FIVE DIMENSIONAL NON-LATTICE SPHERE PACKINGS

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1. The densest lattice packings of spheres in Euclidean spaces E_n of n dimensions are known for $n \leq 8$ (for full references see [6]). However, it is not known for any $n \geq 3$ whether there can be any non-lattice sphere packing with density exceeding that of the corresponding densest lattice packing. Barlow's description [1] of a non-lattice packing in E_3 with the same density as the densest lattice packing serves to show that the possibility of a denser non-lattice packing is not absurd <u>prima facie</u>. In this note I show that in E_5, as in E_3, non-lattice packings are possible with the same density as the densest lattice packings. The construction gives three distinct non-lattice arrangements, with quite different symmetry groups, the symmetries being transitive on the spheres in each case. A remarkable feature is that the configuration of 40 spheres touching any one is the same in all these arrangements, although they are different over larger regions.

My earlier announcements [3, p.60; 4, p.658] of the existence of non-lattice packings in E_5 are somewhat inaccurate, and are superseded and corrected by the present account.

2. The centres of the spheres in the densest lattice packings in E_n, for $n = 3, 4, 5$, are vertices of the honeycomb $h\delta_{n+1}$, whose vertices are alternate vertices of the regular cubic honeycomb δ_{n+1} (for notation see [2]). The cells of the honeycomb $h\delta_{n+1}$ are of two kinds. Each omitted vertex of the cubic honeycomb δ_{n+1} is the centre of a cross polytope β_n whose vertices are those of the cubic honeycomb adjacent to the omitted vertex. Alternate vertices of each cube γ_n of the cubic honeycomb form

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a cell $h\gamma_n$ concentric with the cube. As there is a cell $h\gamma_n$ concentric with every cube of the honeycomb, but the cells β_n are concentric with the alternate vertices of the honeycomb, we see that the cells $h\gamma_n$ are on the average twice as numerous as the cells β_n.

For $n = 3$ we have $h\gamma_3 = \alpha_3$, and the honeycomb is the familiar one of tetrahedra and octahedra. For $n = 4$ we have $h\gamma_4 = \beta_4$, and in this case every cell is a β_4 and the honeycomb is the regular honeycomb $\{3, 3, 4, 3\}$. For $n > 4$ $h\gamma_n$ is only semiregular.

3. We remark in passing on coordinates for the regular honeycomb $\{3, 3, 4, 3\}$ in E_4. We consider the following four sets of points, namely those whose coordinates are either all even or all odd, and which have their sum either a multiple of 4 or twice an odd number, in all four combinations. Then any one of these sets form vertices of a honeycomb $\{3, 3, 4, 3\}$, and the points of the other three sets are all centres of the cells of the honeycomb. Also if all four sets of points are taken together, they form vertices of another honeycomb $\{3, 3, 4, 3\}$, differently oriented and of size $2^{-1/2}$ times that of the former honeycombs. From this we see that the four sets of points are all exactly equivalent and are completely symmetrically related to each other. Thus, for example, all six pairs of sets from the four are also exactly equivalent, being mutually transformable by symmetry operations on the smaller honeycomb.

4. In E_5 I take the coordinates of the vertices of the honeycomb $h\delta_6$ to be pentads $(x_0, x_1, x_2, x_3, x_4)$ of even integers whose sum is a multiple of 4; this avoids the subsequent introduction of fractions. Then the centres of the β_5 cells have their coordinates all even with their sum twice an odd number, and the centres of the $h\gamma_5$ cells have their coordinates all odd with no restriction on their sum.

We may divide the honeycomb $h\delta_6$ into layers separated by flats $x_o = 2m$ for integer m. Each such flat cuts the honeycomb $h\delta_6$ in a regular honeycomb $h\delta_5 = \{3, 3, 4, 3\}$ whose cells are all cross polytopes β_4. Any cell $h\gamma_5$ is contained exactly between two adjacent flats, each of which contains a face β_4 of the cell. Each cell β_5 is exactly bisected by a flat into two β_4-pyramids whose common base is a β_4 of the flat. One third of the cells β_4 of the flat, those whose centres have their coordinates all even with their sum twice an odd number, are bases of pairs of pyramids, while two thirds, those whose centres have their last four coordinates all odd, are faces of contact of pairs of $h\gamma_5$ cells.

