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Asymptotics applied to nonlinear boundary-value problems

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We announce results of Landesman-Lazer type for boundary-value problems for ordinary differential equations. Details will appear elsewhere.

We announce results for boundary-value problems. We look for solutions of

$$Lu(t) = g(u(t)) - f(t) \tag{1}$$

on $[0, \pi]$, where $f \in L^\infty[0, \pi]$, $g : R \to R$ is smooth, and $Lu = a(t)u'' + b(t)u' + c(t)u$ is a regular differential operator with smooth coefficients incorporating either Dirichlet or periodic boundary conditions such that $N(L)$, the kernel of L, is non-trivial. (In the case of periodic boundary conditions, we assume that a, b, and c are periodic.) Our results show the importance of asymptotics in the study of these problems.

Suppose that $h \in N(L)\backslash\{0\}$ and $k \in N(L^*)$ such that $\int_0^\pi k^2 = 1$, and let $R_1 = \{f \in L^\infty[0, \pi] : (1) \text{ has a solution}\}$.

1. Non self-adjoint problems

We assume periodic boundary conditions, and assume that $\int_0^\pi a^{-1}b \neq 0$ and h has a zero in $[0, \pi]$. (The other cases behave like the self-

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adjoint case.) Then h spans $N(L)$. We let

$$E_0 = \{v \in L^\infty[0, \pi] : \langle v, k\rangle = 0\} ,$$

where $\langle\ ,\ \rangle$ denotes the usual scalar product. Assume that $\lim_{y\to\pm\infty} g(y) = I^\pm$ exist (and are finite) and that $I^- < g(y) < I^+$ for all y . Let

$$\mu_1 = \int_{h>0} k , \quad \mu_2 = \int_{h<0} k , \quad \beta_1 = I^- \int_0^\pi k^+ + I^+ \int_0^\pi k^- ,$$

$$\beta_2 = I^+ \int_0^\pi k^+ + I^- \int_0^\pi k^- \ \left(\text{where } k^+ = \sup\{k, 0\} \text{ and } k^- = k - k^+\right),$$

$$A_1 = \{\alpha \in R : (I^+\mu_1+I^-\mu_2-\alpha)(I^-\mu_1+I^+\mu_2-\alpha) \le 0\} , \quad A_2 = \operatorname{int} A_1 ,$$

and $A_3(v) = \{\alpha \in R : v+\alpha k \in R_1\}$. We find that $A_1 \subset (\beta_1, \beta_2)$ (since h and k have no common zero). There are examples where $A_2 = \emptyset$ (that is, is empty). However, $A_1 \ne \emptyset$ and $A_2 \ne \emptyset$ if L is "nearly self-adjoint".

THEOREM 1. *(i) If* $v \in E_0$ *, then* $A_3(v) \ne \emptyset$ *, and*

$$A_2 \subseteq A_3(v) \subset (\beta_1, \beta_2) .$$

(ii) $\beta_2 = \sup\{\alpha : \alpha \in A(v), v \in E_0\}$ *and*

$$\beta_1 = \inf\{\alpha : \alpha \in A(v), v \in E_0\} .$$

(iii) $A_3(v)\backslash A_1$ *is relatively closed in* $(\beta_1, \beta_2)\backslash A_1$.

(iv) If $Lu_n = g(u_n) - f_n$ *,* $\{f_n\}$ *is bounded (in* $L^\infty[0, \pi]$ *) and* $\langle u_n, h\rangle \to \infty$ *as* $n \to \infty$ *, then* $\langle f_n, k\rangle \to I^+\mu_1 + I^-\mu_2$ *as* $n \to \infty$ *. (A similar result holds if* $\langle u_n, h\rangle \to -\infty$ *.)*

The result is proved by standard arguments. Part *(ii)* is proved by constructing suitable u_n so that $g(u_n) - Lu_n$ has the required property. If $\sup\{|g'(y)| : y \in R\}$ is sufficiently small, it can be shown that $A_3(v)$ is an interval. A version of Theorem 1 holds much more generally.

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(In the case of a multi-dimensional kernel, we define A_2 by requiring that an appropriate degree be non-zero.)

Theorem 1 suggests that, unlike the self-adjoint case, it is impossible to obtain a simple formula for R_1. A natural question is whether $A_1 \subseteq A_3(v)$. In general this is false, since there is an example where $A_3(v) = A_2$ for some $v \in E_0$. We need the following lemma.

