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Abstract

We prove a conjecture of Kontsevich, which asserts that the iterations of the non-commutative rational map $F_r : (x, y) \to (xyx^{-1}, (1 + y^r)x^{-1})$ are given by non-commutative Laurent polynomials with non-negative integer coefficients.

1. Introduction

Let $K = k(x, y)$ be the skew field of rational functions in the non-commutative variables x and y, where the ground field k is $\mathbb{Q}$ or any field containing $\mathbb{Q}$, for example $\mathbb{Q}(q)$. For any positive integer r, let F_r be the Kontsevich automorphism of K, which is defined by

$$F_r(\lambda) = \lambda \quad \text{for all } \lambda \in k \quad \text{and} \quad F_r : \begin{cases} x \mapsto xyx^{-1}, \\ y \mapsto (1 + y^r)x^{-1}. \end{cases} \tag{1}$$

For more information, see [Kon11].

The main achievement of this paper is the proof of a special case of the following conjecture.

Conjecture 1 (Kontsevich). For all positive integers r_1, r_2 and for all $m \geqslant 0$, the expressions

$$(F_{r_2} \circ F_{r_1})^m(x) \quad \text{and} \quad (F_{r_2} \circ F_{r_1})^m(y)$$

are non-commutative Laurent polynomials in x and y with non-negative integer coefficients.

We shall prove the conjecture in the $r_1 = r_2$ case by providing an explicit combinatorial formula for these expressions as a sum over certain sets of lattice paths β, where each summand is a Laurent monomial given by the weight of the paths in β. As a direct consequence of this formula, we have the following result.

Theorem 1.1. *Conjecture 1 holds whenever $r_1 = r_2$.*

Let us point out that if the variables x and y were commutative, then the automorphism F_r would describe precisely the exchange relations for the mutations in a skew-symmetric cluster algebra $\mathscr{A}_r$ of rank 2, and our above-mentioned formula would be a non-commutative version of a formula for the cluster variables in $\mathscr{A}_r$ that we obtained earlier; see [LS12]. Our non-commutative formula also represents (a slight modification of) the generators of the non-commutative rank-2 cluster algebra introduced by DiFrancesco and Kedem in [DK11, § 8].

In the special cases where $(r_1, r_2) = (2, 2)$, $(4, 1)$ or $(1, 4)$, the conjecture has been proved by DiFrancesco and Kedem in [DK10]. Moreover, the expressions in Conjecture 1 were shown to be

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Laurent polynomials for any choice of (r_1, r_2) by Berenstein and Retakh [BR11] and, earlier, in the $r_1 = r_2$ case by Usnich [Usn10].

2. Main result

Fix a positive integer $r \geqslant 2$.

Definition 2.1. Let $\{c_n\}$ be the sequence defined by the recurrence relation

$$c_n = rc_{n-1} - c_{n-2}$$

with the initial condition $c_1 = 0,\ c_2 = 1$. When $r = 2,\ c_n = n - 1$. When $r > 2$, it is easy to see that

$$\begin{aligned} c_n &= \frac{1}{\sqrt{r^2-4}}\left(\frac{r+\sqrt{r^2-4}}{2}\right)^{n-1} - \frac{1}{\sqrt{r^2-4}}\left(\frac{r-\sqrt{r^2-4}}{2}\right)^{n-1} \\ &= \sum_{i\geqslant 0}(-1)^i\binom{n-2-i}{i}r^{n-2-2i}. \end{aligned}$$

For example, for $r = 3$, the sequence c_n takes the following values:

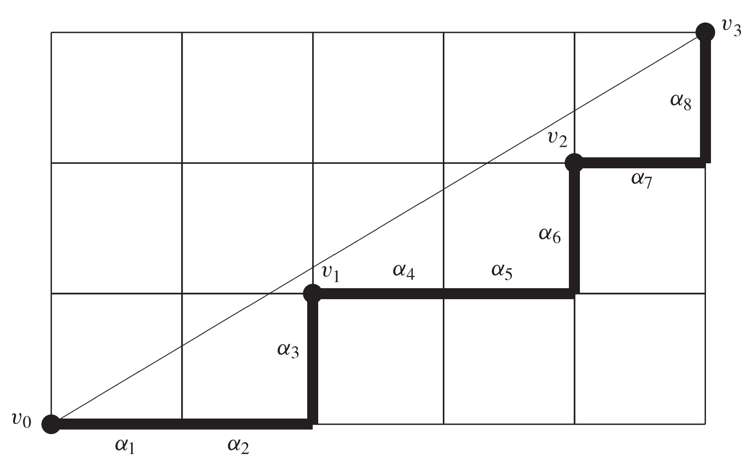
$$0, 1, 3, 8, 21, 55, 144, \ldots.$$

In order to state our theorem, we fix an integer $n \geqslant 4$. Consider a rectangle with vertices $(0, 0)$, $(0, c_{n-2})$, $(c_{n-1} - c_{n-2}, c_{n-2})$ and $(c_{n-1} - c_{n-2}, 0)$. In what follows, by the diagonal we mean the line segment from $(0, 0)$ to $(c_{n-1} - c_{n-2}, c_{n-2})$. A Dyck path is a lattice path from $(0, 0)$ to $(c_{n-1} - c_{n-2}, c_{n-2})$ that proceeds by *north* or *east* steps and never goes above the diagonal.

