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ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF GREATEST COMMON DIVISOR MATRICES

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Abstract. Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary strictly increasing infinite sequence of positive integers. For an integer $n \geq 1$, let $S_n = \{x_1, \ldots, x_n\}$. Let ε be a real number and $q \geq 1$ a given integer. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the power GCD matrix $((x_i, x_j)^\varepsilon)$ having the power $(x_i, x_j)^\varepsilon$ of the greatest common divisor of x_i and x_j as its i, j-entry. We give a nontrivial lower bound depending on x_1 and n for $\lambda_n^{(1)}$ if $\varepsilon > 0$. Especially for $\varepsilon > 1$, this lower bound is given by using the Riemann zeta function. Let $x \geq 1$ be an integer. For a sequence $\{x_i\}_{i=1}^{\infty}$ satisfying that $(x_i, x_j) = x$ for any $i \neq j$ and $\sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$, we show that if $0 < \varepsilon \leq 1$, then $\lim_{n\to\infty} \lambda_n^{(1)} = x_1^\varepsilon - x^\varepsilon$. Let $a \geq 0, b \geq 1$ and $e \geq 0$ be any given integers. For the arithmetic progression $\{x_{i-e+1} = a + bi\}_{i=e}^{\infty}$, we show that if $0 < \varepsilon \leq 1$, then $\lim_{n\to\infty} \lambda_n^{(q)} = 0$. Finally, we show that for any sequence $\{x_i\}_{i=1}^{\infty}$ and any $\varepsilon > 0$, $\lambda_n^{(n-q+1)}$ approaches infinity when n goes to infinity.

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1. Introduction. In 1876, H. Smith [**25**] published his celebrated theorem showing that the determinant of the $n \times n$ matrix $[(i, j)]$, which has the greatest common divisor (i, j) of i and j as its (i, j)-entry, is the product $\prod_{k=1}^{n} \varphi(k)$, where φ is Euler's totient function. Smith also proved that if f is an arithmetical function and $[f(i, j)]$ is the $n \times n$ matrix having f evaluated at the greatest common divisor (i, j) of i and j as its (i, j)-entry, then $\det[f(i, j)] = \prod_{k=1}^{n} (f * \mu)(k)$, where μ is the Möbius function and $f * \mu$ is the Dirichlet convolution of f and μ. In 1972, Apostol [**2**] extended Smith's result. In 1988, McCarthy [**21**] generalized Smith's and Apostol's results to the class of even functions of m (mod r), where m and r are positive integers. A complex-valued function $\beta(m, r)$ is said to be an *even function of* m (mod r) if $\beta(m, r) = \beta((m, r), r)$ for all values of m. The functions considered by Smith and Apostol are even functions of m (mod r). In 1993, Bourque and Ligh [**5**] extended the results of Smith, Apostol, and McCarthy. In 2002, Hong [**12**] generalized the results of Smith, Apostol, McCarthy and Bourque and Ligh to certain classes of arithmetical functions. In 2003, Korkee and Haukkanen [**18**] considered a certain abstract generalization of Smith's determinant.

Let $1 \le x_1 < \cdots < x_n < \cdots$ be a given arbitrary strictly increasing infinite sequence of positive integers. For any integer $n \ge 1$, let

$$S_n = \{x_1, \ldots, x_n\}.$$

Denote by $(f(x_i, x_j))$ the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry and by $(f[x_i, x_j])$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry. The set S_n is said to be *factor closed* if it contains every divisor of x for any $x \in S_n$. From Bourque and Ligh's result [**6**, Theorem 4], we can see that if S_n is a factor-closed set and f is a multiplicative function such that $f \in \mathcal{L}_{S_n}$, where $\mathcal{L}_{S_n}$ is a certain class of arithmetical functions defined by

$$\mathcal{L}_{S_n} := \{f : (f * \mu)(d) \in \mathbf{Z}\backslash\{0\} \text{ whenever } d|\mathrm{lcm}(S_n)\},$$

where $\mathrm{lcm}(S_n)$ means the least common multiple of all elements in S_n, then the matrix $(f(x_i, x_j))$ divides the matrix $(f[x_i, x_j])$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the integers. Hong [**13**] showed that for any *multiple-closed set* S_n (i.e. $y \in S_n$ whenever $x|y|\mathrm{lcm}(S_n)$ for any $x \in S_n$) and for any *divisor chain* S_n (i.e. $x_1|\cdots|x_n$), if f is a completely multiplicative function such that $f \in \mathcal{L}_{S_n}$, then the matrix $(f(x_i, x_j))$ divides the matrix $(f[x_i, x_j])$ in the ring $M_n(\mathbf{Z})$. But such a factorization is no longer true if f is multiplicative.

