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COUNTING COLOURED GRAPHS. III

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1. Introduction. In an earlier paper [**4**], we found an asymptotic expansion for $M_n = M_n(k)$, the number of coloured graphs on n labelled nodes, when n is large. Such a graph is a set of n distinguishable objects called *nodes*, and a set of "edges", that is, undirected pairs of nodes. The nodes are mapped onto k colours. Every pair of nodes of different colours may or may not be joined by an edge, but no edge can join a pair of nodes of the same colour.

We write m_n for the number of these graphs which are connected, F_n for the number which use *all k colours* (i.e., at least one node in each graph is mapped onto each of the k colours), and f_n for the number of connected graphs which use all k colours.

We use A to denote a positive number, not always the same at each occurrence, which is independent of n but which may depend on k. The notation $O(\ )$ refers to the passage of n to infinity and the constants implied are of type A. If x is a positive integer, we write

$$c_x(y) = y(y-1)\dots(y-x+1)/x!,\ c_0(y) = 1.$$

We showed [**4**; **5**] (see also [**3**]) that M_n, F_n, m_n, f_n all have the same asymptotic expansion

$$\left(\frac{k}{n \log 2}\right)^{\frac{1}{2}(k-1)kn2^N}\left\{\sum_{h=0}^{H-1} C_h n^{-h} + O(n^{-H})\right\}$$

for large n, where $K = (k-1)/(2k)$ and $N = Kn^2$. The coefficient C_h is defined in § 2 below and, for $k < 1000$, C_0 is within 2×10^{-6} of unity.

In this paper we consider M_{nq}, the number of these graphs which have just q edges. We call the set of integers $(s_1, s_2, \dots, s_k)$ an *n-set* if

$$(1.1)\qquad s_1 + s_2 + \dots + s_k = n.$$

A *non-negative n-set* is an n set in which none of the s_i is negative. We write

$$\sum_{(n)},\ \sum_{((n))}$$

to denote summation over all non-negative n-sets and over all n-sets, respectively.

In any of our graphs, there are s_1 nodes of colour 1, s_2 of colour 2, and so on, where the s_i form a non-negative n-set. The number of possible edges is then

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$E = \sum s_i s_j$, where the sum is over all i, j such that $1 \leqq i \leqq j \leqq n$. Read [**2**] deduces that

$$M_{nq} = \sum_{(n)} Pc_q(E),$$

where

$$P = n!/(s_1!s_2! \ldots s_k!).$$

He remarks that "it does not appear that this formula is very amenable to manipulation". This seems a very reasonable assessment so far as exact transformation is concerned, but we show here that it is possible to deduce an asymptotic approximation to M_{nq} for large n and all q.

2. Preliminary results. We write

$$K = (k-1)/(2k),$$
$$N = Kn^2,$$
$$R = \sum_{i=1}^{n} (ks_i - n)^2/(2k^2),$$

and a for the least non-negative residue of $n \pmod{k}$. We find that

$$2k^2R = k^2 \sum s_i^2 - kn^2 \tag{2.1}$$

and that $E = N - R$ by (1.1). The smallest value of R for a given n occurs when a of the s_i have the value $[n/k] + 1$ and the remaining $k - a$ have the value $[n/k]$. We call such a set a *minimal n-set*; there are $c_a(k)$ such sets and for each of them R has the value $b = a(k-a)/(2k)$. If we write $Q = N - b$ and $V = R - b$, we see that $\max E = Q$ and that $E = Q - V$. Hence, Q and V are integers and $V > 0$ for all non-minimal n-sets.

LEMMA 1. *There are $O(V^{\frac{1}{2}(k-1)})$ n-sets associated with any positive V.*

For a given R, we have

$$(ks_i - n)^2 \leqq 2k^2R,$$
$$(n/k) - \sqrt{(2R)} \leqq s_i \leqq (n/k) + \sqrt{(2R)},$$

and so there are not more than $AR^{\frac{1}{2}}$ choices of s_i. The lemma follows, since s_k is fixed, once $s_1, \ldots, s_{k-1}$ are chosen, and $R < AV$ if $V \geqq 1$.