Definition 2.2. A Dyck path below the diagonal is said to be maximal if no subpath of any other Dyck path lies above it. The maximal Dyck path, denoted by $\mathscr{D}_n$, consists of $(w_0, \alpha_1, w_1, \ldots, \alpha_{c_{n-1}}, w_{c_{n-1}})$, where $w_0, \ldots, w_{c_{n-1}}$ are vertices and $\alpha_1, \ldots, \alpha_{c_{n-1}}$ are edges, such that $w_0 = (0, 0)$ is the south-west corner of the rectangle, α_i connects w_{i-1} and w_i, and $w_{c_{n-1}} = (c_{n-1} - c_{n-2}, c_{n-2})$ is the north-east corner of the rectangle.

Remark 1. The word obtained from $\mathscr{D}_n$ by forgetting the vertices w_i and replacing each horizontal edge by the letter x and each vertical edge by the letter y is (by definition) the Christoffel word of slope $c_{n-2}/(c_{n-1} - c_{n-2})$.

Example 1. Let $r = 3$ and $n = 5$. Then $\mathscr{D}_5$ is illustrated as follows.



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DEFINITION 2.3. Let v_i be the upper endpoint of the ith vertical edge of $\mathscr{D}_n$. More precisely, let $i_1 < \cdots < i_{c_{n-2}}$ be the sequence of integers such that α_{i_j} is vertical for any $1 \leqslant j \leqslant c_{n-2}$. Define a sequence $v_0, v_1, \ldots, v_{c_{n-2}}$ of vertices by $v_0 = (0, 0)$ and $v_j = w_{i_j}$.

We introduce certain special subpaths called colored subpaths. These colored subpaths are defined by certain slope conditions as follows.

DEFINITION 2.4. For any $i < j$, let $s_{i,j}$ be the slope of the line through v_i and v_j. Let s be the slope of the diagonal, that is, $s = s_{0,c_{n-2}}$.

DEFINITION 2.5 (Colored subpaths). For any $0 \leqslant i < k \leqslant c_{n-2}$, let $\alpha(i, k)$ be the subpath of $\mathscr{D}_n$ defined as follows (for illustrations see Example 2).

(1) If $s_{i,t} \leqslant s$ for all t such that $i < t \leqslant k$, then let $\alpha(i, k)$ be the subpath from v_i to v_k. Each of these subpaths will be called a *blue* subpath; see Example 2.

(2) If $s_{i,t} > s$ for some $i < t \leqslant k$, then:

(2a) if the smallest such t is of the form $i + c_m - wc_{m-1}$ for some integers $3 \leqslant m \leqslant n-1$ and $1 \leqslant w < r-1$, then let $\alpha(i, k)$ be the subpath from v_i to v_k; each of these subpaths will be called a *green* subpath, and when m and w are specified we will say that the subpath is (m, w)-green;

(2b) otherwise, let $\alpha(i, k)$ be the subpath from the immediate predecessor of v_i to v_k; each of these subpaths will be called a *red* subpath.

Note that every pair (i, k) defines exactly one subpath $\alpha(i, k)$. We call these subpaths the *colored subpaths* of $\mathscr{D}_n$. We denote the set of all these subpaths together with the single edges α_i by $\mathscr{P}(\mathscr{D}_n)$, that is,

$$\mathscr{P}(\mathscr{D}_n) = \{\alpha(i, k) \mid 0 \leqslant i < k \leqslant c_{n-2}\} \cup \{\alpha_1, \ldots, \alpha_{c_{n-1}}\}.$$

Now we define a set $\mathscr{F}(\mathscr{D}_n)$ of certain sequences of non-overlapping subpaths of $\mathscr{D}_n$. This set will parametrize the monomials in our expansion formula.

DEFINITION 2.6. Let $\mathscr{F}(\mathscr{D}_n)$ be the collection of all sets $\{\beta_1, \ldots, \beta_t\}$ such that:

- $t \geqslant 0$ and $\beta_j \in \mathscr{P}(\mathscr{D}_n)$ for all $1 \leqslant j \leqslant t$;
- if $j \neq j'$, then β_j and $\beta_{j'}$ have no common edge;
- if $\beta_j = \alpha(i, k)$ and $\beta_{j'} = \alpha(i', k')$, then $i \neq k'$ and $i' \neq k$;
- if β_j is (m, w)-green, then at least one of the $c_{m-1} - wc_{m-2}$ preceding edges of v_i is contained in some $\beta_{j'}$.

For each $\beta \in \mathscr{F}(\mathscr{D}_n)$, we say that α_i is *supported on* β if and only if $\alpha_i \in \beta$ or α_i is contained in some blue, green or red subpath $\beta_j \in \beta$. The *support* of β, denoted by $\mathrm{supp}(\beta)$, is defined to be the union of the α_i that are supported on β.

