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THE COMPLEXITY OF EVERYWHERE DIVERGENT FOURIER SERIES

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ABSTRACT. A natural rank function is defined on the set *DS* of everywhere divergent sequences of continuous functions on the unit circle *T*. The rank function provides a natural measure of the complexity of the sequences in *DS*, and is obtained by associating a well-founded tree with each such sequence. The set *DF* of everywhere divergent Fourier series, and the set *DT* of everywhere divergent trigonometric series with coefficients that tend to zero, can be viewed as natural subsets of *DS*. It is shown that the rank function is a coanalytic norm which is unbounded in ω_1 on *DF*. From this it follows that *DF*, *DT* and *DS* are not Borel subsets of the Polish space *SC*(*T*) of all sequences of continuous functions on *T*. Finally an alternative definition of the rank function is formulated by using nested sequences of closed sets.

1. **Introduction.** In 1906 Fatou [6] asked whether a trigonometric series with coefficients tending to zero must converge on a set of positive measure. (The condition on the coefficients were imposed because it was correctly thought at that time, that a trigonometric series with coefficients not tending to zero can converge only on a set of measure zero.) This question was answered in the negative by Lusin [14] who gave an example in 1911 and showed that it was divergent a.e. (It was later shown by Stechkin [19] that Lusin's example was in fact everywhere divergent.) Not much later Steinhaus [20] gave an example which was everywhere divergent. In the years that follow many more examples were given—among them being one by Hardy and Littlewood [7]. Steinhaus [21] himself later provided a particularly simple example.

However none of these examples were Fourier series. So in 1923 Kolmogorov [11] constructed a Fourier series which was divergent a.e. A few years later Kolomogorov [12] constructed one which was everywhere divergent. Now this latter Fourier series was everywhere unboundedly divergent, so the question arose as to whether there could be a Fourier series which diverges boundedly at each point. Marcinkiewicz [15] then showed that there was a Fourier series which diverges boundedly a.e. (This result is in stark contrast with the Denjoy-Young-Saks theorem which tells us that a function with all four Dini derivates finite a.e. must be differentiable a.e.). However this is the best possible result because of Carlson's theorem on the convergence of Fourier series. Indeed, suppose the Fourier series of f diverges boundedly at each point. Then in some interval I, $S(f)$ and hence f, must be uniformly bounded. But Carlson's theorem implies that $S_n(f)$ must converge a.e. on I, which is a contradiction.

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In another direction one can ask how fast can the coefficients of an everywhere divergent trigonometric series converge to zero. Stechkin [19] showed that it may do so as fast as is permissible. To be more precise, let $\langle r_n \rangle$ be a sequence of positive numbers and put

$$\rho_n = \min\{ r_n : 1 \leq k \leq n\}.$$

Stechkin showed that if $\sum \rho_n^2 = +\infty$, then there exists a sequence $\langle \alpha_n \rangle$ such that the series

$$\sum r_n \cos(nx - \alpha_n)$$

diverges everywhere. One can also ask about the *thinness* of the sequence of coefficients. Here thinness refers to the sizes of the gaps of zeros between successive non-zero terms. It was known (see [2]) that if the sequence of coefficients is lacunary then the series must converge on a dense set. Then Belov [3] showed how to construct examples of everywhere divergent trigonometric series in which the sequence of coefficients falls just short of being lacunary.

The purpose of this paper is to study the structure of the set of functions with everywhere divergent Fourier series. We will also be interested in the set of everywhere divergent trigonometric series with coefficients that tend to zero. So we shall work in the general setting of the Polish space $SC(T)$ of all sequences of continuous functions on the unit circle T. This paper is in many ways similar to [18] and many of the proofs of the early lemmas will be omitted because they are almost identical to those in that paper. Our notation and terminology will be exactly as in that paper. One of our main tools will be well-founded trees. For a concise description of such trees see [16] or [18].