Now let ε be a real number. The $n \times n$ matrix having the power $(x_i, x_j)^\varepsilon$ of the greatest common divisor of x_i and x_j as its i, j-entry is called the *power greatest common divisor (GCD) matrix defined on S_n, denoted by* $((x_i, x_j)^\varepsilon)$*, or abbreviated by* $((S_n)^\varepsilon)$. The matrix having the power $[x_i, x_j]^\varepsilon$ of the least common multiple of x_i and x_j as its i, j-entry is called the *power least common multiple (LCM) matrix, denoted by* $([x_i, x_j]^\varepsilon)$*, or abbreviated by* $[(S_n)^\varepsilon]$. If we let $\varepsilon = 1$, then the power GCD matrix and the power LCM matrix are said to be the *GCD matrix defined on S_n* and the *LCM matrix defined on S_n*, respectively, and *denoted by* (S_n) and $[S_n]$, respectively. In 1989, Beslin and Ligh [**3**] initiated the study of the GCD matrix (S_n) on any set S_n in the direction of structure, determinant and inverse. In particular, they proved that the GCD matrix (S_n) on any set S_n of n distinct positive integers is positive definite. However, the LCM matrix $[S_n]$ on any set S_n is not positive definite in general. It may even not be nonsingular. In fact, Hong [**11**] showed that for any integer $n \ge 8$, there exists a GCD-closed set $S_n = \{x_1, \ldots, x_n\}$ (i.e. one has $(x_i, x_j) \in S_n$ for all $1 \le i, j \le n$) such that the LCM matrix $[S_n]$ on S_n is singular. Note also that recently, Hong [**14**] proved that for any positive integer ε and for any GCD-closed set S_n satisfying $\max_{x \in S_n}\{\nu(x)\} \le 2$, where $\nu(x)$ denotes the number of distinct prime factors of the positive integer x, the power LCM matrix $[(S_n)^\varepsilon]$ on S_n is nonsingular.

For a different form of a power GCD matrix

$$N_n := \left(\frac{(i, j)^{2\varepsilon}}{i^\varepsilon \cdot j^\varepsilon}\right)_{1 \le i, j \le n},$$

Wintner [**26**] proved in 1944 that $\lim\sup_{n \to \infty} \Lambda_n(\varepsilon) < \infty$ if and only if $\varepsilon > 1$, where $\Lambda_n(\varepsilon)$ denotes the largest eigenvalue of the matrix N_n. Let $\lambda_n(\varepsilon)$ denote the smallest eigenvalue of the matrix N_n. Lindqvist and Seip [**19**] in 1998 use the work of [**9**] about Riesz bases to investigate the asymptotic behavior of $\lambda_n(\varepsilon)$ and $\Lambda_n(\varepsilon)$ as $n \to \infty$. In particular, they got a sharp bound for $\lambda_n(\varepsilon)$ and $\Lambda_n(\varepsilon)$. However, for the power GCD matrix $((S_n)^\varepsilon)$ on S_n, the eigenvalues do not seem to be known. In 1993, Bourque and Ligh [**5**] extended Beslin and Ligh's result by showing that for any $\varepsilon > 0$, the power GCD matrix $((S_n)^\varepsilon)$ on S_n is positive definite. From this, one can only conclude that its eigenvalues are positive, but no further information is provided.

In the present paper, our main goal is to consider the asymptotic behavior of the eigenvalues of the power GCD matrix $((S_n)^\varepsilon)$ on S_n. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set $\{x_1, \ldots, x_n\}$. Let $1 \leq q \leq n$ be a fixed integer and $\varepsilon > 0$. Then it follows from Bourque and Ligh's result [**4**] that

$$\lambda_n^{(q)} > 0.$$

On the other hand, by Cauchy's interlacing inequalities (see [**15**]) we have

$$\lambda_{n+1}^{(q)} \leq \lambda_n^{(q)}.$$

Thus the sequence $\{\lambda_n^{(q)}\}_{n=q}^{\infty}$ is a non-increasing infinite sequence of positive real numbers and so it is convergent. Namely, we have the following result.

