.$$

The prediction of fractional Brownian motion is concerned with the computation of

$$(1.1) \qquad E\Big[B_H(T) \mid \sigma\big(B_H(s) : -t_0 \leqslant s \leqslant t_1\big)\Big]$$

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and

$$E\Big[B_H(T) \mid \sigma\big(B_H(s) : -\infty < s \leqslant t_1\big)\Big], \tag{1.2}$$

and their representation using $(B_H(s) : -t_0 \leqslant s \leqslant t_1)$ and $(B_H(s) : -\infty < s \leqslant t_1)$, respectively. The first problem is the prediction from a finite part of time, while the second one is the prediction from an infinite part of time. The problem (1.2) with $0 < H < 1/2$ was solved by Yaglom [**10**], while both problems (1.1) and (1.2) with $1/2 < H < 1$ were solved by Gripenberg and Norros [**3**]. Nuzman and Poor [**8**] introduced a new approach based on Lamperti's transformation, and considered all the cases including the remaining problem (1.1) with $0 < H < 1/2$. The present paper gives a new proof based on an integral equation for (1.1) with $0 < H < 1/2$.

The solution of Gripenberg and Norros [**3**] to (1.1) with $1/2 < H < 1$ is of the form

$$E\Big[B_H(T) \mid \sigma\big(B_H(s) : -t_0 \leqslant s \leqslant t_1\big)\Big] = B_H(t_1) + \int_{-t_0}^{t_1} f(t_0, t_1, T, s) dB_H(s). \tag{1.3}$$

In view of this solution in terms of a stochastic integral with respect to fractional Brownian motion, one tends to believe that the solution to (1.1) with $0 < H < 1/2$ would also be of the form (1.3). However, this is not the case. The solution to (1.1) with $0 < H < 1/2$ is of the form

$$E\Big[B_H(T) \mid \sigma\big(B_H(s) : -t_0 \leqslant s \leqslant t_1\big)\Big] = \int_{-t_0}^{t_1} f(t_0, t_1, T, s) B_H(s)\, ds, \tag{1.4}$$

that is, in terms of an elementary integral.

In our method for obtaining a solution to (1.1) with $0 < H < 1/2$ of the form (1.4), we reduce the problem to a manageable computation by the following equality for f in (1.4):

$$\int_{-t_0}^{t_1} f(t_0, t_1, T, s)\, ds = 1. \tag{1.5}$$

It is found in [1] that the same equality as (1.5) holds for more general processes than fractional Brownian motion with $0 < H < 1/2$.

The solution to (1.1) in the case $0 < H < 1/2$ is given by the following theorem.

Theorem 1. *Let t_0, t_1 and T be as above. We assume $0 < H < 1/2$. Then*

$$\begin{aligned} (1.6)\quad & E\Big[B_H(T) \mid \sigma\big(B_H(s) : -t_0 \leqslant s \leqslant t_1\big)\Big] \\ &= \frac{\sin(\pi((1/2) - H))}{\pi} \int_{-t_0}^{t_1} \Big(\frac{T - t_1}{t_1 - s}\Big)^{(1/2)+H} \Big(\frac{t_0 + s}{T + t_0}\Big)^{(1/2)-H} \frac{1}{T - s} B_H(s)\, ds \\ &\quad + \frac{((1/2) - H)\sin(\pi((1/2) - H))}{\pi(t_0 + t_1)} \Big[\int_0^{(T-t_1)/(T+t_0)} u^{H-(1/2)}(1 - u)^{-2H}\, du\Big] \\ &\qquad \times \int_{-t_0}^{t_1} \Big[\Big(\frac{t_0 + t_1}{t_0 + s}\Big)\Big(\frac{t_0 + t_1}{t_1 - s}\Big)\Big]^{H+(1/2)} B_H(s)\, ds. \end{aligned}$$