For any $\alpha > 0$ we write

$$L(\alpha, n) = \sum_{((n))} e^{-2\alpha R} = \sum_{((n))} \exp\left(-\alpha\left\{\sum_{i=1}^{k} s_i^2 - (n^2/k)\right\}\right).$$

We shall find an asymptotic approximation to M_{nq} in terms of $L(\alpha, n)$, so that we need to evaluate the latter. It is easily verified that $L(\alpha, n + k) = L(\alpha, n)$, so that $L(\alpha, n) = L(\alpha, a)$, where a is the least non-negative residue of $n \pmod{k}$. Hence, $L(\alpha, n)$ depends on α and on a, but not otherwise on n. We see also that $L(\alpha, n)$ is a continuous function of α, for $\alpha > 0$. Using Lemma 1, we have the next lemma almost trivially.

LEMMA 2. *As* $\alpha \to \infty$,

$$L(\alpha, a) \sim c_a(k)e^{-2\alpha b}.$$

We take $\gamma > 0$ and write

$$Z = \sum_{i=1}^{k-1} s_i,$$

$$\Delta = k \sum_{i=1}^{k-1} s_i^2 - Z^2,$$

$$H_k(\gamma, a) = \sum e^{-\gamma\Delta} \cos(2\pi a Z/k),$$

where the sum is extended over all integral values of $s_1, s_2, \ldots, s_{k-1}$, positive, negative or zero. (The coefficient C_0 of § 1 is $H_k(2\pi^2/\log 2, a)$.) In [**5**, Theorem 3], we deduced from [**1**] that

$$L(\alpha, a) = k^{-\frac{1}{2}}(\pi/\alpha)^{\frac{1}{2}(k-1)} H_k(\gamma, a), \tag{2.2}$$

where $\alpha\gamma = \pi^2$. (We were concerned only with the case in which $\alpha = \frac{1}{2}\log j$, where j is a positive integer, but this restriction played no part in the proof and is unnecessary. We require (2.2) here for general positive α.) From this we can deduce another lemma.

LEMMA 3. *As* $\alpha \to 0$, *we have*

$$L(\alpha, a) \sim k^{-\frac{1}{2}}(\pi/\alpha)^{\frac{1}{2}(k-1)}.$$

We shall, however, require the value of $L(\alpha, a)$ for finite positive α. In Lemma 3, we have used the obvious fact that $H_k(\gamma, a) \to 1$ as $\gamma \to \infty$. More precisely, as we have shown in [**5**],

$$\begin{cases} H_2(\gamma, a) = 1 + 2e^{-\frac{1}{2}\gamma} \cos \pi a + O(e^{-2\gamma}), \\ H_3(\gamma, a) = 1 + 6e^{-2\gamma/3} \cos(2\pi a/3) + O(e^{-2\gamma}), \\ H_4(\gamma, a) = 1 + 8e^{-3\gamma/4} \cos \frac{1}{2}\pi a + 6e^{-\gamma} \cos \pi a + O(e^{-2\gamma}). \end{cases} \tag{2.3}$$

Indeed, we gave slightly more complicated formulae valid for all k and for which the error is $O(e^{-9\gamma/2})$.

Thus, we have a very good approximation to $L(\alpha, a)$ when α is small, so that γ is large. As we saw in [**4**], the approximation for small α remained very good when $\alpha = \frac{1}{2}\log 2$ and $k < 1000$, the error involved in taking $H_k(\gamma, a) = 1$ being less than two parts in a million. Indeed, the approximations obtained from (2.2) and (2.3) remain good for $\alpha \leqq \pi$, the proportional error containing a factor $e^{-2\gamma} \leqq e^{-2\pi} < 0.002$. Again, this can be improved by the more complicated formulae in [**5**].