Proposition 1.1. *Let $q \geq 1$ be a given arbitrary integer, $\varepsilon > 0$ and $\{x_i\}_{i=1}^{\infty}$ an arbitrary given strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set $\{x_1, \ldots, x_n\}$. Then the sequence $\{\lambda_n^{(q)}\}_{n=q}^{\infty}$ converges and*

$$\lim_{n\to\infty} \lambda_n^{(q)} \geq 0.$$

Let $x \geq 1$ be an integer. For an arbitrary strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers satisfying that $(x_i, x_j) = x$ for any $i \neq j$ and $\sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$, we show, in section 2, that if $0 < \varepsilon \leq 1$, then $\lim_{n\to\infty}\lambda_n^{(1)} = x_1^\varepsilon - x^\varepsilon$. Let $a \geq 0, b \geq 1$ and $e \geq 0$ be any given integers. In section 3, we show that for the arithmetic progression $\{x_{i-e+1} = a + bi\}_{i=e}^{\infty}$, if $0 < \varepsilon \leq 1$, then $\lim_{n\to\infty}\lambda_n^{(q)} = 0$.

We give in Section 4 a lower bound for the smallest eigenvalue $\lambda_n^{(1)}$ of the power GCD matrix $((S_n)^\varepsilon)$ on any set S_n. This improves the lower bound due to Beslin, Bourque and Ligh. Then we use it to obtain a lower bound for the q-th largest eigenvalue $\lambda_n^{(n-q+1)}$ of the power GCD matrix $((S_n)^\varepsilon)$ on any set S_n for any $\varepsilon > 0$ and any given integer $q \geq 1$. This lower bound then implies that for any $\varepsilon > 0$ and any given integer $q \geq 1$, the q-th largest eigenvalue of the power GCD matrix $((S_n)^\varepsilon)$ on any set S_n tends to infinity as n tends to infinity. The final section contains some remarks and questions.

For a comprehensive review of papers relating to greatest common divisor matrices not presented here, we refer the readers to [**8**]. Throughout this paper, we let E_n denote the $n \times n$ matrix with all entries equal to 1.

2. Some preliminary results. In this section we shall study the asymptotic behavior of the smallest eigenvalue of the power GCD matrix defined on the ordered finite subsequence of an infinite sequence of pairwise relatively prime positive integers. First we state some results on certain symmetric matrices. The following lemma is known.

LEMMA 2.1. *Let $n \geq 1$ be an integer and let $a_1, \ldots, a_n \in \mathbf{F}$ such that $a_i \neq 1$ for all $1 \leq i \leq n$, where $\mathbf{F}$ is an arbitrary field. Then*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & a_1 & \cdots & 1 \\ .. & .. & .. & .. \\ 1 & 1 & \cdots & a_n \end{pmatrix}^{-1} = \begin{pmatrix} 1+\sum_{i=1}^{n}\frac{1}{a_i-1} & -\frac{1}{a_1-1} & -\frac{1}{a_2-1} & \cdots & -\frac{1}{a_n-1} \\ -\frac{1}{a_1-1} & \frac{1}{a_1-1} & 0 & \cdots & 0 \\ -\frac{1}{a_2-1} & 0 & \frac{1}{a_2-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{a_n-1} & 0 & 0 & \cdots & \frac{1}{a_n-1} \end{pmatrix}.$$

Assume now that $\{u_i\}_{i=1}^{\infty}$ is any given strictly increasing infinite sequence of real numbers such that $u_1 > 1$. Let $U_n := E_n + \mathrm{diag}(0, u_1 - 1, \ldots, u_{n-1} - 1)$. It is easy to see that the $n \times n$ matrix U_n is positive definite. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ matrix U_n. Then $\lambda_n^{(1)} > 0$. Now let $\mu_n^{(1)} \leq \cdots \leq \mu_n^{(n)}$ be the eigenvalues of the inverse matrix U_n^{-1}. Then

$$\lambda_n^{(i)} \cdot \mu_n^{(n-i+1)} = 1, \quad 1 \leq i \leq n. \tag{2.1}$$

LEMMA 2.2. *Suppose that $\{u_i\}_{i=1}^{\infty}$ is any given strictly increasing infinite sequence of real numbers such that $u_1 > 1$. Let $U_n = E_n + \mathrm{diag}(0, u_1 - 1, \ldots, u_{n-1} - 1)$ and $\mu_n^{(n)}$ the largest eigenvalue of the inverse matrix U_n^{-1}. Then*

