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GENERIC GATEAUX DIFFERENTIABILITY VIA SMOOTH PERTURBATIONS

PANDO GR. GEORGIEV AND NADIA P. ZLATEVA

We prove that in a Banach space with a Lipschitz uniformly Gateaux smooth bump function, every continuous function which is directionally differentiable on a dense G_δ subset of the space, is Gateaux differentiable on a dense G_δ subset of the space. Applications of this result are given.

The usual applications of variational principles in Banach spaces are to differentiability of real valued functions. For example, the papers [**1**] and [**2**] contain results about Gateaux differentiability on dense sets. An application of Ekeland's variational principle to generic Frechet differentiability is given in the proof of the famous Ekeland-Lebourg theorem (see [**4**]). In [**6**] an application of the smooth variational principle to generic Gateaux differentiability is presented.

In this paper we prove some results about generic Gateaux differentiability of directionally differentiable functions. The tool for proving the main result (Theorem 2) is Proposition 1, which localises precisely the δ-minimum point of the perturbed function. The estimate of this localisation is the same as in the Ekeland variational principle.

Denote by L_f the Lipschitz constant of a Lipschitz function $f : E \to \mathbf{R}$ and by S, $B[x;r]$ (respectively $B(x;r)$) - the unit sphere of E and the closed (respectively open) ball with center x and radius r.

A function $b : E \to \mathbf{R}$ is said to be a *bump function*, if there exists a bounded subset *supp* b, such that $b(x) = 0$ for every $x \notin$ *supp* b.

PROPOSITION 1. *Let b be a bump function, such that supp $b \subset B(0,1)$, $b(0) = 1$ and $0 \leqslant b(x) \leqslant 1$ $\forall x \in E$. Let $f : E \to \mathbf{R} \cup \{+\infty\}$ be a function, bounded below and such that $D(f) = \{x \in E \ : \ f(x) < +\infty\} \neq \emptyset$. Let $\varepsilon > 0$, $\lambda > 0$ be given. Suppose that $y_0 \in X$ satisfies the condition*

$$f(y_0) < \inf_E f + \varepsilon.$$

Then for every $\delta > 0$ there exists a point $x_0 \in E$, such that:

(a) $\quad f(x_0) - \varepsilon b((x_0 - y_0)/\lambda) < \inf_E \{f(x) - \varepsilon b((x - y_0)/\lambda)\} + \delta;$

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(b) $(x_0 - y_0)/\lambda \in supp\ b$;

(c) $\|x_0 - y_0\| < \lambda$.

PROOF: Denote
$$h(x) = f(x) - \varepsilon b\Big(\frac{x - y_0}{\lambda}\Big).$$
Then
$$h(y_0) = f(y_0) - \varepsilon < \inf_E f.$$
Let
$$\delta_1 = \inf_E f - h(y_0).$$
There exists a point $x_0 \in E$ such that
$$h(x_0) < \inf_E h + \min\{\delta, \delta_1\}.$$

Assume that $(x_0 - y_0)/\lambda \notin supp\ b$. Then using that $b((x_0 - y_0)/\lambda) = 0$ we have that
$$\delta_1 > h(x_0) - h(y_0) = f(x_0) - \inf_E f + \delta_1 \geqslant \delta_1,$$
which is a contradiction. So (b) is satisfied. It is clear that (c) follows immediately from (b). ∎

Now we suppose that the bump function b is *uniformly Gateaux differentiable.* This means
$$\lim_{t \to 0} \frac{b(x + th) - b(x)}{t} = \langle \nabla b(x), h \rangle,$$
where $\nabla b(x) \in E^*$ is the Gateaux derivative of b at x, and for every $h \in E$ this limit is uniform with respect to the points $x \in E$.

Recall that the function $f : E \to \mathbf{R}$ is said to be *directionally differentiable at point* x_0 if for every $h \in E$ the one-sided directional derivative
$$f'(x_0; h) = \lim_{t \downarrow 0} \frac{f(x_0 + th) - f(x_0)}{t}$$
exists.