LEMMA 1. *(i) If $v \in E_0$, there is a connected set T of solutions of*

(2) $$Lu = Pg(u) - v$$

(where $Pw = w - \langle w, k\rangle k$) such that $\{\langle u, h\rangle : u \in T\} = R$.

(ii) If $g'(y) \to 0$ as $|y| \to \infty$, then there is a $K > 0$ such that, for each α with $|\alpha| \geq K$, (2) has a unique solution $\alpha h + \Delta(\alpha)$ with $\langle\Delta(\alpha), h\rangle = 0$ and the mapping $\alpha \to \Delta(\alpha)$ is smooth.

The first part is proved by a standard degree argument while the second is proved by combining the argument in the example in [*1*] with the implicit function theorem.

Assume now that $g'(y) \to 0$ as $|y| \to \infty$. By combining a study of the asymptotic behaviour of $t'(\alpha)$ as $|\alpha| \to \infty$ (where

$$t(\alpha) = \int_0^\pi g(\alpha h+\Delta(\alpha))k\ )$$

with Theorem 1 *(iv)*, we can obtain information on $A_3(v)$. For example, if $t'(\alpha) < 0$ for $|\alpha|$ large and $I^+\mu_1 + I^-\mu_2 > I^-\mu_1 + I^+\mu_2$, then $A_1 \subseteq \operatorname{int} A_3(v)$. We illustrate what can be obtained by giving the asymptotic formula for $t'(\alpha)$ as $\alpha \to \infty$ in two cases. If $g'(y) \sim |y|^{-\beta}$ as $|y| \to \infty$, where $\beta < 2$, then $t'(\alpha) \sim \alpha^{-\beta}\int_0^\pi |h|^{-\beta}hk$ as $\alpha \to \infty$. If $y^s g'(y) \to 0$ as $|y| \to \infty$ for some $s > 2$ and h has two zeros x_0, x_1 in $[0, \pi]$, then

$$t'(\alpha) \sim \alpha^{-2}\Big[(\tau_1+\tau_2)\int_{-\infty}^{\infty} g'(y)y\,dy-(s_1\tau_1+s_2\tau_2)(I^+-I^-)\Big] .$$

Here, for $i = 0, 1$, $\tau_i = |h'(x_i)|^{-1}k(x_i)$ and s_i is the value at x_i of a solution of $Lw = P(r-v)$, where $r(t) = I^+$ if $h(t) \geq 0$ and $r(t) = I^-$ otherwise. Note that, in the second case, the asymptotic formula depends on v, while in the first it does not.

We now briefly discuss the case where g is sublinear and $g(y)\operatorname{sgn} y \to \infty$ as $|y| \to \infty$. There is an example where g is odd and $g(y) = |y|^t \operatorname{sgn} y$ for $|y|$ large (where $0 < t < \frac{1}{2}$) but $R_1 \neq L^\infty[0, \pi]$. This contrasts with the self-adjoint case (as in [2]). Sufficient conditions for R_1 to be $L^\infty[0, \pi]$ can easily be found if g has power growth.

Some of the above results can be extended to boundary-value problems for elliptic partial differential equations. It is more difficult to construct counter-examples for this case but it would appear unlikely that this case is better behaved than the case of ordinary differential equations considered above.

2. The case where $g(y) \to 0$ as $|y| \to \infty$ and L is self-adjoint

We consider the case where $h(x) > 0$ on $(0, \pi)$. We assume Dirichlet boundary conditions. (The other case is much easier.) We say that g is *regular* if $g'(y) \leq 0$ for $|y|$ large.

THEOREM 2. *Suppose that $yg(y) \geq 0$ for $|y|$ large and either*

(i) $\int_0^\infty g(y)y\,dy > 0$ *and* $\int_{-\infty}^0 g(y)y\,dy > 0$ *(where the integrals may be infinite), or*

(ii) g *is regular,* $\int_{-\infty}^0 g(y)y\,dy < 0$ *and* $\int_0^\infty g(y)y\,dy < 0$.

Then there exists $\varepsilon : E_0 \to (0, \infty)$ *such that* $\alpha h + v \in R_1$ *if* $|\alpha| \leq \varepsilon(v)$.