$$\mu_n^{(n)} > 1 + \sum_{i=1}^{n-1} \frac{1}{u_i - 1}.$$

Proof. Define an $n \times n$ matrix V_n as follows:

$$V_n = \begin{pmatrix} 1+\sum_{i=1}^{n}\frac{1}{u_i-1} & -\frac{1}{u_1-1} & \cdots & -\frac{1}{u_{n-1}-1} \\ -\frac{1}{u_1-1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{u_{n-1}-1} & 0 & \cdots & 0 \end{pmatrix}.$$

Then the characteristic polynomial of V_n is given by

$$|\lambda I_n - V_n| = \lambda^{n-2}\left(\lambda^2 - \left(1 + \sum_{i=1}^{n-1}\frac{1}{u_i - 1}\right)\lambda - \sum_{i=1}^{n-1}\frac{1}{(u_i - 1)^2}\right).$$

So the largest eigenvalue $\lambda_{\max}(V_n)$ of V_n satisfies

$$\lambda_{\max}(V_n) > 1 + \sum_{i=1}^{n-1}\frac{1}{u_i - 1}. \tag{2.2}$$

By Lemma 2.1 one gets

$$U_n^{-1} = \begin{pmatrix} 1+\sum_{i=1}^{n}\frac{1}{u_i-1} & -\frac{1}{u_1-1} & -\frac{1}{u_2-1} & \cdots & -\frac{1}{u_{n-1}-1} \\ -\frac{1}{u_1-1} & \frac{1}{u_1-1} & 0 & \cdots & 0 \\ -\frac{1}{u_2-1} & 0 & \frac{1}{u_2-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{u_{n-1}-1} & 0 & 0 & \cdots & \frac{1}{u_{n-1}-1} \end{pmatrix}.$$

Replacing all negative terms $-\frac{1}{u_i-1}$ $(1 \le i \le n)$ by $\frac{1}{u_i-1}$ in the matrices V_n and U_n^{-1}, we get the corresponding nonnegative (element-wise) matrices $|V_n|$ and $|U_n^{-1}|$, respectively. Clearly V_n, $|V_n|$ are similar, and U_n^{-1}, $|U_n^{-1}|$ likewise. Thus the spectral radii satisfy

$$\rho(|V_n|) = \rho(V_n) = \lambda_{\max}(V_n)$$

and

$$\rho\left(\left|U_n^{-1}\right|\right) = \rho\left(U_n^{-1}\right) = \mu_n^{(n)}.$$

Since $0 \le |V_n| \le |U_n^{-1}|$, one deduces immediately from the Perron-Frobenius theorem for nonnegative matrices (see, for example, [**15**]) that $\rho(|V_n|) \le \rho(U_n^{-1})$. So we have

$$\lambda_{\max}(V_n) \le \mu_n^{(n)}. \tag{2.3}$$

The result follows from (2.2) and (2.3). □

COROLLARY 2.3. *Suppose that $\{u_i\}_{i=1}^{\infty}$ is any given strictly increasing infinite sequence of real numbers such that $u_1 > 1$ and*

$$\sum_{i=1}^{\infty} \frac{1}{u_i} = \infty. \tag{2.4}$$

Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $U_n = E_n + \mathrm{diag}(0, u_1 - 1, \ldots, u_{n-1} - 1)$. Then

$$\lim_{n\to\infty} \lambda_n^{(1)} = 0.$$

Proof. Let $\mu_n^{(n)}$ be the largest eigenvalue of the inverse matrix U_n^{-1}. By (2.1)

$$\lambda_n^{(1)} = \frac{1}{\mu_n^{(n)}}.$$

It then follows from Lemma 2.2 that for $n \ge 2$,

$$\lambda_n^{(1)} < \frac{1}{1+\sum_{i=1}^{n-1}\frac{1}{u_i-1}} < \frac{1}{1+\sum_{i=1}^{n-1}\frac{1}{u_i}}. \tag{2.5}$$

Since $\lambda_n^{(1)} > 0$ it follows from (2.4) and (2.5) that $\lim_{n\to\infty}\lambda_n^{(1)} = 0$. □