The following theorem extends the main result in [**6**]. Problems in a similar setting are considered in [**9, 5**].

THEOREM 2. *Let the Banach space E admit a Lipschitz uniformly Gateaux differentiable bump function $\tilde{b}$. Then every continuous function f, defined on an open subset $D \subset E$, which is directionally differentiable on a dense G_δ subset G of D, is Gateaux differentiable on a dense G_δ subset of D.*

PROOF: From [**9**, Proposition 2.1] it follows that f is locally Lipschitz on a dense and open subset D_1 of D. Let $U \subset D_1$ be an open subset, such that f is Lipschitz on U. If we prove that f is Gateaux differentiable on a dense G_δ subset of U, then the theorem

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will be proved, having in mind the localisation principle (see [7, Chapter I, Section 10, VI]), stating that a subset P of a topological space is of first Baire category, if for every point $p \in P$ there exists an open set $H \ni p$ such that $P \cap H$ is of first Baire category in H.

Without loss of generality we can assume that $\widetilde{b}(0) \neq 0$. Define the bump function $b : E \to \mathbf{R}$ by

$$b(x) := \tau\big(\widetilde{b}(dx)\big),$$

where $d = \sup_{s \in supp\, \widetilde{b}} \|s\|$, $\tau : \mathbf{R} \to \mathbf{R}^+$ is a differentiable bump function with Lipschitz derivative, such that $\tau\big(\widetilde{b}(0)\big) = \max_{t \in \mathbf{R}} \tau(t) = 1$, $0 \notin supp\ \tau$. Then b is a Lipschitz uniformly Gateaux differentiable bump function, such that $supp\ b \subset B(0;1)$, $b(0) = 1$, $0 \leqslant b(x) \leqslant 1 \ \forall x \in E$.

Define the sets

$$\begin{aligned} X_n \ := \ \Big\{ x \in U : \exists x_n \in E,\ \exists t_n \in \Big(0, \frac{1}{n^2}\Big) : \frac{x - x_n}{\sqrt{t_n}} \in supp\ b,\ B[x_n, 2\sqrt{t_n}] \subset U, \\ f(x) - 2\sqrt{t_n}\, b\Big(\frac{x - x_n}{\sqrt{t_n}}\Big) < \inf_{z \in B[x_n, 2\sqrt{t_n}]} \Big\{ f(z) - 2\sqrt{t_n}\, b\Big(\frac{z - x_n}{\sqrt{t_n}}\Big)\Big\} + t_n^2 \ \Big\}. \end{aligned}$$

Since f and b are continuous functions, the sets X_n are open. We shall prove that X_n is dense in U.

Let $x^* \in U$ and $\varepsilon_0 > 0$ be fixed. For every $n \geqslant 1$, choose $\varepsilon \in (0,\ \min\{\varepsilon_0,\ 1/n\})$ in such a way that $B[x^*; \varepsilon] \subset U$. We may assume without loss of generality that the Lipschitz constant L_f of f on U is less than 1. Then

$$f(x^*) \leqslant \inf_{z \in B[x^*, \varepsilon]} \{ f(z) + L \|x^* - z\| \} \ < \ \inf_{z \in B[x^*, \varepsilon]} f(z) + \varepsilon$$

and we can apply Proposition 1 with $\lambda = \varepsilon/2$, $\delta = \varepsilon^4/16$ for the lower semicontinuous function f_0 defined by

$$f_0(x) = \begin{cases} f(x) & x \in B[x^*, \varepsilon], \\ +\infty & x \notin B[x^*, \varepsilon]. \end{cases}$$

Therefore there exists a point $y_n \in X$ such that

$$f(y_n) - \varepsilon b\Big(2\frac{y_n - x^*}{\varepsilon}\Big) < \inf_{z \in B[x^*, \varepsilon]} \Big\{ f(z) - \varepsilon b\Big(2\frac{z - x^*}{\varepsilon}\Big)\Big\} + \frac{\varepsilon^4}{2^4};$$

$$\|y_n - x^*\| < \frac{\varepsilon}{2};$$

$$2(y_n - x^*)/\varepsilon \in supp\ b.$$

Then, for $t_n = \varepsilon^2/4$, $x_n = x^*$ we have $y_n \in X_n$ and $y_n \in B[x^*; \varepsilon_0]$, hence the denseness is proved.