Let $L^1(T)$ be the set of all Lebesgue integrable functions on T. A function in $L^1(T)$ (or a trigonometric series) can be naturally identified with the sequence of partial sums of its Fourier series (respectively, the sequence of partial sums). We will denote an element $\langle f_n \rangle$ of $SC(T)$ by f when convenient. Let

$DS = \{f \in SC(T) : f \text{ is everywhere divergent}\}$,

$DZ = \{f \in DS : \|f_{m+1} - f_m\| \text{ tends to zero}\}$,

$DT = \{f \in DZ : f \text{ is the partial sums of a trigonometric series}\}$,

$DF = \{f \in DS : f \text{ is the partial sums of a Fourier series}\}$, and

$FDF = \{f \in L^1(T) : \langle S_n(f) \rangle \in DF\}$.

It is not difficult to see that DS, DZ, DT and DF are all coanalytic subsets of $SC(T)$ and that FDF is also a coanalytic subset of $L^1(T)$. Kechris [9] showed that FDF was a complete coanalytic subset of $L^1(T)$ (and hence it can't be Borel). By the way FDF is also a comeager subset of $L^1(T)$ (see [5, p. 184]). In this paper we investigate a natural rank function on DS. With each sequence f in DS we will associate a well-founded tree $T(f)$ and show that the height of $T(f)$ is always a limit ordinal. The rank of f is then defined as the unique ordinal α such that $|T(f)| = \omega . \alpha$. This rank function provides a natural measure of the complexity of the sequences in DS in the sense that sequences with small rank are easily seen to be divergent and vice versa. The rank function measures in some

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sense the uniformity of the divergence of f. As it turns out, the sequences of rank 1 are unboundedly divergent. Also sequences which are uniformly divergent have rank 1 or 2.

We will also show that the rank function is unbounded in ω_1 on DF and that it has the right properties summarized in the concept of a coanalytic norm. From this and the Boundedness theorem [16, p. 103] we will be able to deduce that DS, DZ, DT and DF are all non-Borel subsets of $SC(T)$ and that FDF is a non-Borel subset of $L^1(T)$.

2. **Definition of the rank function.** In this section we define the rank function and in the next section we will give some of its basic properties. But first we verify the following:

PROPOSITION 1. *DS, DZ, DT and DF are all coanalytic subsets of $SC(T)$. Also FDF is a coanalytic subset of $L^1(T)$.*

PROOF. We will prove the result for DS and explain how the other results can be obtained. To show that DS is coanalytic it will suffice to show that $SC(T)-DS$ is analytic. Now observe that $f \in SC(T) - DS$ if and only if $(\exists x)$ $(\forall m)$ $(\exists n)$ such that

$$(*) \qquad (\forall p, q \geq n)\,(|f_p(x) - f_q(x)| \leq 1/m)$$

Let $E(m,n) = \{(f,x) \in SC(T) \times T : (*) \text{ holds}\}$. Then $E(m,n)$ is closed and so $\bigcap_{m>0} \bigcup_{n>0} E(m,n)$ is Borel. $SC(T) - DS$ is the projection of this Borel set onto $SC(T)$ and hence it is analytic. Thus DS is coanalytic.

To see that DZ, DT and DF are also analytic, we just have to observe that these sets are subsets of DS which satisfy added Borel conditions. Thus DZ, DT and DF are intersections of DS with Borel sets, and so they are coanalytic. Finally FDF is the inverse image of DF under the map from $L^1(T)$ to $SC(T)$ which takes a function to the sequence of the partial of its Fourier series. This map is Borel measurable, so FDF is also coanalytic.

Throughout this section it will be more convenient for us to view T as the interval $[0, 1]$ (with the endpoints identified) rather than $[0, 2\pi]$. So a point $x \in [0, 1]$ will represent the point $e^{2\pi i x}$ on the actual unit circle. Let $Q[T]$ now be the set of all closed intervals in T with rational endpoints, and $\mathbb{N}$ be the set of positive integers. With each f in $SC(T)$ and $M > 0$, we shall associate a tree $T(M,f)$ on $Q[T] \times \mathbb{N}$.

