BULL. AUSTRAL. MATH. SOC. 20M20
VOL. 55 (1997) [453–467]

NILPOTENTS IN SEMIGROUPS OF PARTIAL TRANSFORMATIONS

R.P. SULLIVAN

In 1987, Sullivan determined when a partial transformation α of an infinite set X can be written as a product of nilpotent transformations of the same set: he showed that when this is possible and the cardinal of X is regular then α is a product of 3 or fewer nilpotents with index at most 3. Here, we show that 3 is best possible on both counts, consider the corresponding question when the cardinal of X is singular, and investigate the role of nilpotents with index 2. We also prove that the nilpotent-generated semigroup is idempotent-generated but not conversely.

1. INTRODUCTION

Throughout this paper X will denote an infinite set with cardinal m, and if n is any cardinal then n' will denote the *successor* of n (that is, the least cardinal greater than n). All notation and terminology will be from [**1**] and [**3**] unless specified otherwise. In particular, $\mathcal{T}(X)$ denotes the full transformation semigroup on X. If $\alpha \in \mathcal{T}(X)$, we let $r(\alpha)$ denote the *rank* of α (that is, $|X\alpha|$) and put

$$\begin{aligned} D(\alpha) &= X \setminus X\alpha, & d(\alpha) &= |D(\alpha)|, \\ S(\alpha) &= \{x \in X : x\alpha \neq x\}, & s(\alpha) &= |S(\alpha)|, \\ C(\alpha) &= \bigcup\{y\alpha^{-1} : \ |y\alpha^{-1}| \geqslant 2\}, & c(\alpha) &= |C(\alpha)|. \end{aligned}$$

The cardinal numbers $d(\alpha)$, $s(\alpha)$ and $c(\alpha)$ are called, respectively, the *defect*, the *shift* and the *collapse* of α and were originally used by Howie [**2**] to characterise the elements of $\mathcal{T}(X)$ that can be written as a product of idempotents in $\mathcal{T}(X)$. In particular, he later showed [**4**] that the set

$$Q_m = \{\alpha \in \mathcal{T}(X) : d(\alpha) = s(\alpha) = c(\alpha) = m\}$$

is an idempotent-generated subsemigroup of $\mathcal{T}(X)$. Later still, in [**6**] Marques considered the Rees quotient semigroup $P_m = Q_m/I_m$ where $I_m = \{\alpha \in Q_m : r(\alpha) < m\}$, an ideal of Q_m.

Received 24 June 1996
This paper was completed while visiting Dr Peter Trotter at the University of Tasmania. I thank him and his colleagues for their generous hospitality during my stay.

Among other things, she proved that for any infinite m, every element of P_m is a product of 4 or fewer idempotents and that 4 is best possible. Then in [**5**] the authors showed that if m is a regular cardinal, the set

$$K_m = \{\alpha \in P_m \colon \ |y\alpha^{-1}| = m \text{ for some } y \in X\} \cup \{0\}$$

equals the subsemigroup of P_m generated by the nilpotents in P_m. And in [**7**] the authors proved that if m is *singular* (that is, non-regular) then the subsemigroup of P_m generated by its nilpotents equals the set

$$L_m = \{\alpha \in P_m \colon \text{ for each } p < m, \text{ there exists } y \in X \text{ such that } |y\alpha^{-1}| > p\} \cup \{0\}.$$

Moreover, from [**5**] and [**7**] we know that each element of K_m and of L_m is a product of 3 or fewer nilpotents with *index* 2 (that is, $\lambda \neq 0$ and $\lambda^2 = 0$) and that 3 is best possible.

Let $\mathcal{P}(X)$ denote the semigroup of all partial transformations of X and if $\alpha \in \mathcal{P}(X)$, write $g(\alpha) = |X \setminus \operatorname{dom}\alpha|$ and call this the *gap* in α. In [**9**, Corollary 3], I proved that if m is regular then the set

$$\begin{aligned} \mathcal{L}(X) = \{\alpha \in \mathcal{P}(X) \colon d(\alpha) = m,\ g(\alpha) \geqslant 1, \\ \text{and } |y\alpha^{-1} \cup (X \setminus \operatorname{dom}\alpha)| = m \text{ for some } y \in X\} \end{aligned}$$

is the subsemigroup of $\mathcal{P}(X)$ generated by the nilpotents in $\mathcal{P}(X)$. Moreover, in this case, $\mathcal{L}(X)$ is regular and each of its elements equals a product of 3 or fewer nilpotents with index at most 3. In this paper, we show that $\mathcal{L}(X)$ is idempotent-generated; and provide bounds on the number of nilpotents (and their indices) required to express each element of $\mathcal{L}(X)$ as a product of nilpotents: we show, for example, that both the product 3 and the index 3 just mentioned are best possible. We also investigate analogous questions when m is singular.

2. Nilpotents as generators: the regular cardinal case

We extend the convention introduced in [**1**, vol.2, p.241]: namely, if $\alpha \in \mathcal{P}(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i \colon i \in I\}$, and that $X\alpha = \operatorname{ran}\alpha = \{x_i\}$, $A_i = x_i\alpha^{-1}$ and $\operatorname{dom}\alpha = \bigcup A_i$.

To compare the results in [**9**, Section 3] with those in [**5**] and [**7**], we let $\phi \notin X$, put $X^\phi = X \cup \phi$ and

$$F_\phi = \{\alpha \in \mathcal{T}(X^\phi) \colon \phi\alpha = \phi\},$$

and define $\theta\colon \mathcal{P}(X) \to F_\phi$, $\alpha \to \alpha\theta$, where $x(\alpha\theta) = x\alpha$ if $x \in \operatorname{dom}\alpha$ and $x(\alpha\theta) = \phi$ otherwise. Clearly, θ is an isomorphism and, when m is regular, the image of $\mathcal{L}(X)$ under θ is the semigroup

$$L_\phi = \{\alpha \in F_\phi\colon d(\alpha) = m,\ |\phi\alpha^{-1}| \geqslant 2, \text{ and } |y\alpha^{-1}| = m \text{ for some } y \in X^\phi\}.$$

Note that L_ϕ contains the ideal $I_\phi = \{\alpha \in L_\phi\colon r(\alpha) < m\}$ and the Rees quotient semigroup L_ϕ/I_ϕ can be identified with $\{\alpha \in L_\phi\colon r(\alpha) = m\} \cup 0$, where 0 represents the zero of L_ϕ/I_ϕ. In this way, L_ϕ/I_ϕ can be regarded as the semigroup

$$K_m(\phi) = \{\alpha \in K_m(X^\phi)\colon \phi\alpha = \phi \text{ and } |\phi\alpha^{-1}| \geqslant 2\} \cup 0.$$