5. To form non-lattice arrangements, we take layers as obtained in § 4 and fit them together so that the β_4 cells in their bounding flats correspond but the β_4 cells of the different types identified in § 4 do not. Thus one third of the β_4 cells in the flat bounding a layer, those which are bases of pyramids in that layer, are fitted on to faces of $h\gamma_5$ cells of the adjacent layer, while, of the two thirds which are faces of $h\gamma_5$ cells of the layer, half are fitted on to pyramids of the adjacent layer and half on to $h\gamma_5$ cells. There is a choice at each stage of building up a space filling, namely to which half of the $h\gamma_5$ cells of each layer we fit the pyramids of the next layer, but the pairs of layers which may be so formed are congruent (this follows from the equivalence of sets of centres of β_4 cells in the packing in the flat as given in § 3).

There are, however, two different ways of fitting any three layers together. At any flat, half of the $h\gamma_5$ cells on one side fit on to $h\gamma_5$ cells on the other and half fit on to pyramids. The middle of three layers may be of two types. Either the $h\gamma_5$ cells which fit on to pyramids on one side fit on to pyramids on the other, and those which fit on to $h\gamma_5$ cells on one side fit on to $h\gamma_5$ cells on the other; call such a layer type S (Same or Similar). Or each $h\gamma_5$ cell of the middle layer fits on to a pyramid on one side and a $h\gamma_5$ cell on the other; call such a layer type D (Different or Dissimilar). Note that the layers themselves are identical, as are pairs of layers; it is not until we consider a layer and both those adjacent to it that any difference emerges.

There are three possible ways of stacking the layers so that uniform packings of spheres result. The centres of the spheres lie in the flats, and the two layers meeting in each flat must be equivalent to the two layers meeting in any other flat. Thus the layers must be all of type S, or all of type D, or of types S and D alternately. We consider these cases in turn.

6. Each centre is at the apex of a pair of pyramids, one in each layer meeting in the flat containing the centre. If every layer is of type S, then each pyramid fits on to a $h\gamma_5$ cell whose opposite face fits on to another pyramid whose apex is another centre. Thus the positions of the centres repeat exactly in every third flat. To assign coordinates, we take the first two flats ($x_0 = 0, 2$) and the layer between from the lattice packing; thus in these two flats the coordinates of the centres are all even with their sum (including x_0) a multiple of 4. In the third flat the centres have their last four coordinates odd. We may choose whether their sum is to be twice an odd or even number; let us choose arbitrarily to make the sum divisible by 4. From the fourth flat onwards, the last four coordinates repeat those of the third previous flat, and we have the following coordinates.

If	$x_0 \equiv 0 \quad 2 \quad 4$	(mod 6)
then	$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv 0 \quad 0 \quad 1$	(mod 2)
and	$x_1 + x_2 + x_3 + x_4 \equiv 0 \quad 2 \quad 0$	(mod 4) .

7. If every layer is of type D, then each pyramid fits on to a $h\gamma_5$ cell whose opposite face fits on to a $h\gamma_5$ cell whose opposite face fits on to a pyramid. Thus the positions of the centres repeat exactly in every fourth flat. To assign coordinates, we take the first three flats as in § 6. The centres in the fourth flat ($x_0 = 6$) do not match those in any of the first three flats, so they have their last four coordinates odd with their sum (including $x_0 = 6$) divisible by 4. From the fifth flat onwards, the last four coordinates repeat those of the fourth previous flat, and we

have the following coordinates.

If	$x_0 \equiv$	0	2	4	6	(mod 8)
then	$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv$	0	0	1	1	(mod 2)
and	$x_1 + x_2 + x_3 + x_4 \equiv$	0	2	0	2	(mod 4) .