COROLLARY 2.4. *Let* $\{r_i\}_{i=1}^{\infty}$ *be a strictly increasing infinite sequence of real numbers satisfying* $r_1 \geq 1$ *and*

$$\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.$$

Let $\lambda_n^{(1)}$ *denote the smallest eigenvalue of the* $n \times n$ *matrix* $E_n + \mathrm{diag}(r_1 - 1, \ldots, r_n - 1)$. *Then*

$$\lim_{n\to\infty} \lambda_n^{(1)} = r_1 - 1.$$

Proof. Let

$$R_n := E_n + \mathrm{diag}(r_1 - 1, \ldots, r_n - 1).$$

Then

$$R_n = (r_1 - 1)I_n + E_n + \mathrm{diag}(0, (r_2 - r_1 + 1) - 1, \ldots, (r_n - r_1 + 1) - 1).$$

Since $r_2 - r_1 + 1 > 1$ and

$$\sum_{i=2}^{\infty} \frac{1}{r_i - r_1 + 1} = \infty,$$

the result follows immediately from Corollary 2.3. □

The following is the main result of this section.

THEOREM 2.5. *Let* $0 < \varepsilon \leq 1$ *and* x *a positive integer. Let* $\{x_i\}_{i=1}^{\infty}$ *be a strictly increasing infinite sequence of positive integers satisfying the following conditions.*

(i) *For every* $i \neq j$, $(x_i, x_j) = x$;

(ii) $\sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$.

If $\lambda_n^{(1)}$ *is the smallest eigenvalue of the* $n \times n$ *power GCD matrix* $((x_i, x_j)^{\varepsilon})$ *defined on the set* $\{x_1, \ldots, x_n\}$, *then*

$$\lim_{n\to\infty} \lambda_n^{(1)} = x_1^{\varepsilon} - x^{\varepsilon}.$$

Proof. For $i \geq 1$, let $x_i = x \cdot \bar{x}_i$. Note that $x > 0$. Then from (i) and (ii) we can easily deduce that for any $i \neq j$, $(\bar{x}_i, \bar{x}_j) = 1$ and

$$\sum_{i=1}^{\infty} \frac{1}{\bar{x}_i} = \infty. \tag{2.6}$$

Obviously we have

$$((\bar{x}_i, \bar{x}_j)^{\varepsilon}) = E_n + \mathrm{diag}\big(\bar{x}_1^{\varepsilon} - 1, \ldots, \bar{x}_n^{\varepsilon} - 1\big).$$

For $1 \le i \le n$, let $r_i = \bar{x}_i^{\varepsilon}$. Then $r_1 = \bar{x}_1^{\varepsilon} \ge 1$ and for every $1 \le i \le n$, $r_i \le \bar{x}_i$. Hence we have by (2.6)

$$\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty.$$

Let $\bar{\lambda}_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power GCD matrix $((\bar{x}_i, \bar{x}_j)^{\varepsilon})$ defined on the set $\{\bar{x}_1, \dots, \bar{x}_n\}$. Thus by Corollary 2.4 we have $\lim_{n\to\infty} \bar{\lambda}_n^{(1)} = \bar{x}_1^{\varepsilon} - 1$. The result then follows immediately from the fact that $\lambda_n^{(1)} = x^{\varepsilon} \cdot \bar{\lambda}_n^{(1)}$. □

3. Arithmetic progressions and the q-th smallest eigenvalue. In this section, we turn our attention to arithmetic progressions. First we introduce the following concept.

DEFINITION. Let e and r be positive integers. Let $X = \{x_1, \dots, x_e\}$ and $Y = \{y_1, \dots, y_r\}$ be two sets of distinct positive integers. Then we define the *tensor product* (*set*) of X and Y, denoted by $X \odot Y$, by

$$X \odot Y := \{x_1y_1, \dots, x_1y_r, x_2y_1, \dots, x_2y_r, \dots, x_ey_1, \dots, x_ey_r\}.$$

REMARK. It must be pointed out that the elements in the tensor product set are not necessarily arranged in increasing order. For example, let $X = \{1, 2, 3\}$ and $Y = \{3, 5\}$. Then $X \odot Y = \{3, 5, 6, 10, 9, 15\}$. We note also that the elements in the tensor product set are not necessarily distinct. For example, let $X = \{2, 3\}$ and $Y = \{4, 6\}$. Then $X \odot Y = \{8, 12, 12, 18\}$.