In [5] the authors showed that every non-zero $\alpha \in K_m(X^\phi)$ equals a product of nilpotents in $K_m(X^\phi)$ with index 2. This is also true of $K_m(\phi)$. For, if $\alpha, \beta \in K_m(X^\phi)$ and $\phi\alpha\beta = \phi$ then there exist $\alpha' \in K_m(X^\phi)$ and $\beta' \in K_m(\phi)$ such that $\alpha\beta = \alpha'\beta'$, $\ker\alpha = \ker\alpha'$ and $\phi\alpha' = \phi$. To see this, suppose $\phi\alpha = a$, $\phi\beta = b$ and consider two cases. If $\phi \notin X\alpha$, we let $x\alpha' = \phi$ for $x \in a\alpha^{-1}$ and $x\alpha' = x\alpha$ otherwise, and let $x\beta' = \phi$ for $x \in \phi\beta^{-1} \cup \phi$ and $x\beta' = x\beta$ otherwise. On the other hand, if $\phi \in X\alpha \setminus a$, we choose $d \notin X\alpha$ and let $x\alpha' = \phi$ for $x \in a\alpha^{-1}$, $x\alpha' = a$ for $x \in \phi\alpha^{-1}$ and $x\alpha' = x\alpha$ otherwise, and let $x\beta' = \phi$ for $x \in (\phi\beta^{-1} \setminus a) \cup \phi \cup d$, $x\beta' = b$ for $x \in (b\beta^{-1} \setminus \phi) \cup a$ and $x\beta' = x\beta$ otherwise. It is now easy to check that, in both cases, α' and β' possess the required properties. Moreover, since there is little difference between the ranges of α and α' (and the kernels of β and β') it is clear that α' is nilpotent in $K_m(X^\phi)$ if and only if α is (likewise for β and β'). Consequently, if $\alpha \in K_m(\phi)$ then α is a product of nilpotents with index 2 in $K_m(X^\phi)$ and, by the foregoing remark, these can be assumed to lie in $K_m(\phi)$.

Having said all this, it will transpire from what follows that L_ϕ, the inverse image of $K_m(\phi)$ under the natural map $L_\phi \to L_\phi/I_\phi$, is not generated by its *nilpotents with index* 2 (that is, $\lambda \in L_\phi$ such that $\lambda^2 = \phi$ but $\lambda \neq \phi$ where, in this context, ϕ denotes the constant transformation in L_ϕ). In particular, it will be clear that if $X = \{a_i\} \cup \{b_i\} \cup x$ then

$$\lambda = \begin{pmatrix} a_i & \{b_i\} & \{x, \phi\} \\ b_i & x & \phi \end{pmatrix}$$

cannot be written as a product of nilpotents in L_ϕ with index 2 but, as already shown, as an element of $K_m(\phi)$ it does equal a product of nilpotents in $K_m(\phi)$ with index 2. In addition, whereas $K_m(\phi)$ is 0-bisimple (compare [**7**, Theorem 2.1]) the same is not true of L_ϕ. That is, very little information about L_ϕ can be obtained directly from [**5**] and [**7**], so we continue to work within $\mathcal{L}(X)$ itself.

The *cofinality* of m, $\operatorname{cf}(m)$, plays a fundamental role in what follows: since it is difficult to find an elementary account of the relevant facts in the literature, we summarise them in the following way, using [**10**, Theorem A.3.9] as our authority.

THEOREM 2.1. *Suppose m is an arbitrary infinite cardinal. Then* $\mathrm{cf}(m)$ *is the least cardinal n such that m can be expressed as a sum of n cardinals each less than m. Hence,* $\mathrm{cf}(m) \leqslant m$ *where equality occurs if and only if m is regular. In particular, both* $\mathrm{cf}(m)$ *and m' are regular cardinals. If m is singular then* $\mathrm{cf}(m)$ *is infinite and m can be expressed as the sum of a strictly increasing sequence of* $\mathrm{cf}(m)$ *cardinals each less than m.*

For convenience, we recall the following result from [**9**, Theorem 3, p.336 and p.341].

THEOREM 2.2. *If m is a regular cardinal then the semigroup $\mathcal{L}(X)$ is regular and each $\alpha \in \mathcal{L}(X)$ equals a product of 3 or fewer nilpotents with index at most 3. Moreover, $\mathcal{L}(X)$ contains the ideal $I_m^* = \{\alpha \in \mathcal{P}(X) \colon r(\alpha) < m\}$.*

The proof of Theorem 2.2 involves two cases: namely whether $g(\alpha) = m$ or $g(\alpha) < m$, and in the first case α can be written as a product of 3 or fewer nilpotents with index 2. We begin by characterising precisely when α is a product of nilpotents with index 2: besides its intrinsic interest, the next result shows that the index 3 in Theorem 2.2 is best possible.

THEOREM 2.3. *Suppose m is an arbitrary infinite cardinal and $\alpha \in \mathcal{P}(X)$ is non-zero. Then α is a product of nilpotents with index 2 if and only if $d(\alpha) = m$ and $g(\alpha) \geqslant r(\alpha)$. Moreover, when this occurs, α is a product of 3 or fewer nilpotents with index 2.*

PROOF: If $\lambda^2 = \emptyset$ then $X\lambda \subseteq X \backslash \mathrm{dom}\,\lambda$: that is, $r(\lambda) \leqslant g(\lambda)$ and, by [**9**, Lemmas 11 and 13] (compare Lemma 3.2 below), $d(\lambda) = m$. Hence, any nilpotent with index 2 satisfies the given conditions. Consequently, if $\lambda_1 \dots \lambda_r$ is a product of such nilpotents then $d(\lambda_1 \dots \lambda_r) = m$ and

$$r(\lambda_1 \dots \lambda_r) \leqslant r(\lambda_1) \leqslant g(\lambda_1) \leqslant g(\lambda_1 \dots \lambda_r)$$

since $d(\beta) \leqslant d(\alpha\beta)$ and $r(\alpha\beta) \leqslant \min(r(\alpha), r(\beta))$ for all $\alpha, \beta \in \mathcal{P}(X)$.

Conversely, suppose α satisfies the given conditions: what follows is essentially the argument in the first paragraph of the proof of [**9**, Theorem 3]. Suppose $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. If $|(X \setminus \mathrm{ran}\,\alpha) \cap (X \setminus \mathrm{dom}\,\alpha)| \geqslant r(\alpha)$ then we can choose $c_i \in (X \setminus \mathrm{ran}\,\alpha) \cap (X \setminus \mathrm{dom}\,\alpha)$ and write

$$\alpha = \begin{pmatrix} A_i \\ c_i \end{pmatrix} \circ \begin{pmatrix} c_i \\ x_i \end{pmatrix}$$

where each transformation on the right is nilpotent with index 2. Suppose instead that $|(X \setminus \mathrm{ran}\,\alpha) \cap (X \setminus \mathrm{dom}\,\alpha)| < r(\alpha)$. In this event, if $r(\alpha)$ is finite, we choose

https://doi.org/10.1017/S0004972700034092 Published online by Cambridge University Press

$c_i \in X \setminus \operatorname{dom}\alpha$ and $d_i \in (X \setminus X\alpha) \setminus \{c_i\}$, and put

$$\alpha = \begin{pmatrix} A_i \\ c_i \end{pmatrix} \circ \begin{pmatrix} c_i \\ d_i \end{pmatrix} \circ \begin{pmatrix} d_i \\ x_i \end{pmatrix}$$

where each transformation on the right is again nilpotent with index 2. On the other hand, if $r(\alpha)$ is infinite then $|X\alpha \cap (X \setminus \operatorname{dom}\alpha)| = r(\alpha)$, so we can choose $c_i \in X\alpha \cap (X \setminus \operatorname{dom}\alpha)$ and $d_i \in X \setminus X\alpha$ to ensure that the above decomposition of α remains valid. ∎

To show that 3 is best possible in the above result, we need to characterise when α is a product of 2 nilpotents with index 2, and for this we need to describe Green's relations on $\mathcal{L}(X)$. The following characterisation of Green's relations on $\mathcal{P}(X)$ is well-known: its proof is entirely similar to that given in [**1**, vol. 1, pp.52–53] for $\mathcal{T}(X)$, and so is omitted.