If $\alpha > \pi$, the series $L(\alpha, a)$ converges rapidly and it is not difficult to see that

$$L(\alpha, a) = c_a(k)e^{-2\alpha b}\{1 + O(e^{-2\alpha})\}$$

and that $e^{-2\alpha} \leqq e^{-2\pi} < 0.002$. Again, we can easily improve the approximation. Thus, for moderate sized k, $L(\alpha, a)$ can be readily evaluated to any reasonable degree of accuracy for finite α.

LEMMA 4. *If* $N - q \to \infty$, *we have*

$$c_q(Q)N^b \sim c_q(N)(N-q)^b.$$

This can be easily verified if we use the well-known asymptotic expansion of the logarithm of the Γ-function in the form that, if $y = O(1)$ and $X \to \infty$, then

$$\log \Gamma(X + y + 1) = (X + y + \tfrac{1}{2}) \log X - X + \tfrac{1}{2} \log(2\pi) + O(1/X). \tag{2.4}$$

If $N - q = h = O(1)$, however, the result of the lemma is true only if $\Gamma(h + 1) = h^b(h - b)!$, which is certainly false for integral $b \geqq 2$, for example, when $k = 16$, $a = 8$.

3. Asymptotic approximation to M_{nq}: statement of results. We write

$$H = k^{n+\frac{1}{2}k}(2\pi n)^{-\frac{1}{2}(k-1)},$$
$$\beta = \tfrac{1}{2} \log(N/(N-q)) + \tfrac{1}{2}(k/n).$$

THEOREM 1. *If* $0 \leqq q < Q$, *then, as* $n \to \infty$,

$$M_{nq} \sim Hc_q(Q)N^b(N-q)^{-b}L(\beta, a).$$

If $q = Q$ *then* $M_{nq} \sim Hc_a(k)$.

This appears a somewhat complicated statement, but that is because it covers all q. From it and the lemmas of the last section we can deduce a series of results for different ranges of q, which are much simpler.

THEOREM 2. *If* $q = o(n)$, *then*

$$M_{nq} \sim k^n c_q(Q) \sim k^n c_q(N).$$

THEOREM 3. *If* $q/n \to \delta > 0$, *then*

$$M_{nq} \sim k^n c_q(Q)\left(\frac{k-1}{k-1+2\delta}\right)^{\frac{1}{2}(k-1)}.$$

THEOREM 4. *If* $n = o(q)$, $q = o(N)$, *then*

$$M_{nq} \sim k^n c_q(Q)\{(k-1)n/(2q)\}^{\frac{1}{2}(k-1)}.$$

THEOREM 5. *If* $N - q = o(N)$, *then*

$$M_{nq} \sim Hc_a(k)c_q(Q). \tag{3.1}$$

THEOREM 6. *If* $q/N \to \delta$ *and* $0 < \delta < 1$, *then*

$$M_{nq} \sim Hc_q(N)L(-\tfrac{1}{2} \log (1 - \delta), a).$$

We write $c = \frac{1}{8}$. We can easily verify that it is sufficient to prove the following two lemmas.

LEMMA 5. *If* $N - q \leqq N^{1-c}$, *then* (3.1) *is true.*

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LEMMA 6. *If* $N - q > N^{1-c}$, *then*

$$M_{nq} \sim Hc_q(N)L(\beta, a).$$

4. Proof of Lemma 5. We need first two preliminary lemmas.

LEMMA 7. *If* $R = o(n^{4/3})$, *then*

$$\log P = \log H - (kR/n) + o(1).$$

If ξ is small, we have

$$(4.1) \qquad (1 - \xi)\log(1 - \xi) = -\xi + \tfrac{1}{2}\xi^2 + 0(\xi^3).$$

We write $\xi_i = (n - ks_i)/n$, so that $\sum \xi_i = 0$, $\sum \xi_i^2 = 2k^2R/n^2$, $\xi_i = o(1)$, and $n\xi_i^3 = o(1)$. Again, $s_i \to \infty$ with n. Hence, by (2.4) and (4.1), we have

$$\begin{aligned}\log(n!) - k\log(s_i!) &= \log H - n\{(1 - \xi_i)\log(1 - \xi_i) + \xi_i\} + o(1) \\ &= \log H - \tfrac{1}{2}n\xi_i^2 + o(1).\end{aligned}$$

The lemma follows when we sum over i.