DEFINITION. The sequence $\langle (I_1, k_1), \ldots, (I_n, k_n) \rangle$ is in $T(M,f)$ if and only if $I_1 = T$, $k_1 = 1$, and for all $i = 2, \ldots, n$ we have

(i) $I_i \in Q[T]$, $|I_i| < 1/i$, $I_i \subseteq I_{i-1}$, $k_{i-1} < k_i$

(ii) $(\forall x \in I_n)$ $(\forall p, q \in [k_{i-1}, k_i])$ we have

$$|f_p(x) - f_q(x)| < M/(i-1).$$

Note that for $n = 1$, the conditions (i) and (ii) are vacuously fulfilled. So $\langle I_1, k_1 \rangle$ is in all the trees $T(M,f)$. Note also that if $M \leq N$ then we have $T(M,f) \subseteq T(N,f)$ directly from the definition. We define the tree $T(f)$ by

$$T(f) = \{\emptyset\} \cup \bigcup_{N \in \mathbb{N}} \{\langle N \rangle^\wedge u : u \text{ is in } T(N,f)\}$$

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PROPOSITION 2. *$f \in DS \Leftrightarrow T(f)$ is well-founded.*

PROOF. "$\Rightarrow$". It will suffice to show that $T(M,f)$ is well- founded for each $M > 0$. Suppose $T(M,f)$ is not well-founded for some $M > 0$. Then we can find a sequence $\langle (I_m, k_m) \rangle$ such that $\langle (I_1, k_1), \ldots, (I_n, k_n) \rangle$ is in $T(M,f)$ for each $n \geq 1$. Let $\{x_0\} = \cap \{I_m : m > 1\}$. We will show that f converges at x_0. Let $\varepsilon > 0$ be given, then choose i such that $M/i < \varepsilon$. Now let $p, q \geq k_i$ be given. Since $\langle k_m \rangle$ is strictly increasing we can find an n such that $k_n > p, q$. So from the definition of the tree $T(M,f)$ we get

$$(\forall x \in I_n)\, (|f_p(x) - f_q(x)| \leq M/i).$$

Since $x_0 \in I_n$, we have in particular

$$|f_p(x_0) - f_q(x_0)| \leq M/i < \varepsilon.$$

Thus f converges at x_0.

"$\Leftarrow$". Suppose f converges at some point x_0. Then we can find a strictly increasing sequence $\langle k_m \rangle$ with $k_1 = 1$ such that

$$|f_p(x_0) - f_q(x_0)| < 1/3m, \text{ for all } p, q \geq k_m.$$

Let M be an integer greater than $2.\sup\{|f_m(x_0)| : m \geq 1\} + 1$. Then we can also choose a nested sequence $\langle I_m \rangle$ of closed intervals of T with $I_1 = T$ and x_0 in each I_m such that condition (i) of the definition of $T(M,f)$ is satisfied and

$$(\forall x \in I_m)\, (\forall r \leq k_m)\, (|f_r(x) - f_r(x_0)| \leq 1/3m).$$

This last requirement can be obtained because it is imposed on a finite number of the continuous functions f_m. We claim that $\langle (I_1, k_1), \ldots, (I_n, k_n) \rangle$ is in $T(M,f)$ for each $n \geq 1$. So fix n. It will suffice to verify condition (ii) in the definition of $T(M,f)$. Now for all x in I_n and all p, q in $[1, k_n]$ we have

$$\begin{aligned}|f_p(x) - f_q(x)| &\leq |f_p(x) - f_p(x_0)| + |f_p(x_0) - f_q(x_0)| + |f_q(x) - f_q(x0)|\\ &\leq 1/3n + |f_p(x_0)| + |f_q(x_0)| + 1/3n\\ &\leq 1/3n + (M-1) + 1/3n\\ &< M/1.\end{aligned}$$

Also for all x in I_n and all p, q in $[k_i, k_n]$ we have

$$\begin{aligned}|f_p(x) - f_q(x)| &\leq |f_p(x) - f_p(x_0)| + |f_p(x_0) - f_q(x_0)| + |f_q(x) - f_q(x_0)|\\ &< 1/3n + 1/3i + 1/3n\\ &< M/i.\end{aligned}$$

Thus condition (ii) is indeed satisfied. This shows that $T(M,f)$ is not well-founded and consequently $T(f)$ is not well-founded.

Our next aim is to show that the height of $T(f)$ is always a limit ordinal. We proceed exactly as in [18].