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Definition 2.7. For each $\beta \in \mathscr{F}(\mathscr{D}_n)$ and each $i \in \{1, \ldots, c_{n-1}\}$, let

$$\beta_{[i]} = \begin{cases} x^{-1}y^{r} & \text{if } \alpha_i \text{ is not supported on } \beta \text{ and } \alpha_i \text{ is horizontal;} \\ x^{-1}y^{r-1} & \text{if } \alpha_i \text{ is not supported on } \beta \text{ and } \alpha_i \text{ is vertical;} \\ x^{-1}y^{0} & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is horizontal;} \\ x^{-1}y^{-1} & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is vertical;} \\ x^{0}y^{0} & \text{if } \alpha_i \text{ is horizontal and } \alpha_i \in \alpha(j,k) \in \beta \text{ for some } j,k; \\ x^{0}y^{-1} & \text{if } \alpha_i \text{ is vertical, } \alpha_{i-r+1} \text{ is horizontal,} \\ & \text{and } \alpha_i, \alpha_{i-r+1} \in \alpha(j,k) \in \beta \text{ for some } j,k; \\ x^{1}y^{-1} & \text{if } \alpha_i \text{ and } \alpha_{i-r+1} \text{ are vertical and } \alpha_i, \alpha_{i-r+1} \in \alpha(j,k) \in \beta \text{ for some } j,k; \\ x^{-1}y^{-1} & \text{if } \alpha_i \text{ is the first (vertical) edge of a red subpath } \alpha(j,k) \text{ in } \beta. \end{cases}$$

Note that the last three cases exhaust all possibilities for α_i being a vertical edge contained in some $\alpha(j,k)$ in β, because if in addition $\alpha_{i-r+1} \notin \alpha(j,k)$, then α_i must be the first vertical edge of a red subpath.

Recall from the introduction that the Kontsevich automorphism F_r is given by

$$F_r : \begin{cases} x \mapsto xyx^{-1}, \\ y \mapsto (1+y^r)x^{-1}. \end{cases} \tag{2}$$

Let F_r^{-1} be the inverse of F_r, namely,

$$F_r^{-1} : \begin{cases} x \mapsto (1+x^r)y^{-1}, \\ y \mapsto yxy^{-1}. \end{cases} \tag{3}$$

Consider a sequence $\{r_n\}_{n\in\mathbb{Z}}$ of positive integers. For any positive integer n, let

$$x_n = (F_{r_n} \circ \cdots \circ F_{r_2} \circ F_{r_1})(x) = F_{r_n}(\cdots F_{r_2}(F_{r_1}(x))\cdots)$$

and

$$y_n = (F_{r_n} \circ \cdots \circ F_{r_2} \circ F_{r_1})(y),$$

and let

$$x_{-n} = (F^{-1}_{r_{-n+1}} \circ \cdots \circ F^{-1}_{r_{-1}} \circ F^{-1}_{r_0})(x) \quad \text{and} \quad y_{-n} = (F^{-1}_{r_{-n+1}} \circ \cdots \circ F^{-1}_{r_{-1}} \circ F^{-1}_{r_0})(y).$$

Let $x_0 = x$ and $y_0 = y$.

Conjecture 1 (Kontsevich). Let r_1 and r_2 be arbitrary positive integers. Assume that $r_{2i+1} = r_1$ and $r_{2i} = r_2$ for every $i \in \mathbb{Z}$. Then, for any integer n, both x_n and y_n are non-commutative Laurent polynomials of x and y with non-negative integer coefficients.

We are now ready to state our main result. For monomials A_i in K, we let $\prod_{i=1}^m A_i$ denote the non-commutative product $A_1A_2\cdots A_m$.

Theorem 2.1. *If $r_n = r$ for all n, then for $n \geqslant 4$,*

$$x_{n-1} = \sum_{\beta \in \mathscr{F}(\mathscr{D}_n)} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1}. \tag{4}$$

Corollary 2.8. *Conjecture 1 holds in the case where $r_1 = r_2$.*

Proof. The theorem implies that x_n, for $n \geqslant 0$, is a non-commutative Laurent polynomial of x and y with non-negative integer coefficients. The statements for x_n with $n < 0$ and y_n then follow from a symmetry argument; see [DK10, § 2.3] or [BR11, Lemma 7]. □

Remark 1. The right-hand side of equation (4) can be written as a double sum as follows:

$$x_{n-1} = \sum_{i_j,k_j} \sum_{\beta} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1}, \tag{5}$$

where the first sum is over all sequences $0 \leqslant i_1 < k_1 < \cdots < i_\ell < k_\ell \leqslant c_{n-2}$ and the second sum is over all $\beta \in \mathscr{F}(\mathscr{D}_n)$ whose colored subpaths are precisely the $\alpha(i_j, k_j)$ for $1 \leqslant j \leqslant \ell$.

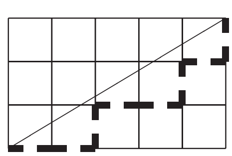
Example 2. Let $r = 3$ and $n = 5$. We use the following presentation for monomials in K:

$$x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_{m-1}}y^{b_{m-1}}x^{a_m}y^{b_m} \longleftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_{m-1} & a_m \\ b_1 & b_2 & \cdots & b_{m-1} & b_m \end{pmatrix}.$$

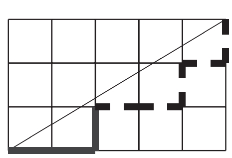
These expressions are not necessarily minimal, i.e. some of the a_i or b_i are allowed to be zero.

The illustrations below show the possible configurations for $\beta \in \mathscr{F}(\mathscr{D}_n)$. If the edge α_i is marked ▬ ▬, then α_i can occur in β. Using the double sum expression of (5), we get that x_{n-1} is the sum of all the sums below. Let $C = xyx^{-1}y^{-1}$.

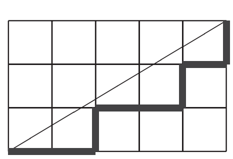
These configurations are grouped according to Remark 1: in the first picture there are no colored subpaths; each of the next four pictures has precisely one blue subpath; the sixth picture has a $(3,1)$-green subpath, forcing the preceding edge to be included in β; the seventh picture has a red subpath; and the last picture has a blue subpath and a red subpath.