Lemma 2.4. *If $\alpha, \beta \in \mathcal{P}(X)$ then*

(a) *$\alpha\mathcal{L}\beta$ if and only if $X\alpha = X\beta$,*
(b) *$\alpha\mathcal{R}\beta$ if and only if $\ker\alpha = \ker\beta$,*
(c) *$\alpha\mathcal{D}\beta$ if and only if $r(\alpha) = r(\beta)$, and*
(d) *$\mathcal{D} = \mathcal{J}$*

The regularity of $\mathcal{L}(X)$ when m is regular was established in [**9**, p.341]. For what follows, we need a more general result.

Lemma 2.5. *If m is an arbitrary infinite cardinal then $\mathcal{L}(X)$ is a regular semigroup.*

Proof: We suppose m is singular and let $\alpha \in \mathcal{L}(X)$. Write $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. Choose $a_i \in A_i$ and define a transformation β by letting $\operatorname{dom}\beta = \{a_i\}$ and $a_i\beta = x_i$ for each $i \in I$. Then $g(\beta) = m$ since $d(\alpha) = m$, and $d(\beta) = m$ whenever $r(\alpha) < m$. If $r(\alpha) = m$ then, by [**9**, Theorem 4], $g(\alpha) = m$ or α is spread over m. In the former case, $d(\beta) = m$; and in the latter case, we know $|\bigcup A_p| = m$ for some $P \subseteq I$ with $|P| = \operatorname{cf}(m)$: that is, $X \setminus X\beta$ contains $\bigcup(A_p \setminus a_p)$, a set with cardinal m. Hence, by [**9**, Corollary 4], β is a product of nilpotents and clearly, $\alpha = \alpha\beta\alpha$. ∎

Since $\mathcal{L}(X)$ is a regular subsemigroup of $\mathcal{P}(X)$, it follows from [**3**, Proposition II.4.5] that the $\mathcal{L}$ and $\mathcal{R}$ relations on $\mathcal{L}(X)$ can be described just as in Lemma 2.4. The reason for noting this fact will be apparent after we quote the following result from [**5**, Lemma 2.5].

Lemma 2.6. *Let T be a regular semigroup with a zero 0. If $a \in T$ and $a = xy$ for some nilpotents x, y in T with index 2 then $a = x_1y_1$ for some nilpotents x_1, y_1 in T with index 2 such that $x_1\mathcal{R}a$ and $y_1\mathcal{L}a$.*

The next result should be compared with [**5**, Proposition 2.4] and [**7**, Lemma 2.2].

THEOREM 2.7. *Suppose m is an arbitrary infinite cardinal and $\alpha \in \mathcal{P}(X)$ is non-zero. Then α is a product of 2 nilpotents with index 2 if and only if $|(X \setminus \operatorname{ran}\alpha) \cap (X \setminus \operatorname{dom}\alpha)| \geqslant r(\alpha)$.*

PROOF: The second paragraph in the proof of Theorem 2.3 shows that if the condition holds then α can be written as a product of two nilpotents with index 2. So, we suppose $\alpha = \lambda\mu \neq \emptyset$ where $\lambda^2 = \mu^2 = \emptyset$. By Lemma 2.6, we can also assume $\ker\lambda = \ker\alpha$ and $X\mu = X\alpha$. Let $X\alpha = \{x_i\}$, $A_i = x_i\alpha^{-1}$ and $A_i\lambda = y_i$. Since $\lambda^2 = \emptyset$, we know $\{y_i\} \subseteq X \setminus \operatorname{dom}\lambda = X \setminus \operatorname{dom}\alpha$. Suppose, for contradiction, that $|(X \setminus \operatorname{ran}\alpha) \cap (X \setminus \operatorname{dom}\alpha)| < r(\alpha)$. Then $\{y_i\} \cap X\alpha \neq \emptyset$ where each $y_i \in \operatorname{dom}\mu$. Hence, we have $X\mu^2 = (X\alpha)\mu \supseteq (\{y_i\} \cap X\alpha)\mu \neq \emptyset$, contradicting $\mu^2 = \emptyset$. ∎

It remains to note that there exist $\alpha \in \mathcal{P}(X)$ with $d(\alpha) = m$ and $g(\alpha) \geqslant r(\alpha)$ but $|(X \setminus \operatorname{ran}\alpha) \cap (X \setminus \operatorname{dom}\alpha)| < r(\alpha)$. To see this, write $X = \{x_i\} \cup \{a_i\} \cup \{b_i\} \cup \{c_j\}$ where $|I| = m > |J|$ and put

$$\alpha = \begin{pmatrix} x_i & a_i \\ x_i & b_i \end{pmatrix}.$$

Note also that there are $\alpha \in \mathcal{P}(X)$ which cannot be written as a product of nilpotents with index 2. For example, if $X = \{a_i\} \cup \{b_i\} \cup x$ then

$$\alpha = \begin{pmatrix} a_i & \{b_i\} \\ b_i & x \end{pmatrix} \tag{1}$$

is a nilpotent with index 3 which does not satisfy the conditions of Theorem 2.3 (the need to consider such nilpotents did not arise in [**5**] and [**7**]).

The second case in the proof of Theorem 2.2 leads to α being written as a product of 3 of fewer nilpotents, the first of which has index 3 and the other two have index 2. We now show this occurs whenever α does not belong to $\mathcal{K}(X)$, *the subsemigroup of $\mathcal{L}(X)$ generated by the nilpotents in $\mathcal{L}(X)$ with index* 2.

