Canad. Math. Bull. Vol. 18 (5), 1975

USING ROLLE'S THEOREM IN EXPONENTIAL FUNCTION-DERIVATIVE APPROXIMATION

BY
L. KEENER[1]

1. **Introduction.** For a continuously differentiable function g defined on an interval $[\alpha, \beta]$, define $\|g\|$ to be the uniform norm of g, i.e. $\|g\|=\sup_{X\in[\alpha,\beta]}|g(x)|$. Define $\|g\|_1$, by $\|g\|_1=\max\{\|g\|, \|g'\|\}$. We call the norm $\|\cdot\|_1$ the function-derivative norm. Using the notation of Werner [3], we define for $n \geq 1$:

$$E_n^+ = \left\{y(x) \mid y(x) = \sum_{j=1}^{n} c_j \exp(\lambda_j x), \quad \lambda_j \text{ real}, \quad c_j \geq 0\right\}$$

$$E_n^0 = \left\{y(x) \mid y(x) = \sum_{j=1}^{n} c_j \exp(\lambda_j x), \quad \lambda_j \text{ real}, \quad c_j \text{ real}\right\}$$

$$E_n = \left\{y(x) \mid y(x) = \sum_{j=1}^{\ell} p_j(x)\exp(\lambda_j x), \quad \lambda_j \text{ real}, \right.$$
$$\left. p_j = \text{a polynomial of degree } \partial p_j \text{ with real coefficients and } k = \sum_{j=1}^{\ell}(\partial p_j+1) \leq n\right\}$$

k is called the degree of the function y. Since $E_n^+\subset E_n$, and $E_n^0\subset E_n$, this definition also applies to elements of these sets.

In the succeeding sections, we will find, by a simple application of Rolle's theorem, a sufficient condition for $y \in E_n^+$ or E_n^0 or E_n to be a best approximation to a given continuously differentiable f, using the function-derivative norm. The following lemma is useful [2].

LEMMA. *Every $y \in E_n$, E_n^o or E_n^+ has at most $n-1$ zeros or else vanishes identically.*

2. Let f and F be continuously differentiable functions on $[\alpha, \beta]$ and let $\varepsilon(x)=f(x)-F(x)$. Let X be a finite set of points $\{x_i\}_{i=0}^m$ such that $\alpha\leq x_0<x_1<\ldots<x_m\leq\beta$, $|\varepsilon(x_i)|=\|F-f\|_1$ for $i=0, 1, \ldots, m$ and such that $\varepsilon(x_i)=-\varepsilon(x_{i+1})$ for $i=0, 1, \ldots, m-1$. Similarly, let Y be a set of points $\{y_i\}_{i=0}^s$ such that $\alpha\leq y_0<y_1<\ldots<y_s\leq\beta$, $|\varepsilon'(y_i)|=\|F-f\|_1 f$ for $i=0, 1, \ldots, s$ and such that $\varepsilon'(y_i)=-\varepsilon'(y_{i+1})$ for $i=0, 1, \ldots, s-1$. Let $\tilde{X}$ be a subset of $X \cup \{\alpha, \beta\}$ that contains at least two elements, where $\tilde{X}=\{\tilde{x}_i\}_{i=0}^p$ and $\tilde{x}_0<\tilde{x}_1<\ldots<\tilde{x}_p$. Define $R_i(F, X, \tilde{X}, Y)$ by

$$R_i(F, X, \tilde{X}, Y) = \max\{\text{card}\{y \in Y: y \in [\tilde{x}_i, \tilde{x}_{i+1}]\}-1$$
$$-\max\{\text{card}\{x \in X: x \in [\tilde{x}_i, \tilde{x}_{i+1}]\}-v_i-u_i, 0\}, 0\}$$

[1] The author is thankful to the referee for his helpful suggestions.

https://doi.org/10.4153/CMB-1975-131-5 Published online by Cambridge University Press

where

$$v_i = \begin{cases} 0 \text{ if } \exists x \in X \text{ such that } x > \tilde{x}_{i+1} \\ 1 \text{ otherwise} \end{cases}$$

$$u_i = \begin{cases} 0 \text{ if } \exists x \in X \text{ such that } x < \tilde{x}_i \\ 1 \text{ otherwise.} \end{cases}$$

Define $T(F, X, \tilde{X}, Y) \equiv \sum_{i=0}^{p-1} R_i(F, X, \tilde{X}, Y) + \max\{m-1, 0\}$. With these definitions we may state:

THEOREM. *Let $F \in E_n^o$, E_n or E_n^+ with degree k. If there are X, $\tilde{X}$, Y such that $T(F, X, \tilde{X}, Y) \geq k+n$, then F is a best approximation to f in norm $\|\cdot\|_1$.*

Proof. Fix X, $\tilde{X}$ and Y so that $T(F, X, \tilde{X}, Y) \geq k+n$, assuming card $X \geq 2$. Suppose F is not a best approximation. Since X contains $m+1$ points, then standard arguments show that the graph of a better approximation F^* would have to intersect the graph of F at at least m points. That is $F - F^* = 0$ at least m times on $[\alpha, \beta]$. By Rolle's Theorem, there are $m-1$ points in $[\alpha, \beta]$ where $F' - (F^*)' = 0$. Call this set of points Z. Now $R_i(F, X, \tilde{X}, Y)$ represents a lower bound for the number of zeros of $F' - (F^*)'$ on $[\tilde{x}_i, \tilde{x}_{i+1}]$ that are not in Z (via the intermediate value theorem.) Thus $T(F, X, \tilde{X}, Y)$ represents a lower bound on the number of zeros of $F' - (F^*)'$ on $[\alpha, \beta]$. But $F' - (F^*)'$ clearly has degree less than or equal to $n+k$, so $F' - (F^*)' = 0$ at at most $k+n-1$ points by the Lemma. But $T(F, X, \tilde{X}, Y) > k+n-1$, a contradiction, so F is a best function-derivative approximation to f. If card $X < 2$, the proof is similar.

This theorem is useful in constructing an example of a function and a best function-derivative approximation to it.

3. EXAMPLE. Consider the following function defined on $[0, \pi+12]$.

$$h(x) = \begin{cases} (1/2) \sin 2x \text{ for } 0 \leq x \leq \pi \\ g(x) \text{ for } \pi \leq x \leq \pi+12 \end{cases}$$

where $g(x)$ is any function defined on $[\pi, \pi+12]$ with the following properties.

1. g is differentiable on $[\pi, \pi+12]$.
2. $|g'(x)| < 1$ for $x \in (\pi, \pi+12]$, $g'(\pi) = 1$.
3. $g(\pi+2) = 1$, $g(\pi+6) = -1$, $g(\pi+10) = 1$, $g(\pi) = 0$.
4. $|g(x)| < 1$ if $x \neq \pi+2$, $\pi+6$ or $\pi+10$.

Let $m(x) = h(x) + \exp(x)$. Then $F(x) = \exp(x)$ is the best approximation to $m(x)$ from E_2^o, in norm $\|\cdot\|_1$. This can be shown as follows: First note that $\varepsilon(x) = m(x) - F(x) = h(x)$. Take $X = \{\pi+2, \pi+6, \pi+10\}$. Then $m = 2$. Note that $\|\varepsilon(x)\|_1 = 1$.

Take $\tilde{X}=\{0, \pi+2\}$ and take $Y=\{0, \pi/2, \pi\}$. Then

$$T(F, X, \tilde{X}, Y)=2-1+\max\{\text{card}\{y \in Y: y \in [0, \pi+2]\}$$
$$-1-\max\{\text{card}\{x \in X: x \in [0, \pi+2]\}-1, 0\}, 0\}$$
$$=1+\max\{2-\max\{0, 0\}, 0\} = 3.$$

Since $n=2$ and $k=1$, $T(F, X, \tilde{X}, Y) \geq n+k$, so by the theorem $\exp(x)$ is a best approximation to $m(x)$ from E_2^o.

4. The example given above is not a degenerate example. That is, $\exp(x)$ is not a best uniform approximation to $m(x)$ from E_n^o and $d(\exp(x))/dx=\exp(x)$ is not a best uniform approximation to $m'(x)$ from $(E_n^o)'=\{y': y \in E_n^o)\}$. This is shown in [1].

The theorem, while stated for approximation by exponential sums can obviously be generalized. Exponential sums offer a convenient example of the application of the idea of the theorem to non-linear approximation. It is also possible (but notationally discouraging) to extend the theorem to the norm $\|\cdot\|_r$, defined by

$$\|g\|_r = \max\{\|g\|, \|g'\|, \ldots, \|g^{(r)}\|\}.$$

References

1. Keener, L. *Doctoral Dissertation*, Rensselaer Polytechnic Institute, 1972.

2. Polya, G. and Szego, G. *Aufgaben und Lehrsätze der Analysis*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

3. Werner, H. "*Tschebyscheff-Approximation with sums of exponentials*", Approximation Theory, A. Talbot ed., Academic Press, London-New York, 1970, pp. 109–136.