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The solution to (1.1) with $0 < H < 1/2$ was first obtained by Nuzman and Poor [8], and the theorem above is essentially the same as their result (Theorem 4.4 in [8]) except for one point. Unlike the theorem above, they do not assume that the interval corresponding to $[-t_0, t_1]$ includes the origin. However, their argument on this point does not seem to be complete. In our notation, they claimed that observation of $B_H(t)$ on $[-t_0, t_1]$ is equivalent to that of $B_H(t - t_0) - B_H(-t_0)$ on $[0, t_0 + t_1]$ (see Section 3.3 in [8]). However, this is true only if $B_H(-t_0)$ is a priori known, that is, $t_0 = 0$, and thus $B_H(-t_0) = 0$. We also remark that, in [8, Theorem 4.4], the factor $\pi^{-1}\sin\big(\pi((1/2)-H)\big)$ or $\pi^{-1}\cos(\pi H)$ is missing.

In [3], the solution to (1.1) with $1/2 < H < 1$ is obtained by reducing the problem to the following type of singular integral equation for the prediction kernel:

$$\int_0^1 F(t)|s-t|^{-\alpha}\,dt = f(s) \qquad (0 < s < 1), \qquad 0 < \alpha < 1. \tag{1.7}$$

In this paper, we obtain (1.6) by reducing the problem to the following type of equation (that is, (2.9) below) for the prediction kernel:

$$\int_0^1 F(t)|s-t|^{-\alpha}\operatorname{sgn}(s-t)\,dt = f(s) \qquad (0 < s < 1), \qquad 0 < \alpha < 1. \tag{1.8}$$

It is interesting to note that both (1.7) and (1.8) were solved in the same paper by Lundgren and Chiang [5] (though the solution to the first equation (1.7) had already been given by Carleman [2]).

2. Proof of Theorem 1

As stated in Section 1, we look for a nonnegative measurable function $h(t) = h(t; t_2, t_3)$ on $(0, t_2)$ satisfying

$$\int_0^{t_2} h(t)\,dt = 1, \tag{2.1}$$

$$E\Big[B_H(T) \mid \sigma\big(B_H(s) : -t_0 \leqslant s \leqslant t_1\big)\Big] = \int_{-t_0}^{t_1} h(t+t_0)B_H(t)\,dt, \tag{2.2}$$

where the positive constants t_2 and t_3 are defined by

$$t_2 := t_0 + t_1, \qquad t_3 := T - t_1.$$

These facts are essential in deriving the manageable form (1.8), hence finally obtaining (1.6).

The equality (2.2) implies

$$E\left[\left\{B(T) - \int_{-t_0}^{t_1} h(t+t_0)B_H(t)\,dt\right\}B_H(s)\right] = 0 \qquad (-t_0 < s < t_1) \tag{2.3}$$

or

$$\text{(2.4)} \quad \int_{-t_0}^{t_1} h(t+t_0)\big(|t|^{2H} + |s|^{2H} - |s-t|^{2H}\big)\,dt \\ = T^{2H} + |s|^{2H} - (T-s)^{2H} \qquad (-t_0 < s < t_1).$$

From this and (2.1), we obtain

$$\text{(2.5)} \qquad \int_{-t_0}^{t_1} h(t+t_0)\big(|t|^{2H} - |s-t|^{2H}\big)\,dt = T^{2H} - (T-s)^{2H} \qquad (-t_0 < s < t_1).$$

Since we have, for $-t_0 < s < t_1$,

$$\int_{-t_0}^{t_1} h(t+t_0)|s-t|^{2H}\,dt = \int_{-t_0}^{s} h(t+t_0)(s-t)^{2H}\,dt + \int_{s}^{t_1} h(t+t_0)(t-s)^{2H}\,dt,$$

formal differentiation of both sides of (2.5) with respect to s yields

$$\text{(2.6)} \qquad \int_{-t_0}^{t_1} h(t+t_0)|s-t|^{2H-1}\operatorname{sgn}(s-t)\,dt = -(T-s)^{2H-1} \qquad (-t_0 < s < t_1)$$

