J. Austral. Math. Soc. (*Series A*) **46** (1989), 177–183

LATTICES OF PSEUDOVARIETIES

P. AGLIANO and J. B. NATION

(Received 11 March 1987)

Communicated by T. E. Hall

Abstract

We consider the lattice of pseudovarieties contained in a given pseudovariety P. It is shown that if the lattice L of subpseudovarieties of P has finite height, then L is isomorphic to the lattice of subvarieties of a locally finite variety. Thus not every finite lattice is isomorphic to a lattice of subpseudovarieties. Moreover, the lattice of subpseudovarieties of P satisfies every positive universal sentence holding in all the lattices of subvarieties of varieties $\mathbf{V}(A)$ generated by algebras $A \in P$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 08 B 15, 08 C 99.

A *pseudovariety* P is a class of finite algebras closed under the formation of homomorphic images, subalgebras and finite direct products: in symbols $\mathbf{HSP}_{\mathrm{fin}}(P) = P$. This concept has been useful in many investigations, particularly in the study of various classes of finite semigroups and monoids; see [2], [3], [7], [8] and, for a more general approach, [5].

We will consider the lattice of subpseudovarieties of a given pseudovariety P. We will show that several recent results about the lattice of subvarieties of a variety have analogs for pseudovarieties.

This investigation originated in a series of discussions between the second author, Kathy Johnston and T. E. Hall at Monash University in August 1986, which produced a direct proof of Corollary 2.6. Hall and Johnston were at tht time working on pseudovarieties of inverse semigroups [8], and Section 5 of that paper contains some interesting results related to the ones herein. C. J. Ash

The second author was supported in part by NSF Grant MCS 83-01087.

has also generalized Corollary 2.6 in a rather different direction (see [8, Remark 5.5]).

1. Preliminaries and forbidden lattices

If K is a class of algebras (of the same type), $\mathbf{V}(K)$ will denote the variety generated by K; if Q is a class of finite algebras $\mathbf{P}(Q) = \mathbf{HSP}_{\mathbf{fin}}(Q)$ will denote the pseudovariety generated by Q.

It is clear that, if V is a variety, the class consisting of the finite algebras in V is a pseudovariety. We will denote this class by $V_{\mathbf{fin}}$ or $V \cap \mathrm{Fin}$. On the other hand it is easy to produce examples of pseudovarieties that are not of this kind, for example

(a) the pseudovariety of finite Abelian groups of square—free exponent,

(b) the pseudovariety of finite semigroups satisfying $x^n = x^{n+1}$ for some n,

(c) the pseudovariety $\mathscr{F}$ where

$$\mathscr{F} = \{F_1 \times \cdots \times F_n \colon F_i \text{ is a finite field, } \operatorname{char} F_i = p \text{ for all } i\}.$$

(d) the pseudovariety of all finite lattices satisfying $SD_\wedge$.

The following lemma (whose proof is straightforward) will be used repeatedly. We recall that a variety V is locally finite if all finitely generated algebras in V are finite.

LEMMA 1.1. *Let V be a locally finite variety and A a finite algebra in V. Suppose there exist a $B \in V$, a family $\{C_\alpha\}_{\alpha<\beta}$ of algebras in V and a surjective homomorphism h such that*

$$A \underset{h}{\leftarrow} B \leq \prod (C_\alpha \colon \alpha < \beta).$$

Then there exist finitely many $\alpha_1, \ldots, \alpha_n$ such that

$$A \underset{h}{\leftarrow} B' \leq C_{\alpha_1} \times \cdots \times C_{\alpha_n}.$$

The following lemma can be found in [1], [6] and [11] and was implicitly stated in [7] for monoids.

LEMMA 1.2. *A class of finite algebras P is a pseudovariety if and only if there exists a directed union of locally finite varieties such that*

$$P = \bigcup (V_i \colon i \in I) \cap \mathrm{Fin}.$$

The proof is straightforward, by taking

$$I = \{S \subseteq P \colon S \text{ is finite}\}$$

https://doi.org/10.1017/S1446788700030640 Published online by Cambridge University Press

and noting that by Lemma 1.1, $\mathbf{V}(S)$ is locally finite for all $S \in I$. If V is a variety let $\mathbf{L_v}(V)$ be the lattice of subvarieties of V, and if P is a pseudovariety let $\mathbf{L_{pv}}(P)$ be the set of pseudovarieties contained in P. It is a routine exercise to prove that $\mathbf{L_{pv}}(P)$ is indeed an algebraic lattice under inclusion, with $\mathbf{HSP_{fin}}$ being the associated closure operator.