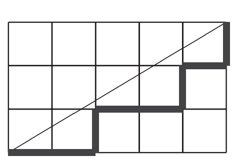
$$\sum_{\beta\subset\{\alpha_1,\ldots,\alpha_8\}} Cx\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1} = A_1$$



$$\sum_{\{\alpha(0,1)\}\subset\beta\subset\{\alpha(0,1)\}\cup\{\alpha_4,\ldots,\alpha_8\}} Cx\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1} = A_2$$



$$\sum_{\{\alpha(0,2)\}\subset\beta\subset\{\alpha(0,2)\}\cup\{\alpha_7,\alpha_8\}} Cx\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1} = A_3$$



$$\sum_{\beta=\{\alpha(0,3)\}} Cx\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1} = A_4$$

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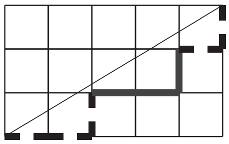
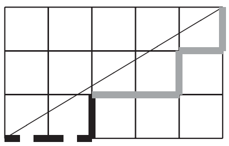
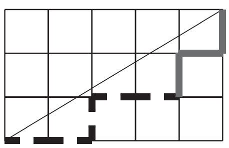


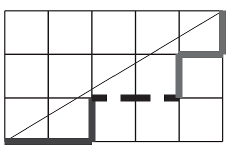
$$\sum_{\{\alpha(1,2)\}\subset\beta\subset\{\alpha(1,2)\}\cup\{\alpha_1,\alpha_2,\alpha_3,\alpha_7,\alpha_8\}} Cx\left(\prod_{i=1}^{c_{n-1}}\beta_{[i]}\right)x^{-1} = A_5$$



$$\sum_{\{\alpha(1,3)\}\cup\{\alpha_3\}\subset\beta\subset\{\alpha(1,3)\}\cup\{\alpha_1,\alpha_2,\alpha_3\}} Cx\left(\prod_{i=1}^{c_{n-1}}\beta_{[i]}\right)x^{-1} = A_6$$



$$\sum_{\{\alpha(2,3)\}\subset\beta\subset\{\alpha(2,3)\}\cup\{\alpha_1,\ldots,\alpha_5\}} Cx\left(\prod_{i=1}^{c_{n-1}}\beta_{[i]}\right)x^{-1} = A_7$$



$$\sum_{\{\alpha(0,1),\alpha(2,3)\}\subset\beta\subset\{\alpha(0,1),\alpha(2,3)\}\cup\{\alpha_4,\alpha_5\}} Cx\left(\prod_{c_{n-1}}^{i=1}\beta_{[i]}\right)x^{-1} = A_8,$$

where

$$A_1 = \sum_{\delta_1,\delta_2,\ldots,\delta_8\in\{0,1\}} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 2-3\delta_1 & 3-3\delta_2 & 2-3\delta_3 & 3-3\delta_4 & 3-3\delta_5 & 2-3\delta_6 & 3-3\delta_7 & 2-3\delta_8 & 0 \end{pmatrix},$$

$$A_2 = \sum_{\delta_4,\delta_5,\ldots,\delta_8\in\{0,1\}} \begin{pmatrix} 1 & -1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 3-3\delta_4 & 3-3\delta_5 & 2-3\delta_6 & 3-3\delta_7 & 2-3\delta_8 & 0 \end{pmatrix},$$

$$A_3 = \sum_{\delta_7,\delta_8\in\{0,1\}} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 & -1 & 3-3\delta_7 & 2-3\delta_8 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$A_5 = \sum_{\delta_1,\delta_2,\delta_3,\delta_7,\delta_8\in\{0,1\}} \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 2-3\delta_1 & 3-3\delta_2 & 2-3\delta_3 & 0 & 0 & -1 & 3-3\delta_7 & 2-3\delta_8 & 0 \end{pmatrix},$$

$$A_6 = \sum_{\delta_1,\delta_2\in\{0,1\}} \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 2-3\delta_1 & 3-3\delta_2 & -1 & 0 & 0 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$A_7 = \sum_{\delta_1,\delta_2,\ldots,\delta_5\in\{0,1\}} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 2-3\delta_1 & 3-3\delta_2 & 2-3\delta_3 & 3-3\delta_4 & 3-3\delta_5 & -1 & 0 & -1 & 0 \end{pmatrix},$$

$$A_8 = \sum_{\delta_4,\delta_5\in\{0,1\}} \begin{pmatrix} 1 & -1 & 0 & 1 & -1 & -1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & 3-3\delta_4 & 3-3\delta_5 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Then $x_4 = \sum_{i=1}^{8} A_i$.

3. Proofs

We need more notation.