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DEFINITION. Let $I \in Q[T]$ and S be a tree on $Q[T] \times \mathbb{N}$. We define the subtree $S \restriction I$ of S to be the set of all

$$\langle (I_1, k_1), (I_2, k_2), \ldots, (I_n, k_n) \rangle$$

in S with $I_2 \subseteq I$. Now let $f \in DS$ and $M > 0$. For each x in T we define

$$|T(M,f) : x| = \min\{ |T(M,f) \restriction I| : x \in I \in Q[T] \}.$$

An argument almost identical to that in [18, Lemma 3] gives

LEMMA 3. *If $|T(M,f)| \geq \omega . \alpha$, then there is an $x \in T$ such that $|T(M,f) : x| \geq \omega . \alpha$.*

DEFINITION. Let S be a tree on $Q[T] \times \mathbb{N}$ and z be a positive integer. We define the subtree $[S]_z$ to be the set of all

$$\langle (I_1, k_1), (I_2, k_2), \ldots, (I_n, k_n) \rangle$$

in S such that there exists $(J_1, h_1), \ldots, (J_z, h_z)$ in $Q(T) \times \mathbb{N}$ such that S also contains

$$\langle (I_1, k_1), (J_1, h_1), \ldots, (J_z, h_z), (I_2, k_2), \ldots, (I_n, k_n) \rangle .$$

We have almost exactly as in [18, Lemma 4]

LEMMA 4. *Suppose $|T(M,f) \restriction I| \geq \omega . \alpha$. Then for each $z \geq 1$, we have $|[T(M,f) \restriction I]_z| \geq \omega . \alpha$.*

PROPOSITION 5. *For each f in DS the height of $T(f)$ is a limit ordinal.*

PROOF. It will suffice to show that $|T(f)| \geq \omega . \alpha + 1$ implies $|T(f)| \geq \omega . (\alpha + 1)$. Suppose $|T(f)| \geq \omega . \alpha + 1$, then for some integer M we must have $|T(M,f)| \geq \omega . \alpha$. By Lemma 3, take x_0 in T such that $|T(M,f) : x_0| \geq \omega . \alpha$. Fix the integer $N \geq 1$, and choose $(I_2, k_2), \ldots, (I_N, k_N)$ in $Q(T) \times \mathbb{N}$ such that $|I_i| \leq 1/i$, $I_i \subseteq I_{i-1}$, $k_i = i$, and x_0 is in I_N. (Here as always $I_1 = T$ and $k_1 = 1$.)

Now define the tree T_N as follows. $\langle (I_1, k_1), \ldots, (I_i, k_i) \rangle$ is in T_N for each $i = 1, \ldots, N-1$, and for each

$$\langle (I_1, k_1), (J_N, k_N), \ldots, (J_z, h_z) \rangle$$

in $[T(M,f) \restriction I_N]_N$ let

$$\langle (I_1, k_1), \ldots, (I_{N-1}, k_{N-1}), (J_N, h_N), \ldots, (J_z h_z) \rangle$$

be in T_N. Then it is easy to see that T_N is a subtree of $T(MN,f)$. But

$$\begin{aligned} |\langle (I_1, k_1) \rangle : T_N| &\geq |\langle (I_1, k_1) \rangle : [T(M,f)| \restriction I_N]_N| + (N-2) \\ &\geq \omega . \alpha + (N-2) \end{aligned}$$

by Lemma 4. Thus

$$T(MN,f) \geq \omega . \alpha + (N-2),$$

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and so

$$\begin{aligned} |T(f)| &= \sup\{\, |T(MN,f)| + 1 : N \geq 1\} \\ &\geq \sup\{\, \omega . \alpha + (N-1) : N \geq 1\} \\ &= \omega . (\alpha + 1). \end{aligned}$$

DEFINITION. For each f in DS we define the rank $|f|$ of to be the unique ordinal α such that $|T(f)| = \omega . \alpha$.