It is clear that in the same way we may construct dense and open sets X_n', coresponding to the function $-f$. From the Baire category theorem the set $X_0 = \Big(\bigcap_{n=1}^{\infty} X_n\Big) \cap \Big(\bigcap_{n=1}^{\infty} X_n'\Big) \cap G$ is dense and G_δ in U.

We shall prove that f is Gateaux differentiable on X_0.

Let $x_0 \in X_0$. By assumption, for every $\varepsilon > 0$ and $h \in E$ there exists $\delta \in (0, \varepsilon)$ such that

$$\frac{b(x+th)-b(x)}{t} - \langle \nabla b(x), h \rangle > \frac{-\varepsilon}{2}$$

for every $x \in supp\ b$ and for every $t \in (0, \delta)$.

For every $\varepsilon > 0$ and such δ, for $n > 1/\delta$ and $h \in S$, since

$$\|x_0 + t_n h - x_n\| \leqslant \|x_0 - x_n\| + t_n < \sqrt{t_n} + \sqrt{t_n} = 2\sqrt{t_n},$$

we have $x_0 + t_n h \in B[x_n, 2\sqrt{t_n}] \subset U$. Then

$$\begin{aligned}\frac{f(x_0+t_nh)-f(x_0)}{t_n} &\geqslant \frac{2\sqrt{t_n}}{t_n}\Big(b\Big(\frac{x_0+t_nh-x_n}{\sqrt{t_n}}\Big) - b\Big(\frac{x_0-x_n}{\sqrt{t_n}}\Big)\Big) - t_n \\ &\geqslant 2\Big(\Big\langle \nabla b\Big(\frac{x_0-x_n}{\sqrt{t_n}}\Big), h\Big\rangle - \frac{\varepsilon}{2}\Big) - t_n \\ &\geqslant \Big\langle 2\nabla b\Big(\frac{x_0-x_n}{\sqrt{t_n}}\Big), h\Big\rangle - \varepsilon - t_n.\end{aligned}$$

Since b is Lipschitz, $\left\|\nabla b\big((x_0 - x_n)/\sqrt{t_n}\big)\right\| \leqslant L_b$ and we can choose a w^*-converging generalised subsequence from the sequence $\left\{\nabla b\big((x_0 - x_n)/\sqrt{t_n}\big)\right\}_{n\geqslant 1}$, whose w^*-limit is denoted by $b_1^*/2$.

After passing to limits, we obtain

$$f'(x_0; h) \geqslant \langle b_1^*, h\rangle - \varepsilon$$

and since this is valid for every $\varepsilon > 0$ and $h \in S$ we have

$$f'(x_0; h) \geqslant \langle b_1^*, h\rangle, \quad \forall h \in E.$$

By repeating this reasoning for the function $(-f)$ we obtain that there exists $b_2^* \in E^*$ such that

$$\langle b_2^*, h\rangle \geqslant f'(x_0; h) \geqslant \langle b_1^*, h\rangle, \quad \forall h \in E.$$

Hence $b_1^* = b_2^* = \nabla f(x_0)$ and the proof is completed. ∎

Since the convex functions are directionally differentiable in the interior of their domain, the above theorem implies that Banach spaces with a uniformly Gateaux differentiable Lipschitz bump function are weak Asplund spaces.

In the following Lemma we shall show that a Banach space with uniformly Gateaux differentiable norm admits a Lipschitz uniformly Gateaux differentiable bump function.