It is not hard to see that the lattices $\mathbf{L_v}$ and $\mathbf{L_{pv}}$ can be quite different even for the same variety of algebras. Let Ab be the variety of abelian groups. Then $\mathbf{L_v}(Ab)$ is isomorphic to the lattice $\mathbb{N}$ of the positive integers under division with the largest element adjoined, while $\mathbf{L_{pv}}(Ab_{\mathrm{fin}})$ is isomorphic to the ideal lattice of $\mathbb{N}$.

We recall that if L is a lattice, the height $h(a)$ of $a \in L$ is the length of the shortest maximal chain in $a/0$; we say that P is a *pseudovariety of finite height* if P has finite height in $\mathbf{L_{pv}}(P)$.

LEMMA 1.3. *If P is a pseudovariety of finite height, then P is generated by finitely many finite algebras.*

PROOF. We induct on the height of P. If $h(P) = 0$, then P is the trivial pseudovariety generated by a one-element algebra. Assume the statement true for any height $< n$ and let $h(P) = n$. Then P covers Q, where $h(Q) < n$ and Q is generated by finitely many finite algebras. Let $A \in Q - P$; then $P < P(Q \cup \{A\}) \leq P$, and since P covers Q we have $P = P(Q \cup \{A\})$. Then P itself is finitely generated, and the lemma is proved.

The next lemma connects pseudovarieties and locally finite varieties.

LEMMA 1.4. *Let V be locally finite. Consider the maps*

$$\varphi\colon \mathbf{L_v}(V) \to \mathbf{L_{pv}}(V_{\mathrm{fin}})$$

via $\varphi(U) = U \cap \mathrm{Fin}$, *and* $\psi\colon \mathbf{L_{pv}}(V_{\mathrm{fin}}) \to \mathbf{L_v}(V)$ *via* $\psi(Q) = \mathbf{V}(Q)$. *Then*

(i) *for all pseudovarieties* $Q \leq V_{\mathrm{fin}}$, $Q = \varphi\psi(Q) = \mathbf{V}(Q) \cap \mathrm{Fin}$,

(ii) *for all varieties* $U \leq V$, $U = \psi\varphi(U) = \mathbf{V}(U \cap \mathrm{Fin})$,

(iii) φ *and* ψ *are lattice isomorphisms.*

PROOF. (i) Clearly $Q \subseteq V(Q) \cap \mathrm{Fin}$. If A is a finite algebra in $V(Q)$, then there exist an algebra $B \in V(Q)$, a family $\{C_\alpha\}_{\alpha<\beta}$ of algebras in Q and a surjective homomorphism h such that

$$A \overset{h}{\leftarrow} B \leq \prod (C_\alpha\colon \alpha < \beta).$$

Since V it is locally finite and A is finite, it can be assumed by Lemma 1.1 that β is finite and all the C_α are finite. Then $A \in \mathbf{HSP_{fin}}(Q) = Q$ and we are done.

(ii) is obvious since any $U \leq V$ is itself locally finite, and is therefore generated by its finite members.

https://doi.org/10.1017/S1446788700030640 Published online by Cambridge University Press

(iii) follows from (i) and (ii), since a one-to-one map f of a lattice L onto a lattice M is an isomorphism if and only if f and its inverse are order-preserving.

The main theorem of this section is now easy to prove.

THEOREM 1.5. *If P is a pseudovariety of finite height, then the lattice of pseudovarieties $\mathbf{L_{pv}}(P)$ is isomorphic to the lattice of subvarieties $\mathbf{L_v}(\mathbf{V}(P))$.*

PROOF. By Lemma 1.3, P is generated by finitely many finite algebras; hence $\mathbf{V}(P)$ is locally finite. So Lemma 1.4 applies, and we can conclude that $\mathbf{L_{pv}}(P) \cong \mathbf{L_v}(\mathbf{V}(P))$.

The following corollary is a direct consequence of the theorem.

COROLLARY 1.6. *If L is a lattice of finite height, forbidden as lattice of subvarieties for a locally finite variety, then it is also forbidden as $\mathbf{L_{pv}}(P)$ for any pseudovariety P.*

A large class of lattices forbidden for locally finite varieties (the so-called "tight" lattices) can be found in Mckenzie's paper [10].

2. A representation theorem

The results of the previous section lead to the question: what properties of $\mathbf{L_v}(V)$ are inherited by $\mathbf{L_{pv}}(V_{\mathrm{fin}})$ (not assuming local finiteness)? This section provides some answers to this question.