THEOREM 2.8. *If m is regular and $\alpha \notin \mathcal{K}(X)$ then α is the product of 3 or fewer nilpotents, the first of which has index 3 and lies outside $\mathcal{K}(X)$ and the other two have index 2.*

PROOF: If $\alpha \notin \mathcal{K}(X)$ then $g(\alpha) < r(\alpha)$ and so $g(\alpha) < m$. Hence, by [**9**, Corollary 3], some $z\alpha^{-1}$ has cardinal m. Then the second paragraph in the proof of [**9**, Theorem 3], shows that α is a product of 3 nilpotents, the first having index 3 and the other two having index 2, and clearly the first cannot belong to $\mathcal{K}(X)$. ∎

It is often possible to do better in the above result and write $\alpha \notin \mathcal{K}(X)$ as a product of just two nilpotents, the first having index 3 and the second having index at most 3. The next result characterises when this occurs and at the same time shows that the product 3 is best possible in Theorem 2.2.

https://doi.org/10.1017/S0004972700034092 Published online by Cambridge University Press

THEOREM 2.9. *Suppose m is regular and $\alpha \notin \mathcal{K}(X)$. Then α is a product of two nilpotents with index at most 3 if and only if there exists $z \in X$ such that $|z\alpha^{-1} \cap (X \setminus X\alpha)| \geqslant r(\alpha)$ and $(X \setminus \operatorname{dom}\alpha) \setminus z$ is non-empty.*

PROOF: Suppose $\alpha = \lambda\mu$ where λ and μ are nilpotent with index at most 3. If λ has index 2 then $g(\lambda) \geqslant r(\lambda)$: an argument similar to that in the first paragraph in the proof of Theorem 2.3 then shows that α must satisfy the same inequality, in which case $\alpha \in \mathcal{K}(X)$. Hence, λ must have index 3. Let $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. If $r(\alpha) < m$ then $|(X \setminus X\alpha) \cap A_i| = m$ for some $i \in I$ since m is regular and, by supposition, $|(X \setminus X\alpha) \cap (X \setminus \operatorname{dom}\alpha)| < m$ and $d(\alpha) = m$. Hence, we now assume $r(\alpha) = m$.

Let $B_i = x_i\mu^{-1}$ and note this is non-empty since $X\alpha \subseteq X\mu$. Let $\ker\lambda = \{D_j\}$ where $|J| = m$ (since $r(\lambda) \geqslant r(\alpha) = m$). Then each A_i is a union of some D_j and $A_i\lambda \subseteq B_i$ for each $i \in I$. Fix some $b_i \in A_i\lambda$ and write $b_i\lambda^{-1} = D_i$. Note that $\{b_i\} \subseteq \operatorname{dom}\mu$, and μ maps $\{b_i\}$ in a one-to-one fashion onto $\{x_i\}$. Now, since $\alpha \notin \mathcal{K}(X)$ and $X \setminus \operatorname{dom}\lambda \subseteq X \setminus \operatorname{dom}\alpha$, $|\{b_i\} \cap X \setminus \operatorname{dom}\lambda| < m$ and hence $|\bigcup(\{b_i\} \cap D_j)| = m$. If $\{b_i\} \cap D_j \neq \emptyset$ for m of the D_j then $|\{b_i\}\lambda| = m$: that is, $r(\lambda^2) = m$, contradicting the fact that $X\lambda^2 \subseteq X \setminus \operatorname{dom}\lambda$ which has cardinal less than m. Hence, if $K = \{j \in J : \{b_i\} \cap D_j \neq \emptyset\}$ then $|K| < m$. But then $\bigcup(\{b_i\} \cap D_k)$ has cardinal m and so $|\{b_i\} \cap D_0| = m$ for some index $0 \in K$ (since m is regular). Note that D_0 is contained in some A_i since $g(\alpha) < m$. Write $\{b_i\} \cap D_0 = \{b_p\}$ and suppose, for contradiction, that $|(X \setminus X\alpha) \cap A_i| < m$ for all $i \in I$. Then in particular, $|(X \setminus X\alpha) \cap \{b_p\}| < m$ and so $X\alpha \cap \{b_p\} = \{b_{p1}\}$ say, has cardinal m. Note that $\{b_{p1}\}\mu$ has cardinal m. Since $\{b_{p1}\} \subseteq \{x_i\}$, by the choice of the b_i we know there exist $c_p \in \{b_i\}$ such that $c_p\mu = b_{p1}$. We now repeat the foregoing argument with $\{c_p\}$ replacing $\{b_i\}$. That is, $\{c_p\}$ must intersect less than m of the D_j and so there is an index $1 \in J$ such that $|\{c_p\} \cap D_1| = m$. Once again, note that D_1 is contained in some A_i and if $\{c_p\} \cap D_1 = \{d_p\}$ then $X\alpha \cap \{d_p\} = \{b_{p2}\}$ say, has cardinal m. Then $\{b_{p2}\}\mu^2$ has cardinal m and we need only repeat the argument one more time to reach a contradiction (since μ has index at most 3).

In the last two paragraphs we have shown that $|(X \setminus X\alpha) \cap A_0| \geqslant r(\alpha)$ for some index $0 \in I$. We now prove that we can assume $(X \setminus \operatorname{dom}\alpha) \setminus x_0$ is non-empty. For, suppose $X \setminus \operatorname{dom}\alpha = x_0$ (recall that $g(\alpha) \geqslant 1$). Then $\emptyset \neq (X\lambda)\lambda \subseteq X \setminus \operatorname{dom}\lambda = x_0$ implies that $x_0\lambda^{-1} = Y$ say, contains $X\lambda$ and so its cardinal is at least $r(\alpha)$. In addition, since $g(\alpha) < r(\alpha)$, x_0 must belong to $\operatorname{dom}\mu$, $x_0\mu = x_1$ say. If $|X\lambda \cap (X \setminus X\alpha)| \geqslant r(\alpha)$ then $|A_1 \cap (X \setminus X\alpha)| \geqslant r(\alpha)$ since $A_1 = x_1\alpha^{-1}$ contains Y. Since $x_1 \neq x_0$ (μ is nilpotent) and $X \setminus \operatorname{dom}\alpha = x_0$ by supposition, the set A_1 possesses the desired properties. Thus, we assume $|X\lambda \cap (X \setminus X\alpha)| < r(\alpha)$ and deduce that $X\lambda \cap X\alpha = \{x_p\}$ say, is non-empty (possibly $r(\alpha)$ is finite). But then there exist $e_p \in X\lambda \subseteq Y$ such that $e_p\mu = x_p$ and $|\{e_p\} \cap (X \setminus X\alpha)| < r(\alpha)$ implies $\{e_p\} \cap X\alpha = \{x_q\}$ say, is non-empty. Once again, there exist $e_q \in X\lambda$ such that $e_q\mu = x_q$ and now $\{e_q\}\mu^2 = (\{e_p\} \cap X\alpha)\mu$ is non-empty (since each $e_p \in \operatorname{dom}\mu$). Clearly, this argument can be repeated once more to find that $\mu^3 \neq \emptyset$, a contradiction.

Conversely, suppose $Y = z\alpha^{-1}$ and $|Y \cap (X \setminus X\alpha)| \geqslant r(\alpha)$. Let $X\alpha \setminus z = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. Choose $b_i \in Y \cap (X \setminus X\alpha)$ and $c \in (X \setminus \operatorname{dom}\alpha) \setminus z$, and note that

$$\alpha = \begin{pmatrix} A_i & Y \\ b_i & c \end{pmatrix} \circ \begin{pmatrix} b_i & c \\ x_i & z \end{pmatrix}$$

where the first transformation on the right is nilpotent with index 3 and the second is nilpotent with index at most 3 (one x_i may equal c). ∎

It remains to note that there exist $\alpha \notin \mathcal{K}(X)$ which do not satisfy the conditions of Theorem 2.9: for example, the transformation defined in (1).