Since $Q(T) \times \mathbb{N}$ is countable, $T(f)$ is a countable tree, so $|f| < \omega_1$. The same argument in [18, Proposition 6] gives

PROPOSITION 6. *$|.| : DS \longrightarrow \omega_1$ is a coanalytic norm.*

3. **Basic properties of the rank function.**

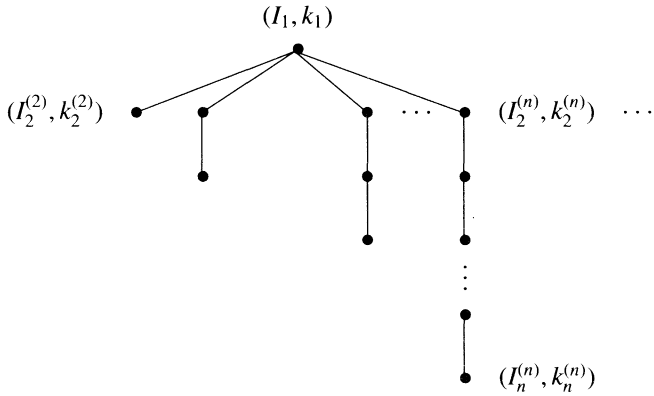
DEFINITION. Let f be in $SC(T)$. We define the *amplitude of divergence* of f at a point x_0 in T by

$$A(f; x_0) = \limsup_{p,q \longrightarrow \infty} |f_p(x_0) - f_q(x_0)|$$

It is clear that f diverges at x_0 if and only if $A(f; x_0) > 0$. We say that f *diverges unboundedly* if for each $x \in T$ we have $A(f; x) = +\infty$.

PROPOSITION 7. *If f diverges unboundedly then $|f| = 1$.*

PROOF. Suppose $|f| > 1$. Then $|T(f)| \geq \omega . 2$ and so for some positive integer M we must have $|T(M,f)| \geq \omega + 1$. Hence $T(M,f)$ has a subtree as shown below.



Let x_n be the midpoint of $I_n^{(n)}$, and take x_0 to be a limit point of the sequence $\langle x_n \rangle$. We claim that $A(f; x_0) < +\infty$, and so f does not diverge unboundedly. Let p, q be arbitrarily given. To prove our claim it will suffice to show that $|f_p(x_0) - f_q(x_0)| \leq M$. So take $\delta > 0$ and choose $n > p, q$ such that

$$|f_p(x_n) - f_p(x_0)| < \delta \text{ and } |f_q(x_n) - f_q(x_0)| < \delta .$$

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These latter conditions can be satisfied because f_p and f_q are continuous. Observe also that $p, q < k_n^{(n)}$ because $n \le k_n^{(n)}$. Now from the definition of $T(M,f)$ we have

$$(\forall x \in I_n^{(n)})\ (\forall r, s \in [1, k_n^{(n)}])\ (|f_r(x) - f_s(x)| \le M).$$

So in particular $|f_p(x_n) - f_q(x_n)| \le M$. Thus

$$\begin{aligned} |f_p(x_0) - f_q(x_0)| &\le |f_p(x_0) - f_p(x_n)| + |f_p(x_n) - f_q(x_n)| + |f_q(x_n) - f_q(x_0)| \\ &< M + 2\delta . \end{aligned}$$

Since this is true for all $\delta > 0$, we get $|f_p(x_0) - f_q(x_0)| \le M$.

REMARK. The converse or Proposition 7 is not true. A simple counterexample is the sequence $f_m(x) \equiv (-1)^m$. A sequence is *strongly uniformly divergent* on the set A if

$$(\exists M > 0)\ (\exists n)\ (\forall m)\ (\exists x \in A)\ (\exists p, q \in [m, m+n])$$
$$\text{such that } |f_p(x) - f_q(x)| \ge M.$$

It can be shown that $|f| = 1$ if and only if f is strongly uniformly divergent on the set of all points x with $A(f; x) < +\infty$; but this would take us away from our main concern. We are, after all, interested mostly in DF which is a subset of DZ, and we have

PROPOSITION 8. *If $f \in DZ$, then $|f| = 1$ if and only if f diverges unboundedly.*

PROOF. In view of Proposition 7 it will suffice to show that if $|f| = 1$ then f diverges unboundedly. Suppose f does not diverge unboundedly. Then we can find an x_0 with $A(f; x_0) < +\infty$. Let M be an integer greater than $2.\sup\{\,|f_m(x)| : m \ge 1\} + 1$. We claim that $|T(M,f)| \ge \omega$, and consequently $|f| \ge 2$.