(the validity of this formal calculation should not be of concern at this stage). We define $\alpha \in (0,1)$, $a \in (0,\infty)$, and $g(t) = g(t; t_2, t_3)$, respectively, by

$$\text{(2.7)} \qquad \alpha := 1 - 2H, \qquad a := t_3/t_2,$$

$$\text{(2.8)} \qquad g(t) := t_2 h\big(t_2(1-t)\big) \qquad (0 < t < 1).$$

Then, by the substitutions $t' = (t_1 - t)/t_2$ and $s' = (t_1 - s)/t_2$, we obtain

$$\text{(2.9)} \qquad \int_0^1 g(t)|s-t|^{-\alpha}\operatorname{sgn}(s-t)\,dt = (a+s)^{-\alpha} \qquad (0 < s < 1).$$

By [5, (18)], the general solution $g(t)$ to (2.9) is given by

$$\text{(2.10)} \qquad g(t) = c_1 t^{(\alpha/2)-1}(1-t)^{(\alpha/2)-1} + g_0(t) \qquad (0 < t < 1),$$

where c_1 is an arbitrary constant and $g_0(t)$ is given by, for $0 < t < 1$,

$$\frac{\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi\Gamma(\alpha/2)^2}\frac{d}{dt}t^{\alpha/2}\int_t^1 s^{-\alpha}(s-t)^{(\alpha/2)-1}\left\{\int_0^s u^{\alpha/2}(s-u)^{(\alpha/2)-1}(a+u)^{-\alpha}\,du\right\}ds.$$

By the change of variables $v = u/s$, we have

$$\int_0^s u^{\alpha/2}(s-u)^{(\alpha/2)-1}(a+u)^{-\alpha}\,du = \left(\frac{s}{a}\right)^{\alpha}\int_0^1 v^{\alpha/2}(1-v)^{(\alpha/2)-1}\left(1+\frac{s}{a}v\right)^{-\alpha}dv \\ = \frac{\Gamma(\alpha/2)^2}{2\Gamma(\alpha)}\left(\frac{s}{a}\right)^{\alpha}F\left(\alpha, \frac{\alpha}{2}+1; \alpha+1; -\frac{s}{a}\right),$$

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where $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ is the hypergeometric function. Thus, for $0 < t < 1$,

$$(2.11)\quad g_0(t) = \frac{\sin(\pi\alpha/2)}{2\pi}\frac{d}{dt}t^{\alpha/2}\int_t^1 s^{-\alpha}(s-t)^{(\alpha/2)-1}\left(\frac{s}{a}\right)^{\alpha}F\left(\alpha, \frac{\alpha}{2}+1; \alpha+1; -\frac{s}{a}\right)ds.$$

From (2.11), we easily find that $\int_0^1 g_0(t)\,dt = 0$. Since we have $\int_0^{t_2} h(t)\,dt = \int_0^1 g(t)\,dt$, the condition (2.1) implies

$$(2.12)\qquad c_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2}.$$

We now obtain an explicit expression for $g_0(t)$ using (2.11). By the formulas

$$\frac{d}{dz}\left[z^a F(a,b;c;z)\right] = az^{a-1}F(a+1,b;c;z),$$
$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z),$$

we see that

$$\frac{d}{ds}\left[\left(\frac{s}{a}\right)^{\alpha}F\left(\alpha,\frac{\alpha}{2}+1;\alpha+1;-\frac{s}{a}\right)\right] = \frac{\alpha}{a}\left(\frac{s}{a}\right)^{\alpha-1}\left(1+\frac{s}{a}\right)^{-(\alpha/2)-1}.$$