3. Nilpotents as generators: the singular cardinal case

Throughout this section m will be a singular infinite cardinal. In this context, we say $\alpha \in \mathcal{P}(X)$ is *spread* over its rank if, for each $p < r(\alpha)$, some $z\alpha^{-1}$ has cardinal greater than p. In [**7**] the authors showed that, when m is singular, the set

$$L_m = \{\alpha \in P_m \colon \alpha \text{ is spread over } m\} \cup \{0\}$$

equals the subsemigroup of P_m generated by the nilpotents of P_m. Moreover, each $\alpha \in L_m$ is a product of 3 or fewer nilpotents with index 2 in P_m, and 3 is best possible. This is comparable with the following result from [**9**, Theorem 4]: note that the proof in [**9**, p.340] involves a nilpotent λ which is stated to have index 3 but in fact has index 4.

Theorem 3.1. *Suppose m is singular and $\alpha \in \mathcal{P}(X)$. Then $\alpha \in \mathcal{L}(X)$ if and only if $g(\alpha) \geqslant 1$, $d(\alpha) = m$ and either $g(\alpha) \geqslant r(\alpha)$ or α is spread over its rank. Moreover, when this occurs, α can be written as a product of 4 or fewer nilpotents with index at most 4.*

In Section 2, we characterised when $\alpha \in \mathcal{P}(X)$ is a product of nilpotents with index 2 and X is an arbitrary infinite set. Since some nilpotents with index 3 lie in $\mathcal{K}(X)$, we shall determine when $\alpha \in \mathcal{P}(X)$ equals a product of nilpotents with index *at most* 3. In order to do this, it will be important to know that $d(\lambda) = m$ for any nilpotent λ. We therefore begin by giving a proof of this fact that is simpler than the one in [**9**, Lemma 13].

Lemma 3.2. *If m is singular and λ is a nilpotent then $d(\lambda) = m$ and hence, any product of nilpotents has defect m.*

Proof: If $r(\lambda) < m$ then $d(\lambda) = m$. So, we assume $r(\lambda) = m$. Then, by the first paragraph in the proof of [**9**, Lemma 13], either $g(\lambda) = m$ or λ is spread over m. If the former occurs, we follow the third paragraph in the proof of [**9**, Lemma 11] (with $\operatorname{cf}(m)$ replaced by m throughout) to conclude that $d(\lambda) = m$. Hence, we assume λ is spread over m and suppose, for contradiction, that $d(\lambda) < m$. Let $X\lambda = \{x_i\}$ and $A_i = x_i\lambda^{-1}$. Since $\aleph_0$ is regular, we therefore know there exists A_0 with $|A_0| > \max(\aleph_0, d(\lambda))$. Write $A_0 = B$ and $|B| = n > \aleph_0$. If $|X\lambda \cap B| < n$ then $n = |(X \setminus X\lambda) \cap B| \leqslant d(\lambda)$, a contradiction. Hence, $|X\lambda \cap B| = n$ and $B\lambda \neq \emptyset$. Let $J = \{i \in I\colon x_i \in B\}$, so $|J| = n$. Choose $a_j \in A_j$ and suppose $|X\lambda \cap \{a_j\}| < n$. Then $n = |(X \setminus X\alpha) \cap \{a_j\}| \leqslant d(\lambda)$, a contradiction as before. So $|X\lambda \cap \{a_j\}| = n$ and $\{a_j\}\lambda^2 \neq \emptyset$. Repeating the argument, we let $K = \{i \in I\colon x_i \in \{a_j\}\}$, so $|K| = n$. If $a_k \in A_k$ then $|X\lambda \cap \{a_k\}| < n$ provides a contradiction, so $|X\lambda \cap \{a_k\}| = n$ and $\{a_j\}\lambda^3 \neq \emptyset$. Clearly, this cannot stop: that is, $\lambda^r \neq \emptyset$ for all $r \geqslant 1$, contradicting the fact that λ is nilpotent. Hence, $d(\lambda) = m$ as required. $\square$

We can now turn to the proof of the following result.

Theorem 3.3. *Suppose m is singular and $\alpha \in \mathcal{P}(X)$. Then α is a product of nilpotents with index at most 3 if and only if $g(\alpha) \geqslant 1$, $d(\alpha) = m$ and*

(a) $g(\alpha) \geqslant r(\alpha)$, *or*
(b) $|z\alpha^{-1}| \geqslant r(\alpha)$ *for some $z \in X$, or*
(c) $g(\alpha) \geqslant \operatorname{cf}(m)$ *and α is spread over its rank.*

Moreover, when one of (a) – (c) occurs, α can be written as a product of 3 or fewer nilpotents, the first of which may have index 3 and the others have index 2.

Proof: If λ is a nilpotent and $r(\lambda) < m$ then $d(\lambda) = m$; and if $r(\lambda) = m > \operatorname{cf}(m)$ then $d(\lambda) = m$ by [**9**, Lemma 13]. Also, if λ has index 2 then $X\lambda \subseteq X \setminus \operatorname{dom}\lambda$ implies that (a) is true. Suppose instead that λ has index 3 and neither (a) nor (b) hold. Then $r(\lambda) = m$ since $|X| \leqslant r(\lambda) + r(\lambda)^2$ by supposition. Hence, by [**9**, Lemma 13], λ is spread over m. Let $\ker\lambda = \{A_i\}$ and $J = \{i \in I\colon X\lambda \cap A_i \neq \emptyset\}$. If $g(\lambda) < \operatorname{cf}(m)$ then $|J| < \operatorname{cf}(m)$ since $X\lambda^2 \subseteq X \setminus \operatorname{dom}\lambda$ and $|J| \leqslant |X\lambda^2|$. In addition, $|X\lambda \cap \operatorname{dom}\lambda| = m$: that is, $|\bigcup(X\lambda \cap A_j)| = m$ where $|J| < \operatorname{cf}(m)$. It follows that not every $X\lambda \cap A_j$ can have cardinal less than m (otherwise we invalidate a property of $\operatorname{cf}(m)$: see Theorem 2.1) and so some $z\lambda^{-1}$ has cardinal m, contradicting our original supposition. Therefore, $g(\lambda) \geqslant \operatorname{cf}(m)$ and part (c) holds. That is, nilpotents with index 2 or 3 satisfy the specified conditions. Now suppose α is a product of such nilpotents and write $\alpha = \lambda\beta$ where λ is one of them. If $g(\alpha) < r(\alpha)$ then $g(\lambda) < r(\lambda)$, so λ must satisfy (b) or (c). Suppose some $|z\lambda^{-1}| \geqslant r(\lambda) \geqslant r(\alpha)$. If $z \notin \operatorname{dom}\beta$

then $r(\alpha) \leqslant |z\lambda^{-1}| \leqslant g(\alpha) < r(\alpha)$ is a contradiction. So, $z \in \operatorname{dom}\beta$ and then $|(z\beta)\alpha^{-1}| \geqslant r(\alpha)$. On the other hand, if $g(\lambda) \geqslant \operatorname{cf}(m)$ then $r(\alpha) > g(\alpha) \geqslant \operatorname{cf}(m)$ and so, by [**9**, Theorem 4], α is spread over its rank.