Lemma 3. *Let the Banach space E have a uniformly Gateaux differentiable norm. Then E admits a Lipschitz uniformly Gateaux differentiable bump function.*

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PROOF: Let $\|.\|$ be an uniformly Gateaux differentiable norm on E and let $r : \mathbf{R} \to \mathbf{R}$ be a function with Lipschitz derivative, such that $r = 0$ on $(-\infty, 1] \cup [3, +\infty)$ and $r(2) \neq 0$. Denote by $L_{r'}$ the Lipschitz constant of r' and $|r'| = \sup_{t \in [1,3]} | r'(t) |$. Then the function

$$b(x) := r(\|x\|)$$

is Lipschitz and uniformly Gateaux differentiable. Indeed, let $h \in S$, $\varepsilon > 0$ be fixed and $0 < \gamma < \varepsilon/(2\,|r'|)$. There exists $t_0 < \varepsilon/L_{r'}$ such that for every $0 < t < t_0$ and $y \in S$

$$\frac{\|y + th\| - 1}{t} - \langle \|y\|', h \rangle < \gamma.$$

From the definition of the function r it is clear that it is enough to restrict our consideration only to $x \in E$ such that $1 \leqslant \|x\| \leqslant 3$. Let $t \in (0, t_0/2)$. By the mean value theorem we have: (where $\alpha = \alpha(x,t)$ is between $\|x\|$ and $\|x + th\|$)

$$\begin{aligned}
&\left| \frac{b(x+th) - b(x)}{t} - \langle b'(x), h \rangle \right| \\
&\qquad = \left| \frac{r(\|x+th\|) - r(\|x\|)}{t} - r'(\|x\|)\langle \|x\|', h \rangle \right| \\
&\qquad = \left| r'(\alpha) \frac{\|x+th\| - \|x\|}{t} - r'(\|x\|)\langle \|x\|', h \rangle \right| \\
&\qquad = \left| r'(\alpha) \left[\frac{\left\| \frac{x}{\|x\|} + \frac{t}{\|x\|} h \right\| - 1}{\frac{t}{\|x\|}} \pm \langle \|x\|', h \rangle \right] - r'(\|x\|)\langle \|x\|', h \rangle \right| \\
&\qquad \leqslant |r'|\gamma + L_{r'} \mid \alpha - \|x\| \mid \leqslant |r'|\gamma + \frac{t_0}{2} L_{r'} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

∎

Since every separable Banach space admits a uniformly Gateaux differentiable norm (see [**2**, Corollary 6.9 (i))], for such spaces Theorem 2 is valid.

Now we shall give some applications.

Let A be an arbitrary non-void set and $\{g_\alpha : E \to \mathbf{R},\ \alpha \in A\}$ be a family of functions. Define the function $f : E \to \mathbf{R}$ by $f(x) = \inf_{\alpha \in A} g_\alpha(x)$.

We need the following lemma.

LEMMA 4. *Let the family $\{g_\alpha : E \to \mathbf{R},\ \alpha \in A\}$ be such that*

(i) *for every $\alpha \in A$, g_α is K-Lipschitz;*

(ii) *for every $x \in E$, $h \in S$, there exists $\varepsilon > 0$ such that the limit*

$$g'_\alpha(x; h) = \lim_{t \downarrow 0} \frac{g_\alpha(x+th) - g_\alpha(x)}{t}$$

exists and it is uniform with respect to $\alpha \in M_\varepsilon(x)$, where

$$M_\varepsilon(x) := \{\alpha \in A \ : \ g_\alpha(x) \leqslant f(x) + \varepsilon\}.$$

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Then the function $f(x) = \inf_{\alpha \in A} g_\alpha(x)$ *is K-Lipschitz,* $f'(x;h)$ *exists and it is equal to* $\sup_{\varepsilon>0} \inf_{\alpha \in M_\varepsilon(x)} g'_\alpha(x;h)$.

