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TWO NECESSARY AND SUFFICIENT CONDITIONS FOR MÖBIUS SUBGROUPS TO BE g-DISCONTINUOUS

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In this paper, two necessary and sufficient conditions for *Möbius subgroups* to be g-discontinuous are obtained. These are generalisations of Lehner's and Larcher's corresponding results.

1. INTRODUCTION

Let $\mathcal{M}$ be the *Möbius group* consisting of all sense-preserving *Möbius transformations* acting on the extended complex plane $\widehat{\mathbb{C}}$, that is, $\mathcal{M} = \{ g : g(z) = (az+b)/(cz+d), \forall z \in \widehat{\mathbb{C}}; a,b,c,d \in \mathbb{C}, ad-bc=1\}/\{\pm I\}$, where I denotes the 2×2 identity matrix.

A subgroup G of $\mathcal{M}$ is called discrete if and only if no infinite sequence consisting of distinct elements of G converges to the identity *id*. G is said to be normal in a domain D provided that every infinite sequence of G contains a subsequence of distinct elements converging uniformly on compact subsets of D to a limit function (the function can be ∞) which is univalent or a constant since every element of $\mathcal{M}$ is univalent, see [**5**] for example.

It is well known that every element g of $\mathcal{M}$ acting on $\widehat{\mathbb{C}}$ has a natural extension acting on $\overline{\mathbb{H}}^3$, which is called the Poincaré extension of g and denoted by $\widetilde{g}$. For a subgroup G of $\mathcal{M}$, we let $\widetilde{G} = \{\widetilde{g} : g \in G\}$. We call G elementary if $\widetilde{G}$ has a finite $\widetilde{G}$-orbit in $\overline{\mathbb{H}}^3$ (that is, there exists some $x \in \overline{\mathbb{H}}^3$ such that the set $\widetilde{G}(x) = \{\widetilde{g}(x) : g \in G\}$ is finite); otherwise we say G non-elementary.

A nontrivial element $g \in \mathcal{M}$ is called loxodromic if g has two fixed points in $\widehat{\mathbb{C}}$ and $\widetilde{g}$ has no fixed point in $\mathbb{H}^3$; parabolic if g has only one fixed point in $\widehat{\mathbb{C}}$; elliptic if $\widetilde{g}$ has some (in fact, infinitely many) fixed points in $\mathbb{H}^3$. Obviously, in the case of g being parabolic, $\widetilde{g}$ has no fixed point in $\mathbb{H}^3$. By [**1**], we know

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Lemma 1. *If G is non-elementary, then G contains infinitely many loxodromic elements, no two of which have a common fixed point.*

Let $G \subset \mathcal{M}$ be a nontrivial elementary subgroup. We say that G is of elliptic type if each nontrivial element of G is elliptic; parabolic (or loxodromic) type if G contains a parabolic (or loxodromic) element and all its non-elliptic nontrivial elements have a common fixed point set.

Lemma 2. *Let G be elementary. Then G is of mixed type if and only if G contains a parabolic element and a loxodromic element with a fixed point in common; or equivalently if and only if G contains two loxodromic elements with only one fixed point in common.*

A point z_0 in the extended complex plane $\widehat{\mathbb{C}}$ is called a limit point with respect to a subgroup G of $\mathcal{M}$ if $\widetilde{g}_n(x) \to z_0$ for some sequence $\{g_n\}$ of distinct elements of G and some fixed point $x \in \mathbb{H}^3$. The set of all limit points of G is denoted by $\Lambda(G)$, that is,

$$\Lambda(G) = \widehat{\mathbb{C}} \cap \mathrm{cl}\big(\widetilde{G}(x)\big),$$

where "cl" means "closure" and $x \in \mathbb{H}^3$. Since elements of G preserve the hyperbolic metric of $\mathbb{H}^3$, this definition is independent of the choice of x. Obviously if G contains a loxodromic element, then its fixed point set is contained in $\Lambda(G)$. By Lemmas 1 and 2 and [**6**], we know

Lemma 3.

(1) *$\Lambda(G)$ is G-invariant and closed;*

(2) *$\Lambda(G)$ is a perfect set if G is a non-elementary group or an elementary group of mixed type.*

Let $\Omega'(G) = \widehat{\mathbb{C}} - \Lambda(G)$.