By Theorem 2.3, the converse certainly holds whenever (a) holds. For the other two possibilites, let $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. If some $|A_0| \geqslant r(\alpha)$, write $J = I \setminus 0$ and consider two sub-cases. If $|A_0 \cap X \setminus X\alpha| = m$, choose distinct y_j, z_j, $c \in A_0 \setminus X\alpha$ as well as $b \notin \operatorname{dom}\alpha$ and note that

$$\alpha = \begin{pmatrix} A_j & A_0 \\ y_j & b \end{pmatrix} \circ \begin{pmatrix} y_j & b \\ z_j & c \end{pmatrix} \circ \begin{pmatrix} z_j & c \\ x_j & x_0 \end{pmatrix}$$

where the first transformation on the right is nilpotent with index 3 and the other two are nilpotent with index 2. On the other hand, if $|A_0 \cap X \setminus X\alpha| < m$ then $|(X \setminus A_0) \cap (X \setminus X\alpha)| = m$, so we can choose z_j and c, each different from b, inside $(X \setminus A_0) \cap (X \setminus X\alpha)$. Then, if $y_j \in A_0$, the above decomposition of α will have the same features as before.

Now suppose (c) holds. Since m is singular, it is the sum of $\operatorname{cf}(m)$ cardinals $k_p < m$ and, for each p, some A_p has cardinal greater than k_p. That is, $|\bigcup A_p| = m$ and we again consider two sub-cases. Put $Q = I \setminus P$. If $|(\bigcup A_p) \cap (X \setminus X\alpha)| = m$, choose distinct y_q, z_q, $z_p \in (\bigcup A_p) \cap (X \setminus X\alpha)$ as well as $y_p \notin \operatorname{dom}\alpha$. Then

$$\alpha = \begin{pmatrix} A_q & A_p \\ y_q & y_p \end{pmatrix} \circ \begin{pmatrix} y_q & y_p \\ z_q & z_p \end{pmatrix} \circ \begin{pmatrix} z_q & z_p \\ x_q & x_p \end{pmatrix}$$

where the first trasnformation on the right is nilpotent with index 3 and the other two are nilpotent with index 2. If instead $|(\bigcup A_p) \cap (X \setminus X\alpha)| < m$ then $|(\bigcup A_p) \cap X\alpha| = m$. So, we can choose $y_q \in (\bigcup A_p) \cap X\alpha$ and z_q, $z_p \in (X \setminus X\alpha) \setminus \{y_p\}$ to ensure that the above decomposition of α remains valid. ∎

To show the product 3 is best possible in the above result, we consider the transformation α defined in (1). By Theorem 3.3 (b), α certainly belongs $\mathcal{O}(X)$, *the subsemigroup of* $\mathcal{L}(X)$ *generated by all nilpotents with index at most* 3. Suppose $\alpha = \lambda\mu$ where λ, μ are nilpotents with index at most 3. If λ has index 2 then $r(\lambda) \leqslant g(\lambda)$ and so $m = r(\alpha) \leqslant g(\alpha) = 1$, a contradiction. Hence, λ has index 3, $X \setminus \operatorname{dom}\lambda = \{x\}$, and λ acts on $\{a_i\}$ in a one-to-one fashion. Let $a_i\lambda = c_i$, $\{b_i\}\lambda = \{c_j\}$ and $c_j\lambda^{-1} = A_j$. Then $x \notin \{c_j\}$ since $\{c_j\}\mu = x$ and $x\mu \neq x$ (μ is nilpotent). Hence, $A_j\lambda^2 \neq \emptyset$ but $(A_j\lambda^2)\lambda = \emptyset$: that is, $x = c_1$ for some index $1 \in I$ and $c_j = a_1$ for each $j \in J$. Consequently, $|J| = 1$ and $c_k \neq x$ for all $k \in K = I \setminus 1$. If $c_k \in \{a_k\} \cup \{b_i\}$ then $a_k\lambda^3 = c_k\lambda^2 \in (\{c_k\} \cup a_1)\lambda \neq \emptyset$ is a contradiction. Thus, $c_k = a_1$ for all k and this contradiction finally proves that α cannot be written as a product of just two nilpotents with index at most 3.

Before leaving this section, we show that there are $\alpha \in \mathcal{L}(X)$ which do not satisfy the conditions of Theorem 3.3. For this, choose $Y = \{x_q\} \subseteq X$ with $|Q| = \mathrm{cf}\,(m)$ as well as some $z \in X \setminus Y$. Then we can write $X \setminus (Y \cup z) = B \cup C$ where $|B| = |C| = m$. Since m is singular, there is a a partition $\{A_q\}$ of C where each $|A_q| < m$. Finally, let $\{A_p\}$ be a partition for B, choose $x_p \in C$ and put

$$\alpha = \begin{pmatrix} A_p & A_q & Y \\ x_p & x_q & z \end{pmatrix} \tag{2}$$

Clearly this is a nilpotent with index 4 that does not satisfy the conditions of Theorem 3.3; so, the index 4 in Theorem 3.1 is best possible.

4. Nilpotents as products of idempotents

We now turn to the question of whether $\mathcal{L}(X)$ is idempotent-generated. In [**8**, Section 4], the authors characterised when $\alpha \in \mathcal{P}(X)$ is a product of idempotents in $\mathcal{P}(X)$ by extending the notion of collapse and shift as follows.

$$C^*(\alpha) = C(\alpha) \cup X \setminus \mathrm{dom}\,\alpha \qquad c^*(\alpha) = |C^*(\alpha)|$$
$$S^*(\alpha) = S(\alpha) \cup X \setminus \mathrm{dom}\,\alpha \qquad s^*(\alpha) = |S^*(\alpha)|$$

If m is regular and $\alpha \in \mathcal{L}(X)$ then $d(\alpha) = m$ and either $g(\alpha) = m$ or some $z\alpha^{-1}$ has cardinal m: that is, $c^*(\alpha) = m$ and it follows that $s^*(\alpha) = m$. Hence, by [**8**, Theorem 8], every element of $\mathcal{L}(X)$ is a product of idempotents in $\mathcal{P}(X)$: the problem is whether these idempotents can be chosen from $\mathcal{L}(X)$ itself. Note, for example, that if $X = \{a_i\} \cup \{b_i\}$ and

$$\delta = \begin{pmatrix} \{a_i, b_i\} \\ a_i \end{pmatrix}$$

then δ is an idempotent which lies outside $\mathcal{L}(X)$. As a first step in answering this problem, we now determine when nilpotents in $\mathcal{L}(X)$ with index 2 can be written as a product of idempotents in $\mathcal{L}(X)$.