Fix n and take $I_1 = T$ and $k_1 = 1$. Since $f \in DZ$ we can choose $k_2 > 1$ such that for all $m \ge k_2$

$$\|f_{m+1} - f_m\| \le M/3n^2.$$

By the continuity of the f_m's we can also choose I_2 such that $|I_2| < 1/n$ and

$$(\forall x \in I_2)\ (\forall m \in [1, k_2 + n])\ (|f_m(x) - f_m(x_0)| \le M/3n).$$

Now let $I_i = I_2$ and $k_i = k_2 + (i-2)$ for each $i = 3, \ldots, n$. Then it is easy to check as in the second half of the proof of Proposition 2 that

$$\langle (I_1, k_1), (I_2, k_2), \ldots, (I_n, k_n) \rangle \in T(M,f).$$

So $|T(M,f)| \ge n$. Since this is true for each integer n, we get $|T(M,f)| \ge \omega$ as claimed.

PROPOSITION 9. *If there is a $c > 0$ such that $A(f; x) \ge c$ for each $x \in T$, then $|f| \le 2$.*

PROOF. Suppose $|f| > 2$. Then $|T(f)| \ge \omega.3$ and so we can find a positive integer M such that $|T(M,f)| \ge \omega.2 + 1$. Hence $T(M,f)$ must have a subtree as shown below.

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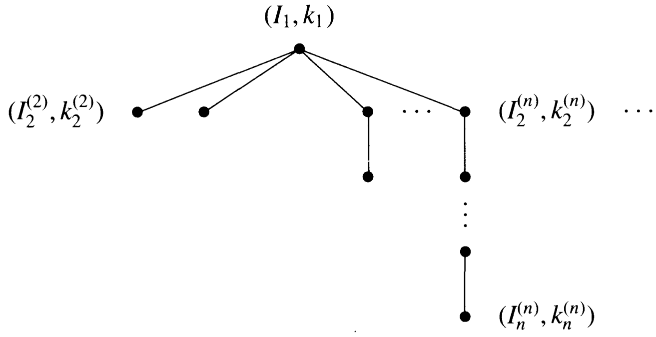
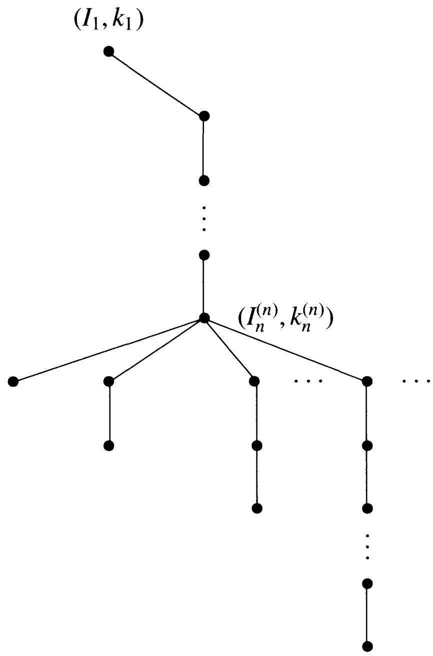
Here each of the nodes $\langle (I_1, k_1), \ldots, (I_n^{(n)}, k_n^{(n)})\rangle$ is of height ω in $\mathrm{T}(M,f)$





Fix n, and consider the subtree of $T(M,f)$ which passes through the node

$$\langle (I_1, k_1), \ldots, (I_n^{(n)}, k_n^{(n)})\rangle$$

as shown above. The same argument in Proposition 7 shows that there is a point $x_n \in I_n^{(n)}$ such that $A(f; x_n) \leq M/n$. But this is true for each integer n, so for a large enough n we would get $A(f; x_n) < c$, which contradicts the hypothesis of the Theorem. Hence the results follows.