PROOF: First let us note that

$$\forall x_0 \in E,\ \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } M_\delta(x) \subset M_\varepsilon(x_0),\ \forall x \in B(x_0;\delta). \tag{1}$$

Indeed, from (i) we have $|g_\alpha(x) - g_\alpha(x_0)| \leqslant K\|x - x_0\|$, for every $\alpha \in A$ and $|f(x) - f(x_0)| \leqslant K\|x - x_0\|$. Then, for $\delta = \min\{\varepsilon/4K, \varepsilon/2\}$, $x \in B(x_0;\delta)$ and $\alpha \in M_\delta(x)$ we have

$$\begin{aligned} g_\alpha(x_0) &\leqslant g_\alpha(x) + K\|x - x_0\| \leqslant f(x) + \delta + K\|x - x_0\| \\ &\leqslant f(x_0) + \delta + 2K\|x - x_0\| \leqslant f(x_0) + \varepsilon, \end{aligned}$$

which means $\alpha \in M_\varepsilon(x_0)$, and (1) is proved.

Let $x \in E$ be fixed. It is enough to consider only the case when $\|h\| = 1$. Denote $a := \sup_{\varepsilon>0} \inf_{\alpha \in M_\varepsilon(x)} g'_\alpha(x;h)$. From (ii), for every $\varepsilon > 0$, there exist $\varepsilon_0 > 0$ and $t_0 = t_0(\varepsilon)$ such that

$$\left|\frac{g_\alpha(x+th) - g_\alpha(x)}{t} - g'_\alpha(x;h)\right| < \varepsilon/3$$

for every $\alpha \in M_{\varepsilon_0}(x)$ and $t \in (0, t_0)$.

Let $t_1 \in (0, t_0)$ and $0 < \varepsilon_1 < \min\{t_1\varepsilon/3, \varepsilon_0\}$. There exists $\alpha_1 \in M_{\varepsilon_1}(x)$ such that

$$\inf_{\alpha \in M_{\varepsilon_1}(x)} g'_\alpha(x;h) > g'_{\alpha_1}(x;h) - \varepsilon/3.$$

Since $\varepsilon_1 < \varepsilon_0$, we have $M_{\varepsilon_1}(x) \subset M_{\varepsilon_0}(x)$ and:

$$\begin{aligned} \frac{f(x+t_1h) - f(x)}{t_1} &\leqslant \frac{g_{\alpha_1}(x+t_1h) - g_{\alpha_1}(x) - \varepsilon_1}{t_1} \\ &< g'_{\alpha_1}(x;h) + \frac{\varepsilon}{3} + \frac{\varepsilon_1}{t_1} \\ &< \inf_{\alpha \in M_{\varepsilon_1}(x)} g'_\alpha(x;h) + \varepsilon \leqslant a + \varepsilon. \end{aligned}$$

Hence

$$\limsup_{t \downarrow 0} \frac{f(x+th) - f(x)}{t} \leqslant a + \varepsilon$$

and since $\varepsilon > 0$ is arbitrary small,

$$\limsup_{t \downarrow 0} \frac{f(x+th) - f(x)}{t} \leqslant a.$$

Now we shall prove the oposite inequality. Let $\varepsilon_3 > 0$ be such that $\inf_{\alpha \in M_{\varepsilon_3(x)}} g'_\alpha(x;h) > a - \varepsilon/3$. From (1), there exists $\delta > 0$ such that $M_\delta(x') \subset M_{\varepsilon_3}(x)$ whenever $\|x' - x\| < \delta$.

Let $0 < t_2 < \min\{t_0, \delta\}$, $0 < \varepsilon_2 < \min\{t_2\varepsilon/3, \varepsilon_0, \delta\}$ and $\alpha_2 \in M_{\varepsilon_2}(x + t_2h)$. Then $M_{\varepsilon_2}(x + t_2h) \subset M_\delta(x + t_2h) \subset M_{\varepsilon_3}(x)$ and we can write:

$$\begin{aligned}\frac{f(x+t_2h)-f(x)}{t_2} &\geqslant \frac{g_{\alpha_2}(x+t_2h)-g_{\alpha_2}(x)-\varepsilon_2}{t_2}\\ &\geqslant g'_{\alpha_2}(x;h)-\frac{\varepsilon}{3}-\frac{\varepsilon_2}{t_2}\\ &> \inf\{g'_\alpha(x;h):\alpha\in M_{\varepsilon_2}(x+t_2h)\}-2\frac{\varepsilon}{3}\\ &\geqslant \inf_{\alpha\in M_{\varepsilon_3}(x)} g'_\alpha(x;h)\ -2\frac{\varepsilon}{3} > a-\varepsilon.\end{aligned}$$

