A NOTE ON MATHIEU FUNCTIONS

by M. BELL

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The Mathieu functions of integral order [1] are the solutions with period π or 2π of the equation

$$\frac{d^2y}{dz^2}+(a-2q\cos 2z)y=0. \qquad (1)$$

The eigenvalues associated with the functions ce_N and se_N, where N is a positive integer, denoted by a_N and b_N respectively, reduce to

$$a_N=b_N=N^2,$$

when q is zero. The quantities a_N and b_N can be expanded in powers of q, but the explicit construction of high order coefficients is very tedious. In some applications the quantity of most interest is a_N-b_N, which may be called the " width of the unstable zone ". It is the object of this note to derive a general formula for the leading term in the expansion of this quantity, namely

$$a_N-b_N=\frac{q^N}{2^{2N-3}\{(N-1)!\}^2}. \qquad (2)$$

Suppose first that N is an odd integer. Then there is an expansion

$$\mathrm{ce}_N(z)=\sum_{n=1,3,5\ldots}\alpha_N^n\phi_n, \qquad (3)$$

where

$$\phi_n=\surd(2/\pi)\cos nz. \qquad (4)$$

These functions ϕ satisfy

$$\frac{d^2\phi_n}{dz^2}+n^2\phi_n=0 \qquad (5)$$

and

$$\int_0^\pi \phi_n\phi_m\,dz=\delta_{nm}. \qquad (6)$$

On substituting (3) in (1), one obtains the algebraic equations

$$(a_N-l^2)\alpha_N^l=\sum_m\{lm\}\alpha_N^m, \qquad (7)$$

where

$$\{lm\}=\int_0^\pi[\phi_l\phi_m\,2q\cos 2z]\,dz. \qquad (8)$$

Explicitly,

$$\begin{aligned}&\{11\}=q,\\ &\{lm\}=q \quad \text{if } |l-m|=2, \qquad (9)\\ &\{lm\}=0 \quad \text{otherwise.}\end{aligned}$$

The equation (7) is solved by the method, well-known in mathematical physics, of Brillouin [2] and Wigner [3]. Imposing the normalisation $\alpha_N^N=1$ (to all orders in q), one may rewrite (7) as

$$\alpha_N^l = \frac{1}{a_N - l^2} \sum_m \{lm\} \alpha_N^m \quad (l \neq N), \quad \text{.......(10)}$$

$$a_N - N^2 = \sum_m \{Nm\} \alpha_N^m \quad (l = N). \quad \text{.......(11)}$$

For $q=0$ one has, of course, $\alpha_N^l = \delta_{Nl}$; with this as starting point, and treating a_N as a known parameter, one solves equation (10) by iteration. When the result is substituted into equation (11), one finds that

$$a_N - N^2 = \{NN\} + \Sigma \{Nl\} \frac{1}{a_N - l^2} \{lN\} + \ldots$$
$$+ \Sigma \{Nl\} \frac{1}{a_N - l^2} \{lm\} \frac{1}{a_N - m^2} \{mn\} + \ldots + \frac{1}{a_N - w^2} \{wN\} + \ldots, \quad \text{.......(12)}$$

where the summations are over all odd integral values, except N itself, of the intermediate indices $l, m, n \ldots$. Equation (12) involves the unknown a_N on the right-hand side; solution by iteration yields an explicit power series for a_N. The first n terms of the power series are determined entirely by the first n terms on the right-hand side of (12).

For the functions se_N there is an expansion of the form (3), the functions ϕ_n being replaced by

$$\phi'_n = \sqrt{(2/\pi)} \sin nz. \quad \text{.......(13)}$$

These satisfy relations analogous to (5) and (6), and we find an equation like (12), except that a_N is everywhere replaced by b_N and $\{lm\}$ by

$$\{lm\}' = \int_0^\pi [\phi'_l \phi'_m 2q \cos 2z]\, dz.$$

Explicitly,

$$\begin{aligned} \{11\}' &= -q, \\ \{lm\}' &= q \quad \text{if } |l-m| = 2, \\ \{lm\}' &= 0 \quad \text{otherwise}. \end{aligned} \quad \text{.......(14)}$$

The equations for a_N and b_N differ only through the difference between $\{11\}$ and $\{11\}'$. Moreover it is clear from (12) and (14) that these quantities cannot effectively appear until the Nth terms on the right-hand sides, nor therefore until the Nth terms in the power series, so that the explicit series for $a_N - N^2$ and $b_N - N^2$ will be identical for terms of order lower than q^N. Examining the Nth terms, one finds that, to lowest order in q,

$$a_N - b_N = \frac{2q^N}{[N^2-(N-2)^2]^2 [N^2-(N-4)^2]^2 \ldots [N^2-1]^2}, \quad \text{.......(15)}$$

where $a_N = b_N = N^2$ is used as a sufficient approximation in the denominator. The resulting expression reduces, with a little manipulation, to (2).

When N is an even integer the expansions used are

$$\mathrm{ce}_N(z) = \sum_{n=0,2,4\ldots} \alpha_N^n \phi_n, \quad \mathrm{se}_N(z) = \sum_{n=2,4,\ldots} \alpha_N'^n \phi'_n, \quad \text{.......(16)}$$

with

$$\phi_0 = 1/\sqrt{\pi}, \qquad \phi_n = \sqrt{(2/\pi)} \cos nz, \qquad \phi'_n = \sqrt{(2/\pi)} \sin nz.$$

We again obtain expressions similar to (12), except that now the intermediate indices are even rather than odd integers, including zero for ce_N but not for se_N. We now find for the latter

$$\{lm\}' = q \quad \text{if } |l - m| = 2,$$
$$\{lm\}' = 0 \quad \text{otherwise.}$$

The quantities $\{lm\}$ appropriate to ce_N are the same as $\{lm\}'$ except that there now occur $\{l0\}$ and $\{0l\}$, which are zero except for

$$\{20\} = \{02\} = \surd 2q. \qquad \text{...(17)}$$

Again $(a_N - N^2)$ and $(b_N - N^2)$ do not differ until the Nth term of the series, when the quantities (17) first effectively appear, and one readily finds in lowest order

$$a_N - b_N = \frac{2q^N}{[N^2 - (N-2)^2]^2 \, [N^2 - (N-4)^2]^2 \ldots N^2},$$

which again reduces to (2).

The formula gives a good approximation to the width only for sufficiently small q. On comparing with the tables in reference [1] it is found that for $q = 1$, the error in the estimation of $a_N - b_N$ is 1·6, 10·1, 2·0, 0·7, and 0·1 per cent. for $N = 1$, 2, 3, 4, 5 respectively.

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