DEFINITION 3.1. For integers u, n with $3 \leqslant u \leqslant n-1$, let

$$\mathcal{T}^{\geqslant u}(\mathscr{D}_n) := \left\{ \{\beta_1, \ldots, \beta_t\} \;\middle|\; \begin{array}{l} \bullet\ t \geqslant 1 \text{ and } \beta_j \in \mathscr{P}(\mathscr{D}_n) \text{ for all } 1 \leqslant j \leqslant t; \\ \bullet\ \text{if } j \neq j', \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge;} \\ \bullet\ \text{if } \beta_j = \alpha(i,k) \text{ and } \beta_{j'} = \alpha(i',k'), \\ \quad \text{then } i \neq k' \text{ and } i' \neq k; \\ \bullet\ \text{there exist integers } j, w, m \text{ with } m \geqslant u \text{ such that} \\ \quad \beta_j \text{ is } (m,w)\text{-green and none of the } c_{m-1} - wc_{m-2} \\ \quad \text{preceding edges of } v_i \text{ is contained in any } \beta_{j'} \end{array} \right\}.$$

DEFINITION 3.2. Let

$$\widetilde{\mathscr{F}}(\mathscr{D}_n) := \left\{ \{\beta_1, \ldots, \beta_t\} \;\middle|\; \begin{array}{l} \bullet\ t \geqslant 0 \text{ and } \beta_j \in \mathscr{P}(\mathscr{D}_n) \text{ for all } 1 \leqslant j \leqslant t; \\ \bullet\ \text{if } j \neq j', \text{ then } \beta_j \text{ and } \beta_{j'} \text{ have no common edge;} \\ \bullet\ \text{if } \beta_j = \alpha(i,k) \text{ and } \beta_{j'} = \alpha(i',k'), \\ \quad \text{then } i \neq k' \text{ and } i' \neq k \end{array} \right\}.$$

Note that

$$\mathscr{F}(\mathscr{D}_n) = \widetilde{\mathscr{F}}(\mathscr{D}_n) \backslash \mathcal{T}^{\geqslant 3}(\mathscr{D}_n). \tag{6}$$

LEMMA 3.3. *If $m \geqslant n-1$, then there do not exist i, w (with $1 \leqslant w < r-1$) such that $\min\{t \mid i < t \leqslant c_{n-2},\ s_{i,t} > s\}$ is of the form $i + c_m - wc_{m-1}$. In particular, for any $n \geqslant 4$, the set $\mathcal{T}^{\geqslant n-1}(\mathscr{D}_n)$ is empty.*

Proof. If $m \geqslant n-1$ and $\min\{t \mid i < t \leqslant c_{n-2},\ s_{i,t} > s\} = i + c_m - wc_{m-1}$, then $\min\{t \mid i < t \leqslant c_{n-2},\ s_{i,t} > s\} \geqslant c_{n-1} - wc_{n-2}$, which would be greater than c_{n-2} because $w \leqslant r-2$. But this is a contradiction, because $v_{c_{n-2}}$ is the highest vertex in $\mathscr{D}_n$. □

Let $z_2 = x_2$ and

$$z_{n-1} = \sum_{\beta \in \widetilde{\mathscr{F}}(\mathscr{D}_n)} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1} \tag{7}$$

for $n \geqslant 4$.

LEMMA 3.4. *Let $n \geqslant 3$. Then*

$$z_n = F(z_{n-1}) + \sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_{n+1}) \backslash \mathcal{T}^{\geqslant 4}(\mathscr{D}_{n+1})} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_n} \beta_{[i]}\right)x^{-1}.$$

LEMMA 3.5. *Let $u \geqslant 3$ and $n \geqslant u+2$. Then*

$$F\left(\sum_{\beta \in \mathcal{T}^{\geqslant u}(\mathscr{D}_n) \backslash \mathcal{T}^{\geqslant u+1}(\mathscr{D}_n)} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\right)x^{-1}\right)$$
$$= \sum_{\beta \in \mathcal{T}^{\geqslant u+1}(\mathscr{D}_{n+1}) \backslash \mathcal{T}^{\geqslant u+2}(\mathscr{D}_{n+1})} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_n} \beta_{[i]}\right)x^{-1}.$$

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Lemma 3.6. *Let $n \geqslant 4$. Then*

$$x_{n-1} = z_{n-1} - \sum_{m=5}^{n} F^{n-m}\Bigg(\sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_m)\setminus \mathcal{T}^{\geqslant 4}(\mathscr{D}_m)} xyx^{-1}y^{-1}x\Bigg(\prod_{i=1}^{c_{m-1}} \beta_{[i]}\Bigg)x^{-1}\Bigg)$$
$$= \sum_{\beta \in \mathscr{F}(\mathscr{D}_n)} xyx^{-1}y^{-1}x\Bigg(\prod_{i=1}^{c_{n-1}} \beta_{[i]}\Bigg)x^{-1}. \qquad (8)$$

The proof of Lemma 3.4 will be independent of the proofs of Lemmas 3.5 and 3.6. We prove Lemmas 3.5 and 3.6 by the following induction:

$$\begin{aligned}[\text{Lemma 3.5 holds true for } n \leqslant d] &\implies [\text{Lemma 3.6 holds true for } n \leqslant d+1]\\ &\implies [\text{Lemma 3.5 holds true for } n \leqslant d+1]\\ &\implies [\text{Lemma 3.6 holds true for } n \leqslant d+2] \cdots . \qquad (9)\end{aligned}$$

Proof of Lemma 3.6. This proof is a straightforward adaptation of [LS12, proof of Lemma 19]. We use induction on n. It is easy to show that $x_3 = z_3$. Assume that (8) holds for n, and let $C = xyx^{-1}y^{-1}$.