LEMMA 8. *If, for a non-negative* n*-set, we have* $R > n^{1+c}$, *then*

$$\log P < \log H - kn^c + 0(1).$$

Let B_h be a non-minimal, non-negative n-set and let R_h, P_h be the corresponding values of R and P. Without loss of generality, we may take $s_1 \leqq s_2 \leqq \ldots \leqq s_k$. Since B_h is non-minimal, we have $s_k - 2 \geqq s_1 \geqq 0$. We construct B_{h+1} by replacing s_1 by $s_1 + 1$ and s_k by $s_k - 1$. It follows that $P_{h+1} = P_h s_k/(s_1 + 1) > P_h$ and from (2.1) that $R_h - R_{h+1} = s_k - s_1 - 1$, and so

$$(4.2) \qquad 1 \leqq R_h - R_{h+1} < n.$$

If we take B_1 to be the n-set of our lemma, we can construct a sequence of non-negative n-sets, viz. $B_1, B_2, \ldots, B_t$, by the above process. The P_h sequence is steadily increasing and the R_h sequence steadily decreasing, both in the strict sense. The B-sequence will come to an end at B_t, a minimal n-set. But, by (4.2), at least one member of the sequence (say B_j) will have $R_j = n^{1+c} + O(n) = o(n^{4/3})$. Hence, by Lemma 7,

$$\log P_1 < \log P_j = \log H - kn^c + O(1),$$

and this is Lemma 8.

If $q \leqq Q - V$, we have

$$(4.3) \qquad \frac{c_q(Q - V)}{c_q(Q)} = \frac{(Q - q)\ldots(Q - q - V + 1)}{Q(Q - 1)\ldots(Q - V + 1)} \leqq \frac{(Q - q)^V}{Q^V},$$

and otherwise $c_q(Q - V) = 0$.

We can now prove Lemma 5. We take $N - q \leqq N^{1-c}$ and deduce from (4.3) that

$$c_q(Q - V)/c_q(Q) \leqq AN^{-cV}.$$

For each of the $c_a(k)$ minimal n-sets, we have $R = b$, and so $P \sim H$, by Lemma 7. For all other non-negative n-sets, $P < AH$, by Lemmas 7 and 8. Hence, by Lemma 1,

$$M_{nq} - Hc_a(k)c_q(Q) \leqq AHc_q(Q)\sum_{V \geqq 1} V^{\frac{1}{2}(k-1)}N^{-cV} < AHc_q(Q)N^{-c},$$

and Lemma 5 follows.

5. Proof of Lemma 6. We write

$$J = \min(n^{c+1}, n^{c+2}/q),$$

$\sum_1$ to denote summation over all n-sets (necessarily non-negative) for which $V \leqq J$, and

$$\sum_2 = \sum_{(n)} - \sum_1, \sum_3 = \sum_{((n))} - \sum_1.$$

We also write

$$E_1 = \sum_1 \{Pc_q(N - R) - Hc_q(N)e^{-2\beta R}\},$$
$$E_2 = \sum_2 Pc_q(N - R),\ E_3 = Hc_q(N)\sum_3 e^{-2\beta R},$$

so that

$$M_{nq} - Hc_q(N)L(\beta, n) = E_1 + E_2 - E_3. \tag{5.1}$$

We have $N - q \geqq N^{1-c}$, and so $q \leqq N - N^{1-c}$. We remark that $L(\beta, n) > Ae^{-2\beta b}$, and that

$$\beta \leqq A + \tfrac{1}{2}\log(N/(N - q)) \leqq A + \tfrac{1}{2}\log N^c \leqq A + c\log n.$$