Hence $\liminf_{t\downarrow 0}(f(x+th)-f(x))/t \geqslant a-\varepsilon$ and since $\varepsilon > 0$ is arbitrary small, $\liminf_{t\downarrow 0}(f(x+th)-f(x))/t \geqslant a$ and the proof is completed. ∎

If A is a non-empty subset of a E, the *distance function* to A is defined by $dist(x, A) := \inf_{a\in A}\|x-a\|$.

Corollary 5. *Let the norm of the Banach space E be such that $\forall h \in S$, $\forall\varepsilon > 0$, $\exists\delta > 0$:*

$$\frac{\|x+th\|-\|x\|}{t} - \|.\|'(x;h) < \varepsilon,\ \forall t\in(0,\delta),\ \forall x\in S,$$

where S is the unit sphere of E. Then for every non-empty closed subset $A \subset E$ the distance function $dist(., A)$ is directionally differentiable on $E \setminus A$.

The following theorem is a direct consequence of Theorem 2 and Lemma 4, and can be considered as a generalisation of a "Gateaux" version of the Ekeland-Lebourg theorem [4] (see also [8], where another variant is announced).

Theorem 6. *If the Banach space E admits a Lipschitz uniformly Gateaux differentiable bump function, then the function $f(x) = \inf_{\alpha\in A} g_\alpha(x)$, where for the family $\{g_\alpha,\ \alpha \in A\}$ the assumptions* (i) *and* (ii) *from Lemma* 4 *hold, is Gateaux differentiable on a dense G_δ subset of E.*

Recall that from an uniformly Gateaux differentiable norm we can construct a uniformly Gateaux differentiable Lipschitz bump function (see Lemma 3) and therefore, we have the following fact, announced in [8].

Corollary 7. *In a Banach space E with an uniformly Gateaux differentiable norm, any distance function is Gateaux differentiable on a dense G_δ subset of E.*

References

[1] J.M. Borwein and D. Preiss, 'A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions', *Trans. Amer. Math. Soc.* **303** (1987), 517–527.

https://doi.org/10.1017/S0004972700031208 Published online by Cambridge University Press

[2] R. Deville, G. Godefroy and V. Zizler, 'A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions', *J. Funct. Anal.* **111** (1993), 197–212.

[3] I. Ekeland, 'Nonconvex minimization problems', *Bull. Amer. Math. Soc.* **1** (1979), 443–474.

[4] I. Ekeland and G. Lebourg, 'Generic Frechet differentiability and perturbed optimization problems in Banach spaces', *Trans. Amer. Math. Soc.* **224** (1976), 193–216.

[5] M. Fabian and D. Preiss, 'On intermediate differentiability of Lipschitz functions on certain spaces', *Proc. Amer. Math. Soc.* **113** (1991), 733–740.

[6] P.G. Georgiev, 'The smooth variational principle and generic differentiability', *Bull. Austral. Math. Soc.* **43** (1991), 169–175.

[7] K. Kuratowski, *Topology I* (Academic Press, New York and London, 1966).

[8] L. Zajicek, 'A generalization of an Ekeland-Lebourg theorem and the differentiability of distance functions', in *Proc. 11-th Winter School on Abstract Analysis*, Suppl. Rend. Circolo Mat. di Palermo, Ser. II **3**, 1984.

[9] N.V. Zhivkov, 'Generic Gateaux differentiability of directionally differentiable mappings', *Rev. Roumaine Math. Pures Appl.* **32** (1987), 179–188.

Department of Mathematics and Informatics
University of Sofia
5 James Bourchier Blvd.
1126 Sofia
Bulgaria