At the end of [**7**, Section 2], the authors noted that $K_m(Y)$ forms a semigroup for any infinite cardinal $m = |Y|$. And, with this generality in [**7**, Proposition 3.4], they characterised when a nilpotent with index 2 in $K_m(Y)$ is a product of two idempotents in $K_m(Y)$. With this in mind, let ω be the composition of the isomorphism $\theta\colon \mathcal{L}(X) \to L_\phi$ and the natural map $L_\phi \to L_\phi \setminus I_\phi = K_m(\phi)$ defined in Section 2. If $\alpha^2 = \emptyset$ in $\mathcal{L}(X)$ and $\alpha = \varepsilon_1\varepsilon_2$ for some idempotents $\varepsilon_1, \varepsilon_2$ in $\mathcal{L}(X)$ then $\alpha\omega$ is a product of two idempotents in $K_m(X^\phi)$. In addition, if $r(\alpha) = m$ then $\alpha\omega$ is a nilpotent with index 2 in $K_m(X^\phi)$. Consequently, by [**7**, Proposition 3.4], $|C(\alpha\omega) \setminus X(\alpha\omega)| = m$. But, since $g(\alpha) \geqslant 1$, we always have $C(\alpha\omega) = C^*(\alpha) \cup \phi$ and clearly $X(\alpha\omega) = X\alpha \cup \phi$. Hence,

https://doi.org/10.1017/S0004972700034092 Published online by Cambridge University Press

$|C^*(\alpha) \setminus X\alpha| = m$ when $r(\alpha) = m$. On the other hand, if $r(\alpha) < m$ then $d(\alpha) = m$ and so $s^*(\alpha) = m$. Therefore, by [**8**, Theorem 8], $c^*(\alpha) = m$ since α is a product of idempotents in $\mathcal{P}(X)$, and so $|C^*(\alpha) \setminus X\alpha| = m$ since $r(\alpha) < m$. That is, we have proved half the following result.

THEOREM 4.1. *Suppose m is an arbitrary infinite cardinal and $\alpha \in \mathcal{P}(X)$ is nilpotent with index 2. Then α is a product of two idempotents in $\mathcal{K}(X)$ if and only if $|C^*(\alpha) \setminus X\alpha| = m$.*

PROOF: It remains to assume the condition holds and deduce that α is a product ot two idempotents. To do this, write

$$\alpha = \begin{pmatrix} B_t & u_q \\ x_t & v_q \end{pmatrix}$$

where each B_t contains at least two elements and $\{x_t\} \cup \{v_q\} \subseteq X \setminus \operatorname{dom}\alpha$. Choose $b_t \in B_t$ and put

$$\varepsilon_1 = \begin{pmatrix} B_t & u_q \\ b_t & u_q \end{pmatrix}$$
$$\varepsilon_2 = \begin{pmatrix} \{b_t, x_t\} & \{u_q, v_q\} \\ x_t & v_q \end{pmatrix}.$$

Then $\ker\varepsilon_1 = \ker\alpha$. Also, $d(\varepsilon_1) = m = g(\varepsilon_2)$ if $|T| = m$ since then $\bigcup(B_t \setminus b_t)$ has cardinal m and is contained in $X \setminus X\varepsilon_1$ as well as $X \setminus \operatorname{dom}\varepsilon_2$. On the other hand, if $|T| < m$ then $(C^*(\alpha) \setminus X\alpha) \setminus \{b_t\}$ has cardinal m and is contained in the same two sets. Also, $d(\varepsilon_2) = d(\alpha) = m$. That is, $\varepsilon_1, \varepsilon_2 \in \mathcal{K}(X)$ and clearly, $\alpha = \varepsilon_1\varepsilon_2$. ∎

Note that if $X = \{a_i\} \cup \{b_i\}$ and α is the transformation with $\operatorname{dom}\alpha = \{a_i\}$ such that $a_i\alpha = b_i$ then α is a nilpotent with index 2 which does not satisfy the condition of Theorem 4.1. Despite this, α is a product of idempotents in $\mathcal{K}(X)$. For, suppose α is any nilpotent with index 2 and rank m, and write $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$. Choose $a_i \in A_i$, and write $\{a_i\} = \{b_i\} \cup \{c_i\} \cup y$ and $X \setminus \operatorname{dom}\alpha = \{d_i\} \cup \{e_i\} \cup z$ (possible since $X\alpha \subseteq X \setminus \operatorname{dom}\alpha$). Then

$$\alpha = \begin{pmatrix} A_i & e_i \\ d_i & z \end{pmatrix} \circ \begin{pmatrix} d_i & e_i \\ b_i & z \end{pmatrix} \circ \begin{pmatrix} b_i & c_i \\ x_i & y \end{pmatrix}$$

where each transformation on the right is a nilpotent with index 2 that satisfies the condition of Theorem 4.1. This leads us to the following result.

THEOREM 4.2. *If m is an arbitrary infinite cardinal then $\mathcal{K}(X)$ is idempotent-generated.*

PROOF: Suppose α is nilpotent with index 2. If $r(\alpha) < m$ then $|(\operatorname{dom}\alpha) \setminus C(\alpha)| < m$, so $c^*(\alpha) = m$ since $X = (\operatorname{dom}\alpha) \setminus C(\alpha) \cup C^*(\alpha)$. Hence, $|C^*(\alpha) \setminus X\alpha| = m$. Therefore, by Theorem 4.1, α is a product of idempotents in $\mathcal{K}(X)$ and, by the above remark, the same conclusion holds if $r(\alpha) = m$. Hence, every element of $\mathcal{K}(X)$ is a product of idempotents in $\mathcal{K}(X)$. ∎

We now proceed to show that $\mathcal{L}(X)$ is also idempotent-generated. To do this, we recall that the proof of Theorem 2.2 (see [**9**, Theorem 3] shows that when m is regular any $\alpha \in \mathcal{L}(X)$ can be written as a product of nilpotents, one of which may have index 3 and take the form:

$$\lambda = \begin{pmatrix} A_i & Y \\ c_i & x \end{pmatrix}$$

where $\{c_i\} \subseteq Y$ and $x \notin \operatorname{dom}\lambda$. Since the other nilpotents in the product have index 2, by Theorem 4.2 it will suffice to prove that the above λ is a product of idempotents in $\mathcal{L}(X)$. So, choose $a_i \in A_i$ and fix an index $0 \in I$. If $r(\lambda) < m$ then $|C^*(\lambda) \setminus X\lambda| = m$, so we can choose $d_i \in (C^*(\lambda) \setminus X\lambda) \setminus (\{a_i\} \cup \{c_0, x\})$ since this set has cardinal m. Now, observe that

$$\lambda = \begin{pmatrix} A_i & Y \\ a_i & c_0 \end{pmatrix} \circ \begin{pmatrix} \{a_i, d_i\} & \{c_0, x\} \\ d_i & x \end{pmatrix} \circ \begin{pmatrix} \{d_i, c_i\} & x \\ c_i & x \end{pmatrix}.$$