Corollary 1. *$\Omega'(G)$ is G-invariant and open.*

We say that G is g-discontinuous if $\Omega'(G) \neq \emptyset$ and G acts g-discontinuously in a domain D provided $D \subset \Omega'(G)$.

Proposition 1. *If G is elementary and not of mixed type, then G must be g-discontinuous.*

It follows from [**1**] that

Lemma 4. *If G is a discrete elementary subgroup of $\mathcal{M}$, then G is one of the following three types.*

(1) *G is finite (that is, elliptic type);*

(2) *G conjugates to a group whose elements are the form*

$$z \longmapsto \omega^k z + n\lambda + m\mu,$$

where $\omega = exp(2\pi i/q)$, λ ($\neq 0$), μ are complex numbers and $Im(\mu/\lambda) \neq 0$ when $\mu \neq 0$, and all k, m, n, q are integers, $0 \leqslant k \leqslant q$ and $q \leqslant 6, q \neq 5$ (that is, parabolic type);

(3) *G conjugates to a group whose elements are of the form*

$$z \longmapsto \omega^k \alpha^n z$$

or

$$z \longmapsto \omega^k \alpha^n / z,$$

where $\omega = exp(2\pi i/q), \alpha$ *is a complex number, and* k, n *are integers,* q *is a positive integer (that is, loxodromic type).*

See [**1**] for more details about subgroups of $\mathcal{M}$.

Let $G \subset \mathcal{M}$. We say that G is discontinuous at $z_0 \in \widehat{\mathbb{C}}$ if there exists a neighbourhood N of z_0 such that

$$g(N) \cap N \neq \emptyset$$

for at most finitely many $g \in G$. z_0 also is called a discontinuous point of G. The set of all discontinuous points of G is denoted by $\Omega(G)$. If $\Omega(G) \neq \emptyset$ then we say that G is discontinuous. Obviously, if G is discontinuous then G is g-discontinuous.

Corollary 2. *Let G be g-discontinuous. Then G is discontinuous if and only if it is discrete.*

It is well-known that the discontinuity of subgroups of $\mathcal{M}$ plays a very important role in the theory of Kleinian groups, see [**1, 3, 4**] et cetera. Hence the problem of under what conditions a subgroup G of $\mathcal{M}$ is discontinuous becomes important and interesting. Many authors have discussed this problem. Among them, Lehner ([**3**]) proved

Theorem A. *A necessary and sufficient condition for G to be discontinuous at a point* z_0 *is that*

(1) *G is discrete,*

(2) *G is a normal family in some disk D containing* z_0.

In ([**2**]) Larcher proved the following theorem.

Theorem B. *Let G be a discrete subgroup of* $\mathcal{M}$. *Then G is discontinuous if and only if there exists an open set D on the extended complex plane and a complex point* z_0 *(finite or infinite) such that no element of G assumes* z_0 *on D.*

In this paper, we consider this problem further. We shall prove the following theorems.

Theorem 1. *Let G be a non-elementary subgroup of* $\mathcal{M}$, *and let D be a domain of* $\widehat{\mathbb{C}}$. *Then G acts g-discontinuously in D if and only if G is normal in D.*

Theorem 2. *Let G be a non-elementary subgroup of* $\mathcal{M}$. *Then G is g-discontinuous if and only if there exist a domain D of* $\widehat{\mathbb{C}}$ *and an complex number* z_0 *(finite or infinite) such that no element of G assumes* z_0 *in D.*

Remark 1. Corollary 2 shows that Theorems 1 and 2 are generalisations of Theorems A and B in the case of G being non-elementary.

REMARK 2. If G is an elementary subgroup of $\mathcal{M}$, then we can easily know that G is discontinuous if and only if G is discrete.

REMARK 3. Let G be elementary. If G is discrete(or discontinuous), then by Lemma 4, we know that it is normal in any domain $D \subset \Omega(G)$, and also we can find D and z_0 such that no element of G takes z_0 in D. The following examples imply that the converse of the above statements are not true. This shows that the hypothesis "G being discrete" in Theorems A and B cannot be removed when G is elementary.