Then

$$\begin{aligned}
x_n &= F(x_{n-1})\\
&\overset{F \text{ is homomorphism}}{=} F(z_{n-1}) - \sum_{m=5}^{n} F^{n-m+1}\Bigg(\sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_m)\setminus \mathcal{T}^{\geqslant 4}(\mathscr{D}_m)} Cx\Bigg(\prod_{i=1}^{c_{m-1}} \beta_{[i]}\Bigg)x^{-1}\Bigg)\\
&\overset{\text{Lemma 3.4}}{=} z_n - \sum_{m=5}^{n+1} F^{n-m+1}\Bigg(\sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_m)\setminus \mathcal{T}^{\geqslant 4}(\mathscr{D}_m)} Cx\Bigg(\prod_{i=1}^{c_{m-1}} \beta_{[i]}\Bigg)x^{-1}\Bigg)\\
&\overset{\text{Lemma 3.5}}{=} z_n - \sum_{m=5}^{n+1} \sum_{\beta \in \mathcal{T}^{\geqslant n-m+4}(\mathscr{D}_{n+1})\setminus \mathcal{T}^{\geqslant n-m+5}(\mathscr{D}_{n+1})} Cx\Bigg(\prod_{i=1}^{c_n} \beta_{[i]}\Bigg)x^{-1}\\
&= z_n - \sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_{n+1})\setminus \mathcal{T}^{\geqslant n}(\mathscr{D}_{n+1})} Cx\Bigg(\prod_{i=1}^{c_n} \beta_{[i]}\Bigg)x^{-1}\\
&\overset{\text{Lemma 3.3}}{=} z_n - \sum_{\beta \in \mathcal{T}^{\geqslant 3}(\mathscr{D}_{n+1})} Cx\Bigg(\prod_{i=1}^{c_n} \beta_{[i]}\Bigg)x^{-1}\\
&\overset{(7)}{=} \sum_{\beta \in \widetilde{\mathscr{F}}(\mathscr{D}_{n+1})\setminus \mathcal{T}^{\geqslant 3}(\mathscr{D}_{n+1})} Cx\Bigg(\prod_{i=1}^{c_n} \beta_{[i]}\Bigg)x^{-1}\\
&\overset{(6)}{=} \sum_{\beta \in \mathscr{F}(\mathscr{D}_{n+1})} Cx\Bigg(\prod_{i=1}^{c_n} \beta_{[i]}\Bigg)x^{-1}.
\end{aligned}$$

In order to prove Lemma 3.4, we need the following notation.

Definition 3.7. The sequence $\{b_{i,j}\}_{i \in \mathbb{Z}_{\geqslant 2}, 1 \leqslant j \leqslant c_i}$ is defined by

$$b_{i,j} = \begin{cases} r & \text{if } \alpha_j \text{ is a horizontal edge of } \mathscr{D}_{i+1},\\ r-1 & \text{if } \alpha_j \text{ is a vertical edge of } \mathscr{D}_{i+1}.\end{cases}$$

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For integers $i \leqslant j$, we denote the set $\{i, i+1, i+2, \ldots, j\}$ by $[i, j]$. We will always identify $[i, j]$ with the subpath given by $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_j)$.

Definition 3.8. We need a function f from {subsets of $[1, c_{n-1}]$} to {subsets of $[1, c_n]$}. For each subset $V \subset [1, c_{n-1}]$, we define $f(V)$ as follows.

If $V = \emptyset$, then $f(\emptyset) = \emptyset$. If $V \neq \emptyset$, then we write V as a disjoint union of maximal connected subsets, $V = \bigsqcup_{i=1}^{j} [e_i, e_i + \ell_i - 1]$ with $\ell_i > 0$ $(1 \leqslant i \leqslant j)$ and $e_i + \ell_i < e_{i+1}$ $(1 \leqslant i \leqslant j-1)$. For each $1 \leqslant i \leqslant j$, let

$$W_i = \left[1 + \sum_{k=1}^{e_i - 1} b_{n-1,k},\ \sum_{k=1}^{e_i+\ell_i-1} b_{n-1,k}\right]$$

and define $f_i(V)$ by

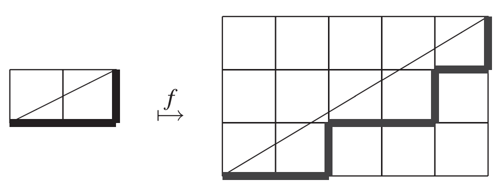
$$f_i(V) := \begin{cases} W_i & \text{if the subpath given by } W_i \text{ is blue or green,} \\ \left\{\sum_{k=1}^{e_i-1} b_{n-1,k}\right\} \cup W_i & \text{otherwise.} \end{cases}$$

Then $f(V)$ is obtained by taking the union of the $f_i(V)$,

$$f(V) := \bigcup_{i=1}^{j} f_i(V).$$

Note that the subpath given by $f_i(V)$ is always one of blue, green or red subpaths, and that every blue, green or red subpath can be realized as the image of a maximal connected interval under f.

Example 3. Let $r = 3$ and $n = 4$. Then $f(\{1, 2, 3\}) = \{1, 2, 3, 4, 5, 6, 7, 8\}$. As illustrated below, the image of the subpath $(\alpha_1, \alpha_2, \alpha_3)$ under f is the subpath $(\alpha_1, \ldots, \alpha_8)$, which is blue.