where the first transformation on the right has non-zero gap and same kernel as λ, so it belongs to $\mathcal{L}(X)$. In addition, the other two transformations on the right have gap equal to m, so they belong to $\mathcal{K}(X)$. On the other hand, if $r(\lambda) = m$, we write $\{c_i\} = \{c_j\} \cup \{c_k\}$ where $|J| = |K| = m$ and note that

$$\lambda = \begin{pmatrix} A_i & Y \\ a_i & c_0 \end{pmatrix} \circ \begin{pmatrix} a_i & \{c_0, x\} \\ a_i & x \end{pmatrix} \circ \begin{pmatrix} \{a_j, c_j\} & a_k & x \\ c_j & a_k & x \end{pmatrix} \circ \begin{pmatrix} c_j & \{a_k, c_k\} & x \\ c_j & c_k & x \end{pmatrix}$$

where, as before, each transformation on the right is idempotent and belongs to $\mathcal{L}(X)$. Note in particular that the above decompositions of λ as a product of idempotents are valid for *any* infinite m. That is, we have proved part of the following result.

THEOREM 4.3. *If m is an arbitrary infinite cardinal then $\mathcal{L}(X)$ is idempotent-generated.*

PROOF: It remains to consider the case when m is singular. In this situation, the proof of Theorem 3.1 (see [**9**, Theorem 4]) shows that any $\alpha \in \mathcal{L}(X)$ can be written as a product of nilpotents, one of which may have index 4 and take the form of (2). But, with the same notation as in (2), we can choose $a_p \in A_p$, $a_q \in A_q$, $y \in Y$ and put

$$\varepsilon_1 = \begin{pmatrix} A_p & A_q & Y \\ a_p & a_q & y \end{pmatrix}$$
$$\varepsilon_2 = \begin{pmatrix} a_p & a_q & y \\ x_p & x_q & z \end{pmatrix}.$$

https://doi.org/10.1017/S0004972700034092 Published online by Cambridge University Press

Then $\ker\varepsilon_1 = \ker\alpha$, and $d(\varepsilon_1) = m = g(\varepsilon_2)$ since $(\bigcup A_q) \setminus \{a_q\}$ has cardinal m. In addition, $d(\varepsilon_2) = d(\alpha) = m$ by Lemma 3.2. That is, ε_1 is an idempotent in $\mathcal{L}(X)$ and, since $\varepsilon_2 \in \mathcal{K}(X)$ by Theorem 2.3, it is a product of idempotents in $\mathcal{K}(X)$ by Theorem 4.2. Since the other nilpotents in the decomposition of α have index at most 3, the result follows from Theorem 4.2 and the above remark. ∎

5. Further observations

A slight modification to the proof of Lemma 2.5 shows that $\mathcal{K}(X)$ is regular. For, with the same notation as used there, if $\alpha \in \mathcal{K}(X)$ then $g(\beta) = d(\alpha) = m$ by [**9**, Corollary 3 or Theorem 4]. In addition, $d(\beta) = m$ by the argument in [**9**, p.341] when m is regular, and by that in the proof of Lemma 2.5 when m is singular. Hence, by [**9**, Corollary 4], β belongs to $\mathcal{K}(X)$. The same argument shows that $\mathcal{O}(X)$ is also regular.

Theorem 5.1. *If X is infinite then $\mathcal{K}(X) \subseteq \mathcal{O}(X) \subseteq \mathcal{L}(X)$ are regular subsemigroups of $\mathcal{P}(X)$, with the second containment being equality when $|X|$ is regular.*

As a matter of interest, we remark that there are transformations with rank less than m at each level in the hierarchy $\mathcal{K}(X) \subseteq \mathcal{O}(X) \subseteq \mathcal{L}(X)$. For, if m is singular, there is a partition $\{A_i\} \cup \{y\}$ of X with $|\bigcup A_i| = m$ but each $|A_i| < m$ and $|I| = \operatorname{cf}(m)$. Then, any transformation having $\{A_i\}$ as its kernel must be spread over its rank $\operatorname{cf}(m)$ and have defect m, in which case it lies in $\mathcal{L}(X) \setminus \mathcal{O}(X)$. And, when m is regular, it is even easier to find transformations belonging to $\mathcal{O}(X) \setminus \mathcal{K}(X)$. In other words, we cannot write $\mathcal{K}(X)$ or $\mathcal{O}(X)$ as the disjoint union of I_m^* (see Theorem 2.2) and another subsemigroup of $\mathcal{L}(X)$.

In the previous three sections, we have often used the characterisation of when an element α of $\mathcal{I}(X)$, the symmetric in inverse semigroup on X, is a product of nilpotents in $\mathcal{I}(X)$ provided in [**9**, Corollary 4]: namely, it occurs if and only if $d(\alpha) = g(\alpha) = m$ where $|X| = m$ is an arbitrary infinite cardinal. And, under these conditions, α is a product of 3 or fewer nilpotents in $\mathcal{I}(X)$ with index 2. An argument identical to that in Theorem 2.7 establishes our next result: it shows that the 3 just mentioned is best possible.

Theorem 5.2. *Suppose m is an arbitrary infinite cardinal and $\alpha \in \mathcal{I}(X)$ is non-zero. Then α is a product of 2 nilpotents in $\mathcal{I}(X)$ with index 2 if and only if $|(X \setminus \operatorname{ran}\alpha) \cap (X \setminus \operatorname{dom}\alpha)| \geqslant r(\alpha)$.*

References

[1] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, Mathematical Surveys **7**, vol 1 and 2 (American Mathematical Society, Providence, RI, 1961 and 1967).

[2] J.M. Howie, 'The subsemigroup generated by the idempotents of a full transformation semigroup', *J. London Math. Soc.* **41** (1966), 707–716.

[3] J.M. Howie, *An introduction to semigroup theory* (Academic Press, London, 1976).

[4] J.M. Howie, 'Some subsemigroups of infinite full transformation semigroups', *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 159–167.

[5] J.M. Howie and M.P.O. Marques-Smith, 'A nilpotent-generated semigroup associated with a semigroup of full transformations', *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), 181–187.

[6] M.P.O. Marques, 'A congruence-free semigroup associated with an infinite cardinal number', *Proc. Roy. Soc. Edinburgh Sect. A* **93** (1983), 245–257.

[7] M.P.O. Marques-Smith and R.P. Sullivan, 'Nilpotents and congruences on semigroups of transformations with fixed rank', *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 399-412.

[8] M.A. Reynolds and R.P. Sullivan, 'The ideal structure of idempotent-generated transformation semigroups', *Proc. Edinburgh Math. Soc.* **28** (1985), 319–331.

[9] R.P. Sullivan, 'Semigroups generated by nilpotent transformations', *J. Algebra* **110** (1987), 324–343.

[10] N.H. Williams, *Combinatorial set theory*, Studies in Logic and the Foundations of Mathematics **91** (North-Holland, Amsterdam, 1977).

Department of Mathematics
University of Western Australia
Nedlands 4009
Australia