On a System of Equations

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1. I propose to prove the following theorem.

With $n > 2$, the $(n + 2)$ equations derived from the matrix

$$\|\Delta\| = \left\| \begin{matrix} \mathbf{a}_1 \cdots \mathbf{a}_n & \mathbf{u} & \mathbf{l} \\ l_1 \ldots l_n & p & 0 \end{matrix} \right\| \text{ with } D = \left| \mathbf{a}_1 \cdots \mathbf{a}_n \right| = 0, \text{ and } S = \left| \begin{matrix} \mathbf{a}_1 \ldots \mathbf{a}_n & \mathbf{u} \\ u_1 \ldots u_n & d \end{matrix} \right| \neq 0,$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{l} = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix} \text{ and } \mathbf{a}_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{bmatrix}, k = 1, 2, \ldots, n,$$

by equating to zero all $(n+1)$-rowed determinants from the matrix $\|\Delta\|$ are equivalent to only two, one of which is linear in l_i $(i = 1, 2, \ldots, n)$ and the other is homogeneous and quadratic in a certain $n - 1$ of l_i $(i = 1, 2, \ldots, n)$; the elements of the matrix are real; $a_{rs} = a_{sr}$ $(r, s = 1, 2, \ldots, n)$ and d is arbitrary.

2. The following notations and symbols will be used. $\|M\|$ will mean a matrix; A_{ij} will denote the algebraic complement of a_{ij} in D; D', the adjoint determinant of D; and

$$S_0 = \left| \begin{matrix} \mathbf{a}_1 \ldots \mathbf{a}_n & \mathbf{l} \\ l_1 \ldots l_n & 0 \end{matrix} \right| \text{ and } S'_0 = \left| \begin{matrix} \mathbf{a}_1 \ldots \mathbf{a}_n & \mathbf{u} \\ l_1 \ldots l_n & p \end{matrix} \right|$$

3. The following generalities may be noted.

3.1 The matrix of the adjoint determinant of a vanishing determinant has the rank one or zero.

3.2 In a symmetric determinant of rank r, there is a non-vanishing r-rowed principal minor.

3.3[1] If, in a matrix with $n + 1$ rows and n columns (or with

[1] See Bôcher, *Introduction to Higher Algebra* (1938), p. 58.

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n rows and $n+1$ columns), two of the determinants of the matrix vanish, then the rank of the given matrix is $n-1$, provided that the matrix common to the two determinants is of rank $n-1$.

4. From the symmetry of D follows that of D'; and as $D=0$, $\| D' \|$ has, according to 3.1, the rank zero or one. $\| D' \|$ cannot, however, have the rank zero, for, developing S according to the theory of bordered determinants, we have

$$D \neq S = - \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} u_i u_j,$$

which would vanish, thereby giving rise to a contradiction, if $\| D' \|$ had the rank zero.

$\| D' \|$ must have, therefore, the rank one. (1)

As D' is symmetric and $\| D' \|$ has the rank one, one of its principal elements say A_{ii}, is, according to 3.2, not equal to zero. We definitely choose i for our future discussion. (2)

From $$0 \neq S = - \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} u_i u_j = - \left(\sum_{j=1}^{n} A_{ij} u_j \right)^2 \Big/ A_{ii},$$

in which we have made use of the equations

$$A_{rq} A_{ii} = A_{iq} A_{ri}, \qquad r, q = 1, 2, \ldots, n$$

(which are derived from the consideration that $\| D' \|$ has the rank one), follows

$$\sum_{j=1}^{n} A_{ij} u_j \neq 0. \tag{3}$$

4.1 Consideration of the equation

$$0 = S_0 = - \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} l_i l_j = - \left(\sum_{=}^{n} A_{ij} l_j \right)^2 \Big/ A_{ii},$$

from the matrix $\| \Delta \|$, leads to the equation

$$\sum_{j=1}^{n} A_{ij} l_j = 0. \tag{4}$$

We note that

$$D = 0, \text{ with } A_{ii} \neq 0 \text{ and } \sum_{j=1}^{n} A_{ij} l_j = 0;$$

consequently the rank of the matrix

$$\| \mathbf{a}_1 \, \mathbf{a}_2 \ldots \mathbf{a}_n \, \mathbf{l} \|$$

is, according to 3.3, $n-1$. (5)

Immediate consequences of (5) are that factorisation is possible, and that the coefficient of p is zero, in the remaining equations from the matrix $\|\Delta\|$; p is, therefore, without influence and can be chosen arbitrarily. (6)

4.2 The equation $S'_0 = 0$ from the matrix $\|\Delta\|$ leads to

$$0 = S'_0 = -\sum_{k=1}^{n} l_k (u_1 A_{k1} + u_2 A_{k2} + \ldots + u_n A_{kn})$$
$$= -\Big(\sum_{j=1}^{n} A_{ij} l_j\Big)\Big(\sum_{j=1}^{n} A_{ij} u_j\Big) \Big/ A_{ii},$$

in which we have made use of the relations

$$A_{rq} A_{ii} = A_{iq} A_{ri}, \qquad r, q = 1, 2, \ldots, n;$$

whence, in consequence of (3), we get the equation (4) (see 4.1); $S'_0 = 0$ is, therefore, not an additional equation.

