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On the ergodicity of geodesic flows

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To the memory of V. M. Alexeyev

Abstract. In this paper we study the ergodic properties of the geodesic flows on compact manifolds of non-positive curvature. We prove that the geodesic flow is ergodic and Bernoulli if there exists a geodesic γ such that there is no parallel Jacobi field along γ orthogonal to $\dot{\gamma}$. In particular, this is true if there exists a tangent vector $\mathbf{v}$ such that the sectional curvature is strictly negative for all two-planes containing $\mathbf{v}$, or if there exists a tangent vector $\mathbf{v}$ such that the second fundamental form of the horosphere determined by $\mathbf{v}$ is definite at the support of $\mathbf{v}$.

Let M be a compact connected smooth n-dimensional Riemannian manifold of non-positive sectional curvature. Denote by g^t the geodesic flow in the unit tangent bundle SM of M. It is well-known that g^t preserves the natural Liouville measure in SM which is the direct product of the Riemannian volume on M and the Lebesgue measure on S^{n-1}.

THEOREM 1. *If there is a geodesic γ such that there is no non-zero parallel Jacobi field along γ orthogonal to $\dot{\gamma}$, then g^t is ergodic and Bernoulli.*

COROLLARY 1. *If there is a tangent vector $\mathbf{v} \in SM$ such that the horosphere determined by $\mathbf{v}$ is strictly convex (i.e., the second fundamental form of the horosphere is definite at the support of $\mathbf{v}$), then the geodesic flow is ergodic and Bernoulli.*

Proof. There is no parallel Jacobi field along the geodesic determined by $\mathbf{v}$.

COROLLARY 2. *If there is a tangent vector $\mathbf{v} \in SM$ such that the sectional curvature is strictly negative for all two-planes containing $\mathbf{v}$, then g^t is ergodic and Bernoulli.*

Proof. If J is a non-zero parallel Jacobi field along a geodesic γ and orthogonal to $\dot{\gamma}$, then the sectional curvature of the plane $(\dot{\gamma}(t), J(t))$ vanishes for all t.

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Proof of theorem 1. Let $\mathbf{v} \in SM$ and $w \in T_{\mathbf{v}}SM$, $w \neq 0$. Define the characteristic exponents of w by the formulae

$$\chi^{+}(\mathbf{v}, w) = \limsup_{t \to \infty} \left(\frac{1}{t} \ln \|dg^t w\| \right),$$

$$\chi^{-}(\mathbf{v}, w) = \limsup_{t \to -\infty} \left(-\frac{1}{t} \ln \|dg^t w\| \right).$$

Let $G_{\mathbf{v}}$ be the vector field on SM corresponding to the geodesic flow and let

$$\Lambda^{+} = \{\mathbf{v} \in SM \mid \chi^{+}(\mathbf{v}, w) \neq 0 \quad \text{for every } w \in T_{\mathbf{v}}SM,\ w \neq 0,\ w \perp G_{\mathbf{v}}\},$$
$$\Lambda^{-} = \{\mathbf{v} \in SM \mid \chi^{-}(\mathbf{v}, w) \neq 0 \quad \text{for every } w \in T_{\mathbf{v}}SM,\ w \neq 0,\ w \perp G_{\mathbf{v}}\}.$$

Taking into account the natural identification of the tangent spaces $T_{\mathbf{v}}SM$ and $T_{-\mathbf{v}}SM$ we get

$$\chi^{+}(-\mathbf{v}, w) = \chi^{-}(\mathbf{v}, w),$$

and hence,

$$\Lambda^{+} = -\Lambda^{-}.$$

It follows from the Oseledeč multiplicative ergodic theorem that $\Lambda^{+} = \Lambda^{-} = \Lambda \pmod 0$, where Λ is the set of regular points of g^t in SM with all exponents non-zero (see [**4**] and [**5**], § 3).

LEMMA 1. *If the hypothesis of theorem* 1 *is satisfied, then* Λ^{+} (*and, hence, also* Λ^{-} *and* Λ) *has positive measure.*

Proof. Let $K(\mathbf{v}, u)$ be the sectional curvature of the plane $(u, \mathbf{v})$, and for $\mathbf{v} \in SM$ let $\gamma_{\mathbf{v}}$ be the geodesic determined by $\mathbf{v}$. A Jacobi field $J(t)$ along a geodesic γ is called asymptotic if $J(t) \perp \dot{\gamma}(t)$ and $\|J(t)\| \leq \|J(0)\|$ for all $t \geq 0$. For every $\mathbf{v} \perp \dot{\gamma}(0)$ there exists a unique asymptotic Jacobi field $J_{\mathbf{v}}(t)$ along $\gamma_{\mathbf{v}}$ such that $J_{\mathbf{v}}(0) = \mathbf{v}$.