THEOREM 2.1. *Let P be a pseudovariety and for $A \in P$ let $\mathbf{L_A} = \mathbf{L_v}(\mathbf{V}(A))$. If $\mathscr{K}(P) = \{L_{\mathbf{A}} : A \in P\}$, then*

$$\mathbf{L_{pv}}(P) \in \mathbf{HSP_u}\{\mathscr{K}(P)\}.$$

We will first show that $\mathbf{L_{pv}}(P)$ is in the variety generated by $\mathscr{K}(P)$. Let M be the direct product $\prod(\mathbf{L_A} : A \in P)$, where it is understood that we take an algebra from each isomorphism class. Let

$$S = \{x \in M : A \in \mathbf{V}(B) \text{ implies } x_A \leq x_B\}.$$

Clearly S is a sublattice of M. Define $\gamma \subseteq S^2$ by setting $(x, y) \in \gamma$ if

(*) $\forall A, C \in P$, $C \in x_A$ implies there exists $B \in P$ such that $A \in \mathbf{V}(B)$ and $C \in y_B$ and

(**) $\forall A, C \in P$, $C \in y_A$ implies there exists $B' \in P$ such that $A \in \mathbf{V}(B')$ and $C \in x_{B'}$.

We would like to prove that $\gamma \in \operatorname{Con} S$. The proof uses the following straightforward lemma.

https://doi.org/10.1017/S1446788700030640 Published online by Cambridge University Press

LEMMA 2.2. *If V is a locally finite variety, $U, W \leq V$ and C is a finite algebra in $U \vee W$, then there are finite algebras $D \in U$ and $E \in W$ such that $C \in \mathbf{HS}(D \times E)$.*

LEMMA 2.3. $\gamma \in \operatorname{Con} S$.

PROOF. It is not hard to check that γ is an equivalence relation on S. Assume now $z \in S$, $x\gamma y$, and $C \in x_A \wedge z_A$. By $(*)$ there is a $B \in P$ such that $C \in y_B$ and $A \in \mathbf{V}(B)$, whence $C \in z_B \geq z_A$. Thus $C \in y_B \wedge z_B$ which proves $(*)$; $(**)$ is similar so we can conclude that $x \wedge z \,\gamma\, y \wedge z$.

If $C \in x_A \vee z_A$ then, by Lemma 2.2, $C \in \mathbf{HS}(D \times E)$ for finite algebras in x_A and z_A respectively. Because $D \in x_A$, $(*)$ yields a $B \in P$ with $A \in \mathbf{V}(P)$ and $D \in y_B$. Since $z \in S$ and $A \in \mathbf{V}(B)$, it follows that $E \in z_B$ and $C \in y_B \vee z_B$. The proof of $(**)$ is similar, so $x \vee z \,\gamma\, y \vee z$.

Define a function $\varphi\colon \mathbf{L_{pv}}(P) \to S$ by setting $\varphi_A(Q) = \mathbf{V}(Q \cap \mathbf{V}(A))$; it is trivial to see that indeed $\varphi(Q) \in S$ for all $Q \leq P$.

LEMMA 2.4. *For all $Q, R \leq P$*
(i) $\varphi(Q \cap R) = \varphi(Q) \cap \varphi(R)$,
(ii) $\varphi(Q) \vee \varphi(R) \leq \varphi(Q \vee R)$,
(iii) $\varphi(Q \vee R) \,\gamma\, \varphi(Q) \vee \varphi(R)$,
(iv) *if $Q \nleq R$ then* $(\varphi(Q), \varphi(R)) \notin \gamma$.

PROOF. (i) and (ii) are immediate. For (iii), assume that C is finite and $C \in \varphi_A(Q \vee R) = \mathbf{V}((Q \vee R) \cap \mathbf{V}(A))$. By Lemma 1.1, $C \in (Q \vee R) \cap \mathbf{V}(A)$, so by Lemma 2.2 there are $D \in Q$, $E \in R$ with $C \in \mathbf{HS}(D \times E)$. Let $B = D \times E \times A$; then $A \in \mathbf{V}(B)$ and $C \in (Q \cap \mathbf{V}(B)) \vee (R \cap \mathbf{V}(B))$, so $C \in \varphi_B(Q) \vee \varphi_B(R)$. Thus $(*)$ is satisfied and $(**)$ follows at once from (ii); hence (iii) is proved.