EXAMPLE 1. Let $G = \langle g \rangle$, $D = \mathbb{C}$ and $z_0 = \infty$, where $g(z) = z\exp(2\pi i\sqrt{2})$. G is normal in D since every infinite sequence in G contains a convergent subsequence. No element takes ∞ in D since for $h \in G$, $h(z) = \infty$ if and only if $z = \infty$. But G is not discontinuous since G is not discrete.

EXAMPLE 2. Let $G = \{g : g(z) = az + b;\ 0 \neq a, b \in \mathbb{C}\}$, $D = \mathbb{C}$ and $z_0 = \infty$.

2. PROOFS OF THE MAIN THEOREMS

At first, we introduce a lemma (see [**6**]) which will be useful in the following proofs.

LEMMA 5. *Let G be a non-elementary subgroup of $\mathcal{M}$, and let D_1 and D_2 be two disjoint open sets both meeting $\Lambda(G)$. Then there exists a loxodromic element g in G with one fixed point in D_1 and the other in D_2.*

Now we come to prove our main results.

PROOF OF THEOREM 1: First we prove the sufficiency. If G does not act g-discontinuously in D, that is $D \cap \Lambda(G) \neq \emptyset$, then by Lemmas 3 and 5, there exists some loxodromic element $g \in G$ with fixed points $\alpha, \beta \in D$ since G is non-elementary. Without loss of generality, we may assume that α is the attractive fixed point of g.

There exists f such that $g^{n_k} \to f$ as $k \to \infty$ local uniformly in D since $\{g^n\}$ is normal in D, where f is univalent or a constant. Let $z_1, z_2 \in D\backslash\{\alpha, \beta\}$ and $z_1 \neq z_2$, we know $g^{n_k}(z_1) \to f(z_1), g^{n_k}(z_2) \to f(z_2)$. Hence $f(z_1) = f(z_2) = \alpha$. It follows that f is a constant and $f = \alpha$. But $g^{n_k}(\beta) \to \beta \neq \alpha$. This is the desired contradiction.

For the necessity, we consider the set $E = \widehat{\mathbb{C}}\backslash G(D)$.

We claim that E contains at least two points. Suppose not, we divide our discussions into two separate cases: E is empty or E contains only one point.

If E is empty, that is, $G(D) = \widehat{\mathbb{C}}$, then $\Lambda(G) = \emptyset$. This is a contradiction since G is non-elementary.

If E contains exactly one point, we may assume that $E = \{\alpha\}$. Then we can conclude that every element of G must fix the point α and so G is elementary. Otherwise, if $g \in G$ and $z_1 = g(\alpha) \neq \alpha$, then there must exist $f \in G$ and $z_2 \in D$ such that $f(z_2) = z_1$. We have $g^{-1}f(z_2) = \alpha$.

The normality of G in D follows from our claim and the following easy fact:

If $\widehat{\mathbb{C}}\backslash G(D)$ contains at least two points, then G is normal in D. $\square$

PROOF OF THEOREM 2: First we prove the sufficiency. For the contrary, we suppose that G does not act g-discontinuously in D. Then, by similar reasoning as in the proof of Theorem 1, we can find a loxodromic element $g \in G$ with fixed points $\alpha, \beta \in D$. Without loss of generality, we may assume that α is its attractive fixed point. That means $f^n(z) \to \alpha \ \forall\ z \in \widehat{\mathbb{C}}\backslash\{\beta\}$. Obviously, $\alpha \neq z_0$ and $\beta \neq z_0$. Hence $g^n(z_0) \to \alpha \in D$. Then $g^{n_0}(z_0) \in D$ for large enough n_0. This implies that $g^{-n_0}\big(g^{n_0}(z_0)\big) = z_0 \in D$. This is a contradiction.

For the proof of the necessity we assume that the group G has the property that for every open set $O \subset \widehat{\mathbb{C}}$ the set $G(O) = \widehat{\mathbb{C}}$. Since G is g-discontinuous, let $z_0 \in \Omega'(G)$. By Corollary 1, we know there exists an open neighbourhood N of z_0 such that $N \subset \Omega'(G)$. Then $G(N) = \widehat{\mathbb{C}}$. By Corollary 1, we know $\Omega'(G) = \widehat{\mathbb{C}}$. This is the desired contradiction since G is non-elementary. $\square$

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