We will show now that there exists a geodesic γ such that

$$K(\dot{\gamma}(t), J(t)) < 0 \quad \text{for a } t = t(J) \geq 0$$

for every asymptotic Jacobi field J along γ. Suppose this is not true. Then for any $\mathbf{v} \in SM$ there is an asymptotic Jacobi field $J_{\mathbf{v}}$ along $\gamma_{\mathbf{v}}$ such that

$$K(\dot{\gamma}_{\mathbf{v}}(t), J_{\mathbf{v}}(t)) = 0 \quad \text{for all } t \geq 0.$$

Set $\mathbf{v}_n = g^{-n}\mathbf{v}$, $n \in \mathbb{N}$. Renormalizing $J_{\mathbf{v}_n}$, if necessary, we can assume that $\|J_{\mathbf{v}_n}(n)\| = 1$. The vectors $J_{\mathbf{v}_n}(n)$ are orthogonal to $\mathbf{v}$, and a subsequence of $\{J_{\mathbf{v}_n}(n)\}$ converges. The asymptotic Jacobi field J along $\gamma_{\mathbf{v}}$ determined by the limit satisfies

$$K(\dot{\gamma}_{\mathbf{v}}(t), J(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

It follows that J is parallel. Indeed, $\langle R(X, \dot{\gamma}_{\mathbf{v}})\dot{\gamma}_{\mathbf{v}}, X\rangle \leq 0$ for every X, since the curvature is non-positive (R is the Riemann tensor). Hence, $\langle R(X, \dot{\gamma}_{\mathbf{v}})\dot{\gamma}_{\mathbf{v}}, X\rangle = 0$ implies $R(X, \dot{\gamma}_{\mathbf{v}})\dot{\gamma}_{\mathbf{v}} = 0$. Therefore, $\nabla^2 J = \nabla^2 J + R(J, \dot{\gamma}_{\mathbf{v}})\dot{\gamma}_{\mathbf{v}} = 0$, and thus ∇J is parallel. Let $\{X_i\}$ be a basis of parallel fields along $\gamma_{\mathbf{v}}$. Then $\nabla J = \sum \alpha_i X_i$ and hence $J = \sum (\alpha_i t + \beta_i) X_i$. Since J is asymptotic, $\alpha_i = 0$ for all i, and J is parallel. Conversely, if J is a parallel Jacobi field along a geodesic γ, then $K(\dot{\gamma}(t), J(t)) = 0$ for all $t \in \mathbb{R}$.

We conclude that for the geodesic γ given by the hypothesis of theorem 1

$$K(\dot\gamma(t), J(t)) < 0 \quad \text{for a } t = t(J) \geq 0$$

for every asymptotic Jacobi field J along γ.

Because J depends continuously on $J(0)$, by compactness there exist $T > 0$ and $b < 0$ such that

$$\int_0^T K(\dot\gamma(t), J(t))\, dt < b < 0$$

for all asymptotic Jacobi fields J along γ. Since the limit of a sequence of asymptotic Jacobi fields is an asymptotic Jacobi field, there is an open neighbourhood U of $\dot\gamma(0)$ in SM such that for every $\mathbf{v} \in U$

$$\int_0^T K(g^t\mathbf{v}, J(t))\, dt < b < 0$$

for every asymptotic Jacobi field J at $\mathbf{v}$.

According to the Birkhoff ergodic theorem, for almost every vector $v \in U$ the trajectory $g^t v$ will return to U regularly, i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \chi_U(g^s\mathbf{v})\, ds > 0,$$

where χ_U is the characteristic function of U. Every such vector $\mathbf{v}$ is contained in the set

$$\Gamma = \left\{\mathbf{v} \in SM \,\middle|\, \limsup_{T\to\infty} \frac{1}{T} \int_0^T K(g^t\mathbf{v}, J(t))\, dt < 0 \text{ for every asymptotic Jacobi field } J\right\}.$$

By theorem 10.5 in [**5**] we have

$$\Gamma \subset \Lambda^+.$$

Therefore, Λ^+ has positive measure. Lemma 1 is proved. □

We shall show now that g^t is ergodic and Bernoulli. In [**6**] Pesin proved that g^t is ergodic and Bernoulli if Γ has positive measure and M satisfies the visibility axiom (see [**2**]). The rest of our argument is a modification of his proof.

LEMMA 2. *The flow g^t is topologically transitive.*

Proof. By assumption there is a geodesic which does not bound a flat strip. By [**1**] (see theorem 4.7), g^t is topologically transitive. The lemma is proved. □

Let H be the universal cover of M.

LEMMA 3. *Let U be a bounded open subset of H, V an open subset of the absolute $H(\infty)$ whose complement has a non-empty interior, and W a neighbourhood in $H \cup H(\infty)$ of a point $z \in H(\infty)$.*

Then there exists an element ϕ of $\pi_1(M)$ such that $\phi(U \cup V) \subset W$.

Proof. This follows immediately from lemma 4.4 in [**1**]. □

For each $v \in SH$ denote by $W^s(\mathbf{v})$ the set of vectors $\mathbf{v}' \in SH$ supported on the horosphere determined by $\mathbf{v}$, perpendicular to the horosphere, and pointing in the same direction as $\mathbf{v}$. Let $W^u(\mathbf{v}) = -W^s(-\mathbf{v})$.