4.3 Since the quantities $A_{i1}, A_{i2}, \ldots, A_{in}$ are not all zero ($A_{ii} \neq 0$, see (2)), the equation

$$\sum_{r=1}^{n} A_{ir} x_r = 0$$

has $n-1$ linearly independent solutions. Evidently, the vectors $\mathbf{a}_k$ ($k = 1, 2, \ldots, n$), of which exactly $n-1$ are linearly independent, form a complete, though redundant, set of solutions. On the other hand, we already have the equation (4) in 4.1, *viz.*,

$$\sum_{r=1}^{n} A_{ir} l_r = 0.$$

Hence
$$\mathbf{l} = \sum_{k=1}^{n} \lambda_k \mathbf{a}_k,$$

where the λ's are scalars.

Considering one of the remaining n determinants, f_j say, from the matrix $\|\Delta\|$, we have

$$f_j = \begin{vmatrix} \mathbf{a}_1 \cdots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} \cdots & \mathbf{a}_n & \mathbf{u} & \mathbf{l} \\ l_1 \ldots & l_{j-1} & l_{j+1} \ldots & l_n & p & 0 \end{vmatrix}$$
$$= \pm \lambda_j \begin{vmatrix} \mathbf{a}_1 \cdots \mathbf{a}_j \cdots \mathbf{a}_n & \mathbf{u} \\ l_1 \ldots l_j \ldots l_n & p \end{vmatrix} \pm Q \mid \mathbf{a}_1 \ldots \mathbf{a}_{j-1}\, \mathbf{a}_{j+1}\, \mathbf{a}_n\, \mathbf{u} \mid,$$

where
$$Q = \sum_{r=1}^{n} \lambda_r l_r. \qquad (7)$$

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The first determinant is S'_0, which vanishes, and the second determinant, *viz.*,

$$| \mathbf{a}_1 \cdots \mathbf{a}_{j-1}\, \mathbf{a}_{j+1} \cdots \mathbf{a}_n\, \mathbf{u} | ,$$

is different from zero when $j = i$: see (3); and so from the equations

$$f_1 = f_2 = \ldots = f_n = 0,$$

we must have $Q = 0$. (7′)

Eliminating from

$$\mathbf{l} = \sum_{r=1}^{n} \lambda_r\, \mathbf{a}_r$$

one of the vectors, say $\mathbf{a}_i$, where

$$\mathbf{a}_i = \sum_{r=1}^{n}{}' \; p_r\, \mathbf{a}_r$$

with
$$a_{ki} = \sum_{r=1}^{n}{}' \, p_r\, a_{kr}, \qquad k = 1, 2, \ldots, n, \tag{8}$$

(the symbol $\sum_{r=1}^{n}{}'$ denoting a summation in which r assumes all values $1, 2, \ldots, n$ excepting i; and the p's are scalars), we have

$$\mathbf{l} = \sum_{r=1}^{n}{}' \, (\lambda_r + \lambda_i\, p_r)\, \mathbf{a}_r,$$

with

$$l_k = \sum_{r=1}^{n}{}' \, (\lambda_r + \lambda_i\, p_r)\, a_{rk}, \qquad k = 1, 2 \ldots, i-1, i+1, \ldots, n, \tag{9}$$

and

$$l_i = \sum_{r=1}^{n}{}' \, (\lambda_r + \lambda_i\, p_r) a_{ri}.$$

With the help of (8) and (9), it is easy to prove that

$$l_i = \sum_{r=1}^{n}{}' \; p_r\, l_r. \tag{10}$$

In consequence of (10), the equation (7′) is reduced to

$$\sum_{r=1}^{n}{}' \, (\lambda_r + \lambda_i\, p_r)\; l_r = 0. \tag{11}$$

Eliminating the $n-1$ constants $\lambda_r + \lambda_i p_r$ $(r = 1, 2, \ldots, i-1, i+1, \ldots, n)$ from the equations (9) and (11), we have finally

$$\begin{vmatrix} \mathbf{a}'_1 & \mathbf{a}'_2 \cdots & \mathbf{a}'_{i-1} & \mathbf{a}_{i+1} \cdots & \mathbf{a}'_n & \mathbf{l}' \\ l_1 & l_2 \ldots & l_{i-1} & l_{i+1} \cdots & l_n & 0 \end{vmatrix} = 0, \tag{12}$$

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where

$$\mathbf{l}' = \begin{bmatrix} l_1 \\ \vdots \\ l_{i-1} \\ l_{i+1} \\ \vdots \\ l_n \end{bmatrix} \text{ and } \mathbf{a}'_k = \begin{bmatrix} a_{1k} \\ \vdots \\ a_{i-1,k} \\ a_{i+1,k} \\ \vdots \\ a_{nk} \end{bmatrix}, \quad k = 1, 2, \ldots, i-1, i+1, \ldots n.$$

The equation (12) is homogeneous and quadratic in $l_1, \ldots, l_{i-1}, l_{i+1} \ldots, l_n$. We have hereby proved that the $n+2$ equations from the matrix $\| \Delta \|$ are equivalent to the equations (4) and (12).

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