Lemma 3.9. *Let $C = xyx^{-1}y^{-1}$. Then $F(C) = C$.*

Proof. We calculate that

$$F(xyx^{-1}y^{-1}) = xyx^{-1}(1 + y^r)x^{-1}xy^{-1}x^{-1}x(1 + y^r)^{-1} = xyx^{-1}y^{-1}. \qquad \square$$

Proof of Lemma 3.4. The idea is the same as in the commutative case [LS12, Lemma 17], that is, we choose any subset, say V, of $\{\alpha_1, \ldots, \alpha_{c_{n-1}}\}$ and consider all β whose support is V. Then one can check that

$$F\Bigg(\sum_{\substack{\beta \in \mathscr{F}(\mathscr{D}_n), \\ \beta:\, \mathrm{supp}(\beta) = V}} Cx \left(\prod_{i=1}^{c_{n-1}} \beta_{[i]} \right) x^{-1} \Bigg) = \sum_{\substack{\beta \in \widetilde{\mathscr{F}}(\mathscr{D}_{n+1}), \\ \beta:\, \text{colored subpaths of } \beta \text{ are precisely} \\ \text{the ones given by } f(V)}} Cx \left(\prod_{i=1}^{c_n} \beta_{[i]} \right) x^{-1}.$$

Then, as in the commutative case, the $\beta \in \widetilde{\mathscr{F}}(\mathscr{D}_{n+1})$ which are not covered by this construction belong to $\mathcal{T}^{\geqslant 3}(\mathscr{D}_{n+1})\backslash\mathcal{T}^{\geqslant 4}(\mathscr{D}_{n+1})$.

For example, if $r=3$, $n=4$ and $V=\emptyset$, then β must be empty, and we get

$$\prod_{i=1}^{3}\beta_{[i]} = x^{-1}y^3x^{-1}y^3x^{-1}y^2.$$

It is straightforward to show that

$$\begin{aligned}&F\big(Cx(x^{-1}y^3x^{-1}y^3x^{-1}y^2)x^{-1}\big)\\ &= \sum_{\beta\subset\{\alpha_1,\ldots,\alpha_8\}} Cx\left(\prod_{i=1}^{8}\beta_{[i]}\right)x^{-1} = A_1,\end{aligned}$$

where A_1 is the same as that defined in Example 2.

If $r=3$, $n=4$ and $V=\{1,2,3\}$, then β is either $\alpha(0,1)$ or $\{\alpha_1,\alpha_2,\alpha_3\}$, and we get $\prod_{i=1}^{3}\beta_{[i]} = x^{-1}y^0x^{-1}y^0x^{-1}y^{-1}$ or $\prod_{i=1}^{3}\beta_{[i]} = x^0y^0x^0y^0x^0y^{-1}$. Then

$$\begin{aligned}&F\Big(Cx\big(x^{-1}y^0x^{-1}y^0x^{-1}y^{-1} + x^0y^0x^0y^0x^0y^{-1}\big)x^{-1}\Big)\\ &= Cxyx^{-1}\big(xy^{-3}x^{-1}x(1+y^3)^{-1} + x(1+y^3)^{-1}\big)xy^{-1}x^{-1}\\ &= Cxy\big(y^{-3}(1+y^3)^{-1} + (1+y^3)^{-1}\big)xy^{-1}x^{-1}\\ &= Cxy\big(y^{-3}(1+y^3)^{-1} + y^{-3}y^3(1+y^3)^{-1}\big)xy^{-1}x^{-1}\\ &= Cxy\big(y^{-3}(1+y^3)(1+y^3)^{-1}\big)xy^{-1}x^{-1}\\ &= Cxy\big(y^{-3}\big)xy^{-1}x^{-1}\\ &= \sum_{\beta=\{\alpha(0,3)\}} Cx\left(\prod_{i=1}^{c_n}\beta_{[i]}\right)x^{-1} = A_4,\end{aligned}$$

where A_4 is the same as that defined in Example 2. □

It remains to prove Lemma 3.5. Here we only sketch the proof, which is a straightforward adaptation of [LS12, proof of Lemma 18].

Proof of Lemma 3.5. We will deal only with the case of $n=u+2$. The case of $n>u+2$ makes use of the same argument. As we use the induction (9), we can assume that

$$x_j = \sum_{\beta\in\mathscr{F}(\mathscr{D}_i)} xyx^{-1}y^{-1}x\left(\prod_{i=1}^{c_{j-1}}\beta_{[i]}\right)x^{-1}$$

for $j\leqslant n$.

Since it is straightforward to check the statement for $n=5$, we assume that $n\geqslant 6$. For any $w\in[1,r-2]$, it is easy to show that the lattice point $(w(c_{n-2}-c_{n-3}), wc_{n-3})$ is below the diagonal from $(0,0)$ to $(c_{n-1}-c_{n-2}, c_{n-2})$ and that the points $(w(c_{n-2}-c_{n-3}), 1+wc_{n-3})$ and $(w(c_{n-2}-c_{n-3})-1, wc_{n-3})$ are above the diagonal. So $(w(c_{n-2}-c_{n-3}), wc_{n-3})$ is one of the vertices v_i on $\mathscr{D}_n$. Actually, $v_{wc_{n-3}} = (w(c_{n-2}-c_{n-3}), wc_{n-3})$. Since $u=n-2$ and $\alpha(wc_{n-3}, c_{n-2})$ is the only $(n-2,w)$-green subpath in $\{\alpha(i,k)\mid 0\leqslant i<k\leqslant c_{n-2}\}$, every $\beta\in\mathcal{T}^{\geqslant u}(\mathscr{D}_n)\backslash\mathcal{T}^{\geqslant u+1}(\mathscr{D}_n)$ must contain the green subpath from $v_{wc_{n-3}}$ to $v_{c_{n-2}}$. Then none of the $c_{n-3}-wc_{n-4}$ preceding edges of $v_{wc_{n-3}}$ is contained in any element $\beta_{j'}$ of β. The green subpath

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from $v_{wc_{n-3}}$ to $v_{c_{n-2}}$ corresponds to the interval $[wc_{n-2}+1, c_{n-1}] \subset [1, c_{n-1}]$. The $c_{n-3} - wc_{n-4}$ preceding edges of $v_{wc_{n-3}}$ are $\alpha_{(rw-1)c_{n-3}+1}, \dots, \alpha_{wc_{n-2}}$.