LEMMA 4. *Let* $\mathbf{v} \in \Lambda$ *and* o *be an open neighbourhood of* $-\mathbf{v}$ *in* $W^u(-\mathbf{v})$. *Then the set* $\{\gamma_{\mathbf{v}'}(\infty) | v' \in o\}$ *contains an open neighbourhood of* $\gamma_v(-\infty)$.

Proof. The geodesic $\gamma_{\mathbf{v}}$ does not bound a flat strip since $\mathbf{v} \in \Lambda$. It follows from lemma 2.2 in [**1**] that for any $x \in H(\infty)$ sufficiently close to $\gamma_{\mathbf{v}}(-\infty)$ there exists a geodesic γ_x such that $\gamma_x(-\infty) = \gamma_{\mathbf{v}}(\infty)$ and $\gamma_x(\infty) = x$. We can parametrize γ_x in such a way that $\dot{\gamma}_x(0) \in W^u(-\mathbf{v})$.

If the assertion of the lemma is not true, then it follows from the above that there exists a sequence of geodesics γ_n such that

(i) $\gamma_n(\infty) = \gamma_{\mathbf{v}}(\infty)$;
(ii) $\gamma_n(-\infty) \to \gamma_{\mathbf{v}}(-\infty)$ as $n \to \infty$;
(iii) $d(\gamma_n(0), \gamma_{\mathbf{v}}(0)) \geq C > 0$.

It follows from [**1**, lemma 2.1] that $\gamma_{\mathbf{v}}$ bounds a flat strip. This is a contradiction. The lemma is proved. □

The rest of the proof of theorem 1 proceeds as in [**6**] (see theorem 9.1).

The main ideas of Pesin's proof are the following. Consider all objects on the universal cover H of M. For almost every $\mathbf{v} \in \Lambda$ the strong stable and unstable manifolds of $\mathbf{v}$ (see [**5**] or [**6**] for the definition) are exactly $W^s(\mathbf{v})$ and $W^u(\mathbf{v})$ (see [**6**, lemma 9.4]). Obviously, Λ^+ consists of entire stable manifolds and Λ^- consists of entire unstable manifolds. Since $\Lambda^+ = \Lambda^- = \Lambda \pmod 0$ by Oseledeč theorem [**4**], Λ consists mod 0 of entire stable and unstable manifolds. Hence, by the absolute continuity of the stable and unstable foliations, for almost every $\mathbf{v} \in \Lambda$ we have $W^u(\mathbf{v}) \subset \Lambda$ and $W^s(\mathbf{v}) \subset \Lambda$ mod 0 with respect to the Lebesgue measure on $W^u(\mathbf{v})$ and $W^s(\mathbf{v})$. The set Λ^+ consists of entire stable manifolds and is g^t-invariant, therefore, Λ^+ consists of entire weak stable manifolds

$$W^{0s}(\mathbf{v}) = \bigcup_{-\infty < t < \infty} W^s(g^t\mathbf{v}).$$

Since the foliation W^{0s} is absolutely continuous on Λ, the set Λ consists mod 0 of entire weak stable manifolds. Call a point $\mathbf{v} \in \Lambda$ 'good' if $W^u(\mathbf{v})$ almost entirely belongs to Λ. Almost every point of Λ is 'good', therefore we can find a leaf $W^{0s}(\mathbf{v})$, $\mathbf{v} \in \Lambda$, which consists mod 0 of 'good' points, i.e.

$$A = \bigcup_{\mathbf{v}' \in W^{0s}(\mathbf{v})} W^u(\mathbf{v}') \subset \Lambda \pmod 0.$$

The set $\bigcup_{\mathbf{v}' \in W^s(\mathbf{v})} \gamma_{\mathbf{v}'}(-\infty)$ contains an open neighbourhood of $\gamma_{\mathbf{v}}(-\infty)$ in $H(\infty)$ (see lemma 4). Lemma 3 shows that $SH \subset \Lambda$ mod 0. The ergodicity now follows from theorem 9.5 in [**5**] since g^t is topologically transitive (lemma 2).

THEOREM 2. *If* Λ^+ *is not empty, then* g^t *is ergodic and Bernoulli.*

Proof. If $\mathbf{v} \in \Lambda^+$, then there is no parallel Jacobi field along the geodesic $\gamma_{\mathbf{v}}$ determined by $\mathbf{v}$. □

COROLLARY 3. *If either of the assumptions of theorem* 1 *or* 2 *is satisfied, then the number of geometrically distinct closed geodesics grows exponentially with the length and vectors determining closed geodesics are dense in the tangent bundle.*

Proof. This follows from [**3**]. The density of closed geodesics also follows from [**1**]. Namely, the set X of $\mathbf{v}$ such that there is no non-zero parallel Jacobi field along $\gamma_{\mathbf{v}}$ orthogonal to $\dot{\gamma}_{\mathbf{v}}$ is open and, as we assume, not empty. Since X is invariant under g^t, it follows from [**1**, theorem 4.7], or from theorem 1 above, that X is dense in SM. By [**1**, theorem 4.7], every $\mathbf{v} \in SM$ is the limit of vectors which determine closed geodesics. The corollary is proved. □

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