On the other hand, by the change of variables $v = (u-t)/t$, we have

$$t^{\alpha/2}\int_t^s u^{-\alpha}(u-t)^{(\alpha/2)-1}\,du = f\left(\frac{s-t}{t}\right),\qquad (t \leqslant s \leqslant 1),$$

where

$$f(x) := \int_0^x (1+v)^{-\alpha}v^{(\alpha/2)-1}dv \qquad (x \geqslant 0).$$

We have

$$\frac{d}{dt}f\left(\frac{s-t}{t}\right) = -t^{(\alpha/2)-1}s^{1-\alpha}(s-t)^{(\alpha/2)-1}\qquad (0 < t < s).$$

Hence, by integration by parts, we obtain, for $0 < t < 1$,

$$(2.13)\quad g_0(t) = -\frac{\sin(\pi\alpha/2)}{2\pi}a^{-\alpha}F\left(\alpha,\frac{\alpha}{2}+1;\alpha+1;-\frac{1}{a}\right)t^{(\alpha/2)-1}(1-t)^{(\alpha/2)-1} + g_1(t),$$

where

$$\begin{aligned} g_1(t) &= -\frac{\sin(\pi\alpha/2)}{2\pi}\frac{d}{dt}\int_t^1 f\left(\frac{s-t}{t}\right)\frac{\alpha}{a}\left(\frac{s}{a}\right)^{\alpha-1}\left(1+\frac{s}{a}\right)^{-(\alpha/2)-1}ds \\ &= -\frac{\sin(\pi\alpha/2)}{2\pi}\int_t^1\left\{\frac{d}{dt}f\left(\frac{s-t}{t}\right)\right\}\frac{\alpha}{a}\left(\frac{s}{a}\right)^{\alpha-1}\left(1+\frac{s}{a}\right)^{-(\alpha/2)-1}ds \\ &= \frac{\alpha\sin(\pi\alpha/2)}{2\pi}\left(\frac{a}{t}\right)^{1-(\alpha/2)}\int_t^1 (s-t)^{(\alpha/2)-1}(a+s)^{-(\alpha/2)-1}\,ds. \end{aligned}$$

By the change of variables $u = (1-s)/(1-t)$ and the equality

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu}\,dt = c^{-\nu}(c-1)^{-\mu}B(\mu,\nu)\qquad (\mu,\nu > 0,\ c > 1)$$

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(see [**7**, Lemma 2.2 (i)]), we see that, for $0 < t < 1$,

$$g_1(t) = \frac{\alpha\sin(\pi\alpha/2)}{2\pi}\Big(\frac{a}{t}\Big)^{1-(\alpha/2)}\frac{1}{1-t}\int_0^1(1-u)^{(\alpha/2)-1}\Big(\frac{a+1}{1-t}-u\Big)^{-(\alpha/2)-1}du$$
$$= \frac{\sin(\pi\alpha/2)}{\pi}\Big(\frac{a}{t}\Big)^{1-(\alpha/2)}\Big(\frac{1-t}{a+1}\Big)^{\alpha/2}\frac{1}{a+t}.$$

From this as well as (2.10) and (2.13), the general solution $g(t)$ to the equation (2.9) is given by

$$g(t) = c_2 t^{(\alpha/2)-1}(1-t)^{(\alpha/2)-1} + \frac{\sin(\pi\alpha/2)}{\pi}\Big(\frac{a}{t}\Big)^{1-(\alpha/2)}\Big(\frac{1-t}{a+1}\Big)^{\alpha/2}\frac{1}{a+t},$$

for $0 < t < 1$, where c_2 is an arbitrary constant.