REMARKS. The converse of Proposition 9 is false even if f is in DF. By using the methods we develop in the next section we can find an f in DF with $|f| = 2$ such that

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$A(f : x_n) \longrightarrow 0$ for some $\langle x_n \rangle$ in T. A sequence is *weakly uniformly divergent* on the set A if

$$(\exists M > 0)\ (\forall m)\ (\exists n)\ (\forall x \in A)\ (\exists p, q \in [m, m+n])$$
$$\text{such that } |f_p(x) - f_q(x)| \geq M.$$

It can be shown that f is weakly uniformly convergent on T if and only if there is a $c > 0$ such that $A(f; x) \geq c$ for each $x \in T$ (see [17]). By using Proposition 9 it is not hard to check that all the natural examples of sequences in DS given in [3], [6], [7], [12], [14], [21] have rank 1 or 2. A little more effort along with Proposition 8 shows that most of these sequences in fact have rank 1 (see [17] for details).

4. **Unboundedness of the rank function.** In this section our goal will be to show the following

PROPOSITION 10. *For each $\alpha < \omega_1$ there is a function $f_\alpha \in FDF$ such that $|f_\alpha| \geq \alpha$.*

PROOF. We prove the result by induction on α. For $\alpha = 1$, there is nothing to prove. For $\alpha = 2$, let f be the function constructed by Kolmogorov (see [13]). It is shown there that the Fourier series of f is unboundedly divergent everywhere. Also let $\mu(x)$ be a $C^{(3)}$ function such that $\mu(\pi) = 0$ and for all the other x's, $0 < \mu(x) \leq 1$. Here the unit circle is viewed as the interval $[0, 2\pi]$.

Now take $\varphi(x)$ to be a function whose Fourier series converges everywhere except at π and all of whose partial sums are bounded by 1. For the construction of such a function see [2, p. 127–128]. We define f_2 by

$$f_2(x) = \mu(x).f(x) + \varphi(x).$$

Since $\mu(x)$ is zero only at $x = \pi$, it follows that the Fourier series of $\mu(x).f(x)$ diverges unboundedly except at $x = \pi$, where it converges. So the Fourier series of f_2 diverges unboundedly except at $x = \pi$, where it diverges boundedly. Thus $|f_2| \geq 2$ by Proposition 7.

For $\alpha = 3$, let J_n be the open interval $(2\pi/n+1, 2\pi/n)$. For each n, let g_n be a scaled copy of f_2 which fits exactly in J_n. Now let h_2 be the function defined by

$$h_2(x) = \begin{cases} g_n(x) & \text{if } x \in J_n \\ 0 & \text{otherwise} \end{cases}$$

and take

$$f_3(x) = \mu(x - \pi).\, h2(x) + \varphi(x - \pi).$$

Then it is easy to see that the Fourier series of f_3 diverges uboundedly except at $x = 0$ and the midpoints of the J_n's, where it diverges boundedly. One can now check that $|f_3| \geq 3$ by going back directly to the definition of the rank function. This verification is rather tedious so we shall omit it.

The induction for the general case is done similarly. If we know the result for α, then $f_{\alpha+1}$ is constructed by using scaled copies of f_α in the intervals J_n. If we know the result

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for all $\alpha < \lambda$, where λ is a limit ordinal, then we first find an increasing sequence $\alpha(n)$ with $\lim \alpha(n) = \lambda$. Then we proceed as before but this time we place a scaled copy of $f_{\alpha(n)}$ in the interval J_n. This completes the proof.

Since our rank function was a coanalytic norm, we get exactly as in Corollary 12 of [18]

COROLLARY .@@ *The sets DS, DZ, DT and DF are all coanalytic but not Borel subsets of SC(T). Also FDF is a coanalytic but not Borel subset of* $L^1(T)$.