Thus we have

$$\begin{aligned}
&\sum_{\beta \in \mathcal{T}^{\geqslant u}(\mathscr{D}_n) \setminus \mathcal{T}^{\geqslant u+1}(\mathscr{D}_n)} Cx \left(\prod_{i=1}^{c_{n-1}} \beta_{[i]} \right) x^{-1} \\
&= \sum_{w=1}^{r-2} \sum_{V \subset [1,(rw-1)c_{n-3}]} \sum_{\substack{\beta :\, \bigcup \beta_i = V \cup [wc_{n-2}+1, c_{n-1}], \\ \beta \ni \alpha(wc_{n-3}, c_{n-2})}} Cx \left(\prod_{i=1}^{c_{n-1}} \beta_{[i]} \right) x^{-1}. \qquad (*)
\end{aligned}$$

We observe that the subpath corresponding to $[1, (rw-1)c_{n-3}]$ consists of $w-1$ copies of $\mathscr{D}_{n-1}$, $r-1$ copies of $\mathscr{D}_{n-2}$, and $w-1$ copies of $\mathscr{D}_{n-3}$. Let $v_{j_0} = (0, 0)$ and let v_{j_i} be the endpoint of each of these copies, that is,

$$\begin{aligned}
v_{j_i} &= v_{ic_{n-3}} \quad \text{for } 1 \leqslant i \leqslant w-1, \\
v_{j_{w-1+i}} &= v_{(w-1)c_{n-3}+ic_{n-4}} \quad \text{for } 1 \leqslant i \leqslant r-1, \\
v_{j_{w+r-2+i}} &= v_{(w-1)c_{n-3}+(r-1)c_{n-4}+ic_{n-5}} \quad \text{for } 1 \leqslant i \leqslant w-1.
\end{aligned}$$

If a (m, w')-green (respectively, blue or red) subpath, say $\alpha(i, k)$, in $[1, (rw-1)c_{n-3}]$ passes through $v_{j_e}, v_{j_{e+1}}, \dots, v_{j_{e+\ell}}$, then $\alpha(i, k)$ can be naturally decomposed into $\alpha(i, j_e)$, $\alpha(j_e, j_{e+1}), \dots, \alpha(j_{e+\ell}, k)$. It is not hard to show that $\alpha(i, j_e)$ is also (m, w')-green (respectively, blue or red) and that $\alpha(j_e, j_{e+1}), \cdots, \alpha(j_{e+\ell}, k)$ are all blue.

Hence

$$\begin{aligned}
(*) &= \sum_{w=1}^{r-2} Cx \left(\sum_{\beta \in \mathscr{F}(\mathscr{D}_{n-1})} \left(\prod_{i=1}^{c_{n-2}} \beta_{[i]} \right) \right)^{w-1} \left(\sum_{\beta \in \mathscr{F}(\mathscr{D}_{n-2})} \left(\prod_{i=1}^{c_{n-3}} \beta_{[i]} \right) \right)^{r-1} \\
&\quad \times \left(\sum_{\beta \in \mathscr{F}(\mathscr{D}_{n-3})} \left(\prod_{i=1}^{c_{n-4}} \beta_{[i]} \right) \right)^{w-1} x^{-1} y x y^{-1} x^{-1} \\
&= \sum_{w=1}^{r-2} Cx (x^{-1} y x y^{-1} x^{-1} x_{n-2} x)^{w-1} (x^{-1} y x y^{-1} x^{-1} x_{n-3} x)^{r-1} \\
&\quad \times (x^{-1} y x y^{-1} x^{-1} x_{n-4} x)^{w-1} x^{-1} y x y^{-1} x^{-1} \\
&= \sum_{w=1}^{r-2} C (C^{-1} x_{n-2})^{w-1} (C^{-1} x_{n-3})^{r-1} (C^{-1} x_{n-4})^{w-1} C^{-1}.
\end{aligned}$$

For the same reason, we get

$$\begin{aligned}
&\sum_{\beta \in \mathcal{T}^{\geqslant u+1}(\mathscr{D}_{n+1}) \setminus \mathcal{T}^{\geqslant u+2}(\mathscr{D}_{n+1})} Cx \left(\prod_{i=1}^{c_n} \beta_{[i]} \right) x^{-1} \\
&= \sum_{w=1}^{r-2} C (C^{-1} x_{n-1})^{w-1} (C^{-1} x_{n-2})^{r-1} (C^{-1} x_{n-3})^{w-1} C^{-1}.
\end{aligned}$$

Since $F(C) = C$, we have

$$F \left(\sum_{\beta \in \mathcal{T}^{\geqslant u}(\mathscr{D}_n) \setminus \mathcal{T}^{\geqslant u+1}(\mathscr{D}_n)} Cx \left(\prod_{i=1}^{c_{n-1}} \beta_{[i]} \right) x^{-1} \right) = \sum_{\beta \in \mathcal{T}^{\geqslant u+1}(\mathscr{D}_{n+1}) \setminus \mathcal{T}^{\geqslant u+2}(\mathscr{D}_{n+1})} Cx \left(\prod_{i=1}^{c_n} \beta_{[i]} \right) x^{-1}. \qquad \square$$

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