By (2.12), under the condition (2.1), c_2 is given by

$$c_2 := \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} - \frac{\sin(\pi\alpha/2)}{2\pi}a^{-\alpha}F\Big(\alpha, \frac{\alpha}{2}+1; \alpha+1; -\frac{1}{a}\Big).$$

However, since we have, for $0 < \mu < 1$, $\nu > 0$ and $c > 1$,

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}(c-t)^{-\mu-\nu+1}\,dt$$
$$= \frac{\pi}{\sin(\pi\mu)} - (\mu+\nu-1)B(\mu,\nu)\int_0^{1-(1/c)} s^{-\mu}(1-s)^{\mu+\nu-2}\,ds$$

(see [**7**, Lemma 2.2 (iii)]), we see that

$$c_2 = \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2}\Big[1 - \frac{\sin(\pi\alpha/2)}{\pi}\int_0^1 s^{(\alpha/2)-1}(1-s)^{\alpha/2}(a+1-s)^{-\alpha}\,ds\Big]$$
$$= \frac{\alpha\sin(\pi\alpha/2)}{2\pi}\int_0^{a/(1+a)} s^{-\alpha/2}(1-s)^{\alpha-1}\,ds.$$

Thus, using (2.7) and (2.8), we finally obtain, for $-t_0 < t < t_1$,

$$h(t) = \frac{\sin(\pi((1/2)-H))}{\pi}\Big(\frac{T-t_1}{t_2-t}\Big)^{(1/2)+H}\Big(\frac{t}{T+t_0}\Big)^{(1/2)-H}\frac{1}{T+t_0-t}$$
$$+ \frac{((1/2)-H)\sin(\pi((1/2)-H))}{\pi t_2}\Big[\int_0^{(T-t_1)/(T+t_0)} u^{H-(1/2)}(1-u)^{-2H}\,du\Big]$$
$$\times\Big[\Big(\frac{t_2}{t}\Big)\Big(\frac{t_2}{t_2-t}\Big)\Big]^{H+(1/2)}, \tag{2.14}$$

which implies (1.6).

Now, for a rigorous proof, we may start with (2.14). Then we have (2.6) and (2.1) by the arguments given above. From (2.6), we see (rigorously this time) that

$$\frac{d}{ds}\phi(s) = 2(T-s)^{2H-1}H \qquad (-t_0 < s < t_1),$$

where

$$\phi(s) := \int_{-t_0}^{t_1} h(t+t_0)\big(|t|^{2H} - |s-t|^{2H}\big)\,dt \qquad (s \in \mathbf{R}).$$

Since $\phi(0) = 0$, we get (2.5). Finally, from (2.5) and (2.1), we obtain (2.4) or (2.3) or (1.6). This completes the proof of Theorem 1.

3. Remarks

1. From the proof in Section 2, we see that

$$\int_0^1 g_1(t)|s-t|^{-\alpha}\mathrm{sgn}(s-t)\,dt = (a+s)^{-\alpha} \qquad (0 < s < 1).$$

This implies the following equality: for $a > 0$, $0 < \alpha < 1$ and $0 < s < 1$,

$$\int_0^1 \frac{t^{(\alpha/2)-1}(1-t)^{\alpha/2}}{a+t}|s-t|^{-\alpha}\mathrm{sgn}(s-t)\,dt = \frac{\pi}{\sin(\pi\alpha/2)}a^{(\alpha/2)-1}(a+1)^{\alpha/2}(a+s)^{-\alpha}.$$

2. Since $E\big[B_H(s)^2\big]^{1/2} = |s|^H$, we easily find that the second term on the right-hand side of (1.6) tends to zero, as $t_0 \to \infty$, in $L^2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the probability space on which $\big(B_H(t)\big)$ is defined. Hence, by letting $t_0 \to \infty$ in (1.6), we obtain the following prediction formula for fractional Brownian motion with $H \in (0, 1/2)$ from the infinite past:

$$\text{(3.1)}\quad E\Big[B_H(T) \mid \sigma\big(B_H(s) : -\infty < s \leqslant t_1\big)\Big] = \frac{\sin(\pi((1/2)-H))}{\pi}\int_{-\infty}^{t_1}\Big(\frac{T-t_1}{t_1-s}\Big)^{(1/2)+H}\frac{1}{T-s}B_H(s)\,ds.$$

This result was given in Yaglom [**10**, (3.41)].

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