Hence,

$$L(\beta, n) > An^{-2bc}. \tag{5.2}$$

If $q \leqq n$, we have $J = n^{c+1}$ and, in $\sum_2$,

$$\log P < \log H - kn^c + o(1),$$

by Lemma 8. Hence

$$\sum_2 p < AHe^{-kn^c}\sum_2 1 \leqq AHn^{k-1}e^{-kn^c},$$

since $\sum_2 1 \leqq n^{k-1}$. Hence,

$$E_2 = o(Hc_q(N)L(\beta, n)), \tag{5.3}$$

by (5.2). If $q > n$, we have $J = n^{c+2}/q$ and, in $\sum_2$,

$$c_q(N - R) \leqq c_q(Q)\{(Q - q)/Q\}^J,$$

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by (4.3). Again,

$$J\log\left(\frac{Q-q}{Q}\right) \leqq -\frac{qJ}{Q} \leqq -\frac{n^{c+2}}{Q} \leqq -An^c.$$

Hence,

$$E_2 \leqq c_q(Q)e^{-An^c}\sum\nolimits_2 p \leqq k^n c_q(Q)e^{-An^c},$$

and (5.3) follows again.

We have also

$$\beta = -\tfrac{1}{2}\log\{(N-q)/N\} + \tfrac{1}{2}(k/n) \geqq An^{-2}(q+n),$$

and so $\beta J > An^c$. Hence, by Lemma 1,

$$\sum_3 e^{-2\beta R} \leqq Ae^{-2\beta b}\sum_{V>J} V^{k-1}\exp(-2\beta V)$$
$$\leqq An^A e^{-2\beta b - An^c},$$

and so, by (5.2),

$$E_3 = o(Hc_q(N)L(\beta, n)). \tag{5.4}$$

To deal with E_1 we need one further lemma.

LEMMA 9. *If* $0 \leqq q \leqq N - N^{1-c}$ *and* $R = o((N-q)^{2/3})$, *then*

$$\log\left(\frac{c_q(N-R)}{c_q(N)}\right) = R\log\left(1 - \frac{q}{N}\right) - \frac{qR^2}{2N(N-q)} + o(1).$$

We have

$$\frac{c_q(N-R)}{c_q(N)} = \frac{\Gamma(N-R+1)\Gamma(N-q+1)}{\Gamma(N-R-q+1)\Gamma(N+1)}.$$

We write $Y = N - q$, $\xi = R/Y$, and

$$\omega(q) = \log\Gamma(Y+1) - \log\Gamma(Y-R+1) - R\log Y + \tfrac{1}{2}R\xi.$$

It is enough to prove that $\omega(q) - \omega(0) = o(1)$. We see that $\xi = o(1)$ and that $Y\xi^3 = o(1)$. Hence, by (2.4) and (4.1),

$$\omega(q) = (Y - R + \tfrac{1}{2})\{\log Y - \log(Y-R)\} - R + \tfrac{1}{2}R\xi + o(1)$$
$$= -Y\{(1-\xi)\log(1-\xi) + \xi - \tfrac{1}{2}\xi^2\} + o(1) = o(1),$$

and the lemma follows.

In $\sum_1$, we have

$$R \leqq b + J \leqq A + \min(n^{c+1}, n^{c+2}/q).$$

Hence

$$R \leqq A + n^{c+1} = o(N^{2(1-c)/3}) = o((N-q)^{2/3}),$$

since $c + 1 < 4(1-c)/3$. Again,

$$\frac{qR^2}{N(N-q)} \leqq o(1) + \frac{qJ^2}{n^{4-2c}} \leqq o(1) + \frac{n^{2c+3}}{n^{4-2c}} = o(1).$$

Hence, in $\sum_1$, by Lemmas 7 and 9,

$$Pc_q(N - R) = Hc_q(N)e^{-2\beta R}\{1 + o(1)\},$$

and so

$$E_1 = o(Hc_q(N)L(\beta, n)).$$

Combining this with (5.1), (5.3), and (5.4), we have Lemma 6.

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