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This is proved by using Lemma 1 *(i)* and by estimating $\int_0^\pi g(\alpha h+w(\alpha))h$ where $|\alpha|$ is large. (Here $\alpha h + w(\alpha) \in T$. Note that w may be multi-valued.) It is more convenient to study $\int_0^\pi \alpha^2 g(\alpha h+w(\alpha))h$. We use changes of variable of the form $u = \alpha h'(0)x$.

Theorem 2 still holds if L is not self-adjoint. Moreover, the result readily generalises to elliptic partial differential equations. (We need to assume that $y^2 g(y) \to 0$ in case *(ii)*.) The methods in §4 of [3] can be used to deduce more precise information on R_1.

The above method can be used if h has zeros in $(0, \pi)$. Theorem 2 holds in this case if $\int_{-\infty}^{\infty} g(y)ydy$ diverges. If $\int_{-\infty}^{\infty} g(y)ydy$ converges and g is regular, there exist $a, b \in R$ and a linear map $l : E_0 \to R$ such that $v + \alpha h \in R_1$ for $|\alpha| \le \varepsilon$ (where $\varepsilon > 0$) *if* $(a-l(v))(b-l(v)) > 0$. (For example, if $\int_{-\infty}^{\infty} g(y)dy = 0$ and $h'(0) = -h'(\pi)$, this last condition becomes $\int_{-\infty}^{\infty} g(y)ydy \ne 0$.) There is an example where $(a-l(v))(b-l(v)) < 0$ and $v \notin R_1$. Our methods can be used to obtain results for periodic boundary conditions and to obtain partial results for elliptic partial differential equations when h has interior zeros (mainly when $N(L)$ is one-dimensional). This is much more complicated due to the greater variety of possible behaviour of h near zeros. We must replace $\int_{-\infty}^{\infty} g(y)ydy$ by $\int_{-\infty}^{\infty} g(y)|y|^{1/m} \operatorname{sgn} y dy$, where $m \ge 1$, m depends on the behaviour of h near its zeros, m may be arbitrarily large, and $m = 1$ in the "generic" case.

3. The case where $g(y) = a \sin y$

We first assume Dirichlet boundary conditions.

THEOREM 3. *Assume that* $h''(x) \ne 0$ *whenever* $h'(x) = 0$ *and that*

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there exists an x_0 *such that* $h'(x_0) = 0$ *and* $h'(x) \neq 0$ *whenever* $x \neq x_0$ *and* $|h(x)| = |h(x_0)|$. *Then there exists* $\varepsilon : E_0 \to (0, \pi)$ *such that* $\alpha h + v \in R_1$ *if* $|\alpha| \leq \varepsilon(v)$. *Moreover, if* $f \in E_0$, (1) *has an infinite number of solutions.*

As before, we prove this by examining the asymptotic behaviour of $\int_0^{\pi} \sin(\alpha h + w(\alpha))h$. It is more convenient to study $\int_0^{\pi} \exp(i\alpha h)s(\alpha)$, where $s(\alpha)(x) = \exp(iw(\alpha)(x))h(x)$. We do this by the method of stationary phase (as in §2.9 of [4]). (Although $s(\alpha)$ depends on α, $s(\alpha)$, $(s(\alpha))'(x)$, $(s(\alpha))''(x)$ are uniformly bounded and $s(\alpha)$ has a limit as $|\alpha| \to \infty$.) We find that $\int_0^{\pi} \sin(\alpha h + w(\alpha))h$ oscillates and tends to zero as $|\alpha| \to \infty$. This gives the result.

The above method still applies if the technical assumptions on h are weakened except that there may be a proper subset of E_0 for which our proof fails. The above method can also be applied to some elliptic partial differential equations in two or three dimensions and to some periodic boundary-value problems when h is non-constant. This last problem behaves a little differently if h is constant. In this case, $Lu = g(u) - f$ has an infinite number of solutions for each f in R_1 and R_1 is still closed but we do not know if $E_0 \subseteq \text{int } R_1$. (It is possible to obtain partial results. For example, $E_0 \subseteq R_1$ if L is self-adjoint.) If $h(x) > 0$ on $(0, \pi)$, the methods of §4 of [3] can be used to obtain more precise information on R_1 .

The results in §§2-3 partially answer a problem raised by Fučík in a lecture at Oberwolfach in 1976.

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