8. The situation is more complex in the third case. Consider first a pyramid in a layer of type D . This fits on to a $h\gamma_5$ in a layer of type S which fits on to another pyramid. Thus the centres in each flat repeat exactly in the flat three away on the side of the flat on which is the layer of type D bounded by it. Now consider a pyramid in a layer of type S . This fits on to a $h\gamma_5$ in a layer of type D which fits on to a $h\gamma_5$ in a layer of type S which fits on to a $h\gamma_5$ in a layer of type D which fits on to a pyramid. Thus the centres in each flat repeat exactly in the flat five away on the side of the flat on which is the layer of type S bounded by it. Since each centre is in a flat which is the boundary of layers one of each type, this means that, in any line of centres perpendicular to the flats, the centres have alternate intervals of three and five layers. Thus if we take the first layer, between $x_0 = 0$ and $x_0 = 2$, to be of type D , we obtain the following coordinates.

If	$x_0 \equiv$	0	2	4	6	8	10	12	14	(mod 16)
then	$x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv$	0	0	1	0	1	1	0	1	(mod 2)
and	$x_1 + x_2 + x_3 + x_4 \equiv$	0	2	0	0	2	0	2	2	(mod 4) .

9. In each of these arrangements, as in the lattice packing, each sphere touches 40 others. Of these, 24 have their centres in the same flat, these centres being the vertices of a regular 24-cell $\{3, 4, 3\}$, and eight have their centres in each of the adjacent flats, these centres being vertices of the pyramids whose common apex is the centre of the chosen sphere. In the non-lattice arrangements, these last 16 spheres are not reflections of each other in the central flat, as the layers were fitted in § 5 to avoid this. As all pairs of layers so fitted are equivalent, the configuration of 40 spheres touching each one is the same whichever of the three arrangements we consider, despite their non-identity over larger regions.

The Dirichlet region for each sphere, which is the set of points which are not closer to another centre than to the centre of the chosen sphere, is a polytope which has one vertex in each adjacent flat, at the centre of the base of each pyramid with apex at the chosen centre. As this does not extend beyond the adjacent flats, it is the same whichever of the three arrangements we consider. We thus have the remarkable case of a convex space-filling polytope which can be used to fill space in distinct discrete arrangements (unlike, say, layers of cubes which can be slid one on another), which have groups of symmetries which are transitive on the polytopes but which are quite different from each other.

10. A description of the lattice and non-lattice packings in E_3 makes an instructive comparison. In the densest lattice packing in E_3 the centres of the spheres are vertices of the honeycomb $h\delta_4$ of tetrahedra and octahedra. This honeycomb may be divided into layers by planes which cut it in the regular triangular tessellation $\{3, 6\}$. Each octahedron is exactly contained between two of the planes, each of which contains a face of it, and each tetrahedron is also contained between two planes, with a face in one and the opposite vertex in the other. Each triangle of the tessellation is the face of contact of a tetrahedron and an octahedron.

We can move the layers so that each triangle of the tessellation is the face of contact either of two tetrahedra or of two octahedra. There is no choice here, as half of the triangles in the the plane bounding a layer are faces of tetrahedra of the layer and half are faces of octahedra of the layer. In order that all spheres should be surrounded equivalently, every plane has to be the boundary of two layers fitted together in the same way; this specifies the non-lattice packing uniquely, and we see that its symmetries are transitive on the spheres.

In the lattice packing each octahedron fits on to tetrahedra in both adjacent layers, and in the uniform non-lattice packing, each octahedron fits on to octahedra in both adjacent layers. It is possible to assemble the layers in such a way that each octahedron fits on to an octahedron in one adjacent layer and a tetrahedron in the other, in an attempted analogy with the packings in E_5. When this is done, it is found that the pairs of layers are not equivalent, alternate planes separating layers fitted as in the lattice and uniform non-lattice arrangements, and the symmetries

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are no longer transitive on all the spheres. This arrangement is one of those considered by Melmore [5], in which the spheres are divided into two sets, whose Dirichlet regions are rhombic or trapezo-rhombic dodecahedra, with symmetries transitive on the spheres of each set. There are several comparable arrangements in E_5 (about 12, depending on the precise conditions imposed), but these do not seem sufficiently interesting to be worth discussing in detail.

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