For (iv) take $C \in Q - R$. If $C \in \mathbf{V}(B)$, then by Lemma 1.1,

$$C \in \mathbf{V}(Q \cap \mathbf{V}(B)) - \mathbf{V}(R \cap \mathbf{V}(B)).$$

Since $C \notin \varphi_B(R)$ for all B, we have $(\varphi(Q), \varphi(R)) \notin \gamma$.

We can summarize all we have gotten so far by

THEOREM 2.5. *If P is a pseudovariety then $\mathbf{L_{pv}}(P) \in \mathbf{HSP}\{\mathscr{K}(P)\}$; in fact $\mathbf{L_{pv}}(P)$ is embedded into S/γ via the function φ described above.*

COROLLARY 2.6. *For any lattice equation ε, if $\mathbf{L_v}(V)$ satisfies ε then $\mathbf{L_{pv}}(V_{\mathrm{fin}})$ satisfies ε.*

COROLLARY 2.7. *If $\mathbf{L_v}(\mathbf{V}(P))$ satisfies $SD_\wedge$ then so does $\mathbf{L_{pv}}(P)$.*

https://doi.org/10.1017/S1446788700030640 Published online by Cambridge University Press

PROOF. If $Q \wedge R_1 = Q \wedge R_2$, then for all $A \in P$

$$\mathbf{V}(Q \wedge R_1 \cap \mathbf{V}(A)) = \mathbf{V}(Q \wedge R_2 \cap \mathbf{V}(A)),$$

implying

$$\mathbf{V}(Q \cap \mathbf{V}(A)) \wedge \mathbf{V}(R_1 \cap \mathbf{V}(A)) = \mathbf{V}(Q \cap \mathbf{V}(A)) \wedge \mathbf{V}(R_2 \cap \mathbf{V}(A)).$$

Since $\mathbf{L}_{\mathbf{v}}(\mathbf{V}(P))$ satisfies $SD_\wedge$,

$$\mathbf{V}(Q\cap\mathbf{V}(A))\wedge\mathbf{V}(R_1\cap\mathbf{V}(A)) = \mathbf{V}(Q\cap\mathbf{V}(A))\wedge[\mathbf{V}(R_1\cap\mathbf{V}(A))\vee\mathbf{V}(R_2\cap\mathbf{V}(A))].$$

Hence

$$\begin{aligned}\varphi_A(Q \wedge R_1) &= \varphi_A(Q) \wedge [\varphi_A(R_1) \vee_A (R_2)]\\ &= \varphi_A(Q) \wedge [\varphi_A(R_1) \vee \varphi(R_2)]_A,\end{aligned}$$

so, by Lemma 2.4(iii)

$$\varphi(Q \wedge R_1) = \varphi(Q) \wedge [\varphi(R_1) \vee \varphi(R_2)]\, \gamma\, \varphi(Q) \wedge \varphi(R_1 \wedge R_2).$$

By Theorem 2.5, $Q \wedge R_1 = Q \wedge (R_1 \vee R_2)$ and $\mathbf{L}_{\mathbf{pv}}(P)$ satisfies $SD_\wedge$.

There is no similar argument for $SD_\vee$, and we conjecture that $SD_\vee$ and Lampe's condition [9], which is similar, are not preserved by $\mathbf{L}_{\mathbf{pv}}(V_{\mathrm{fin}})$.

We can refine Theorem 2.4, to obtain $\mathbf{L}_{\mathbf{pv}}(P) \in \mathbf{HSP_u}\{\mathscr{K}(P)\}$, by the following general argument.

Let $\mathscr{A} = (A_i)_{i\in I}$ be a family of algebras, $N = \prod(A_i \colon i \in I)$, $S \leq N$ and $\gamma \in \operatorname{Con} S$. Let $\mathscr{U}$ be an ultrafilter on I, and let μ be the induced congruence on N, that is $x\mu y$ if $\{i \colon x_i = y_i\} \in \mathscr{U}$. Let $T = \{t \in N\colon$ there exists $x \in S$ with $t\mu x\}$. Note that T is a subalgebra of N. Define $\rho \subseteq T^2$ by $(t, t') \in \rho$ if there are $x, x' \in S$ with $t\mu x\gamma x'\mu t'$.

LEMMA 2.8. *If $\mu \cap S^2 \subseteq \gamma$ then $\rho \in \operatorname{Con} T$ and $S/\gamma \cong T/\rho$. Consequently $S/\gamma \in \mathbf{HSP_u}\{\mathscr{A}\}$.*

PROOF. ρ is clearly reflexive and symmetric. If $t\rho t'\rho t''$, then there are $x, x', y', y'' \in S$ with $t\mu x\gamma x'\mu t'\mu y'\gamma y''\mu t''$. But then $x'\mu y'$, so by hypothesis $x'\gamma y'$. Hence $x\gamma x''$ and $t\rho t''$. Thus ρ is transitive. This relation clearly respects operations, so that $\rho \in \operatorname{Con} T$.