5. **An alternative definition.** In this section we give an alternative definition of the rank function. Let N be a non-negative integer and f be in $SC(T)$. We define the *downward shift* of f by N to be the sequence $\langle g_k \rangle$ defined by

$$g_k = f_{k+N},$$

and will denote it by $f \downarrow N$. For each f in $SC(T)$, each $M > 0$, and each non-negative integer N we will define a sequence

$$P^\alpha(M,f \downarrow N)$$

of closed subsets of T, and a relation

$$S[x, P^\alpha(M,f \downarrow N)](W) = S(W)$$

on the closed subsets of T which reflect the properties of $f \downarrow N$. M is to be thought of as being large, and $S(W)$ is to be interpreted as the relation "W *witnesses* that x is in $P^\alpha(M,f \downarrow N)$."

First we define $P(M,f \downarrow N)$ to be the set of all x in T such that

$$\sup\{\, |f_p(x) - f_q(x)| \; : p, q \geq N\} \leq M.$$

We define $P^\alpha(M,f \downarrow N)$ and $S[x, P^\alpha(M,f \downarrow N)](W)$ by induction as follows. Let

$$P^1(M,f \downarrow N) = P(M,f \downarrow N),$$

and let

$$S[x, P^1(M,f \downarrow N)](W) \text{ hold}$$

if and only if

$$x \in W \cap P(M,f \downarrow N).$$

For a limit ordinal λ, let

$$P^\lambda(M,f \downarrow N) = \cap\{\, P^\alpha(M,f \downarrow N) : \alpha < \lambda\},$$

and let

$$S[x, P^\lambda(M,f \downarrow N)](W) \text{ hold}$$

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if and only if

$$(\forall\alpha < \lambda)\; S[x, P^{\alpha}(M,f \downarrow N)](W) \text{ hold}.$$

Also, let $P^{\alpha+1}(M,f \downarrow N)$ be the set of all $x \in P^{\alpha}(M,f \downarrow N)$ such that for all $I \in Q[T]$ with $x \in \mathrm{int}(I)$ there exist

$$N' \geq 1/|I|,\; y \in I, \text{ and } V \subseteq P(M,f \downarrow N)$$

such that

$$S[y, P^{\alpha}(M.|I|,f \downarrow N+N')](V) \text{ holds}.$$

Finally, let $S[x, P^{\alpha+1}(M,f \downarrow N)](W)$ hold if and only if for all $I \in Q[T]$ with $x \in \mathrm{int}(I)$ there exists

$$N' \geq 1/|I|,\, y \in I, \text{ and } V \subseteq W \cap (PM,f \downarrow N)$$

such that

$$S[y, P^{\alpha}(M.|I|,f \downarrow N+N')](V) \text{ holds}.$$

Observe that our definition was made by simultaneous induction on α, M and N. It is easy to see that $P^{\alpha}(M,f \downarrow N)$ is a closed set. Also if $\alpha < \beta$, then

$$P^{\beta}(M,f \downarrow N) \subseteq P^{\alpha}(M,f \downarrow N);$$

and if $M < M'$, then

$$P^{\alpha}(M,f \downarrow N) \subseteq P^{\alpha}(M',f \downarrow N).$$

When $N = 0$, we shall refer to $P^{\alpha}(M,f \downarrow N)$ simply as $P^{\alpha}(M,f)$.

The following result can then be shown.

PROPOSITION 13. *$f \in DS \Leftrightarrow$ for each $M > 0$, there exists $\alpha < \omega_1$, such that*

$$P^{\alpha}(M,f) = \emptyset.$$

The "$\Leftarrow$" part of this problem is very easy because if f diverges at x_0, then it is easy to verify that for a fixed large enough M, $x_0 \in P^{\alpha}(M,f)$ for all $\alpha < \omega_1$. The other half of the proof is rather complicated so we omit it (see [17] for full details). Proposition 12 allows us to make the following definition, and in the course of the proof the next result can be implicitly obtained.

DEFINITION. For each f in DS, we define $r(f)$ to be the least ordinal α such that

$$P^{\alpha}(M,f) = \emptyset,$$

for all positive integers M.

PROPOSITION 14. *For each $f \in DS$, $|f| = r(f)$.*

CONCLUDING REMARKS. It is clear that our alternative definition is very complicated when compared to those in [1] and [10] (where parametric inductions were used instead of simultaneous induction). But, as in [18], we feel that a simpler definition is not possible because the trees $T(M,f)$ have nested-like structure which makes the use of simultaneous induction necessary.

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