It is an easy exercise to prove that $\psi\colon S/\gamma \to T/\rho$ defined by $\psi(x/\gamma) = x/\rho$ is an isomorphism. Since $\mu \cap T^2 \subseteq \rho$, we have $T/\rho \in \mathbf{HSP_u}(\{\mathscr{A}\})$.

We want to apply Lemma 2.8 to the specific S and γ defined earlier in this section. For $B \in P$, define $\mathscr{S}(B) = \{A \in P\colon B \in \mathbf{V}(A)\}$. Note that $\{\mathscr{S}(B)\colon B \in P\}$ is a filterbase: $\mathscr{S}(B) \neq \varnothing$, and if $B, C \in P$ then $\mathscr{S}(B) \cap \mathscr{S}(C) \supseteq \mathscr{S}(D)$, where $D = B \times C$. Let $\mathscr{U}$ be any ultrafilter containing this filterbase, and let μ be the congruence on M associated to $\mathscr{U}$. We need the following facts.

LEMMA 2.9. (i) *For all* $E \in \mathscr{U}$, $B \in P$ *implies* $E \cap \mathscr{S}(B) \neq \varnothing$.
(ii) $\mu \cap S^2 \subseteq \gamma$.

PROOF. (i) is trivial from the maximality of $\mathscr{U}$. For (ii), let $(x, y) \in \mu \cap S^2$; then $\{C\colon x_C = y_C\} \in \mathscr{U}$. Fix $A \in P$. By (i), $\{C\colon x_C = y_C\} \cap \mathscr{S}(A) \neq \varnothing$ so there is a B such that $A \in \mathbf{V}(B)$ and $x_B = y_B$. If $D \in x_A$ then $x_A \leq x_B = y_B$ since $x \in S$, so $D \in y_B$. If $D \in y_A$, then $y_A \leq y_B = x_B$, so $D \in x_B$. Hence $x\gamma y$ and (ii) is proved.

Lemma 2.9 (ii) allows us to apply Lemma 2.8 to our situation, yielding $S/\gamma \in \mathbf{HSP_u}(\mathscr{K}(P))$. Since $\mathbf{L_{pv}}(P)$ is isomorphic to a sublattice of S/γ, we obtain $\mathbf{L_{pv}}(P) \in \mathbf{HSP_u}\{\mathscr{K}(P)\}$, which is the claim of Theorem 2.1.

COROLLARY 2.10. *Any positive universal sentence holding in* $\mathbf{L_v}(V)$ *is also satisfied by* $\mathbf{L_{pv}}(V_{\text{fin}})$.

References

[1] C. J. Ash, 'Pseudovarieties, generalized varieties and similarly described classes', *J. Algebra* **92** (1985), 104–115.

[2] J. Almeida, 'Power pseudovarieties of semigroups. I, II', *Semigroup Forum* **33** (1986), 357–373; 375–390.

[3] J. Almeida, 'The algebra of implicit operations', preprint.

[4] K. A. Baker and A. W. Hales, 'From a lattice to its ideal lattice', *Algebra Universalis* **4** (1974), 250–258.

[5] J. T. Baldwin and J. Berman, 'Varieties and finite closure conditions', *Colloq. Math.* **35** (1976), 15–20.

[6] B. Banaschewski, 'The Birkhoff Theorem for varieties of finite algebras', *Algebra Universalis* **17** (1983), 360–368.

[7] S. Eilenberg and M. P. Schützenberger, 'On pseudovarieties', *Adv. in Math.* **19** (1976), 413–418.

[8] T. E. Hall and K. G. Johnston, 'The lattice of pseudovarieties of inverse semigroups', *Pacific J. Math.*, to appear.

[9] W. A. Lampe, 'A property of the lattice of equational theories', *Algebra Universalis* **23** (1986), 61–69.

[10] R. McKenzie, 'Finite forbidden lattices', Proceedings of the Universal Algebra and Lattice Theory Conference, Puebla 1982 (Lecture Notes in Math., vol. 1004, Springer-Verlag, pp. 176–205).

[11] J. Reiterman, 'The Birkhoff theorem for finite algebras', *Algebra Universalis* **14** (1982), 1–10.

Department of Mathematics
University of Hawaii at Manoa
Honolulu
Hawaii 96822