LEMMA 3.1. *Let ε be any real number. Let e and r be positive integers. Let $X = \{x_1, \dots, x_e\}$ be a set of e distinct positive integers such that for any $1 \le i \ne j \le e$, $(x_i, x_j) = 1$. Let $Y = \{y_1, \dots, y_r\}$ be a set of r distinct positive integers such that for any $1 \le i \ne j \le r$, $(y_i, y_j) = 1$. Assume that for all $1 \le i \le e$, $1 \le j \le r$, $(x_i, y_j) = 1$. Then the following equality holds:*

$$((X \odot Y)^{\varepsilon}) = ((X)^{\varepsilon}) \otimes ((Y)^{\varepsilon}).$$

Proof. First we have

$$((X)^{\varepsilon}) = \begin{pmatrix} x_1^{\varepsilon} & 1 & \cdots & 1 \\ 1 & x_2^{\varepsilon} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & x_e^{\varepsilon} \end{pmatrix} \quad \text{and} \quad ((Y)^{\varepsilon}) = \begin{pmatrix} y_1^{\varepsilon} & 1 & \cdots & 1 \\ 1 & y_2^{\varepsilon} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & y_r^{\varepsilon} \end{pmatrix}.$$

Since

$$(x_{i_1}y_{j_1},\ x_{i_2}y_{j_2}) = \begin{cases} x_{i_1}y_{j_1} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2, \\ y_{j_1} & \text{if } i_1 \ne i_2 \text{ and } j_1 = j_2, \\ x_{i_1} & \text{if } i_1 = i_2 \text{ and } j_1 \ne j_2, \\ 1 & \text{if } i_1 \ne i_2 \text{ and } j_1 \ne j_2, \end{cases}$$

letting $Y_r = ((Y)^\varepsilon)$ we get

$$((X \odot Y)^\varepsilon) = \begin{pmatrix} x_1^\varepsilon Y_r & Y_r & \cdots & Y_r \\ Y_r & x_2^\varepsilon Y_r & \cdots & Y_r \\ \vdots & \vdots & \vdots & \vdots \\ Y_r & Y_r & \cdots & x_e^\varepsilon Y_r \end{pmatrix}$$

$$= \begin{pmatrix} x_1^\varepsilon & 1 & \cdots & 1 \\ 1 & x_2^\varepsilon & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & x_e^\varepsilon \end{pmatrix} \otimes Y_r$$

$$= ((X)^\varepsilon) \otimes ((Y)^\varepsilon)$$

as desired. □

LEMMA 3.2. *Let $b \geq 1$ be an integer and $0 < \varepsilon \leq 1$. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((1 + bi, 1 + bj)^\varepsilon)$ defined on the set $\{1 + bi\}_{i=0}^{n-1}$. Then for any given integer $q \geq 1$ we have*

$$\lim_{n\to\infty} \lambda_n^{(q)} = 0.$$

Proof. By Dirichlet's theorem (see [**1**], or [**17**]) there are infinitely many primes in the arithmetic progression $\{1 + bi\}_{i=0}^{\infty}$. Let $p_1 < \cdots < p_n < \cdots$ denote the primes in this arithmetic progression. By Mertens' theorem (see [**22**])

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty,$$

and since $0 < \varepsilon \leq 1$, it follows that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^\varepsilon} = \infty. \tag{3.1}$$

For $i \geq 1$, let $\pi_i = p_{q-1+i}$. Then $p_{q-1} < \pi_1 < \pi_2 < \cdots$. Since q is a fixed number, it follows from (3.1) that

$$\sum_{i=1}^{\infty} \frac{1}{\pi_i^\varepsilon} = \infty. \tag{3.2}$$

Now let $r \geq 2$ be an arbitrary integer and let

$$P_q := \{1, p_1, \ldots, p_{q-1}\}, \quad T_r := \{1, \pi_1, \ldots, \pi_{r-1}\}.$$

Consider the tensor product set $P_q \odot T_r$. Note that the entries in the set $P_q \odot T_r$ are *not* arranged in increasing order, but the eigenvalues of the corresponding power GCD matrix do not depend on rearranging those entries. By Lemma 3.1

$$((P_q \odot T_r)^\varepsilon) = ((P_q)^\varepsilon) \otimes ((T_r)^\varepsilon).$$

Let $\mu_q^{(1)} \le \cdots \le \mu_q^{(q)}$ and $\tilde{\lambda}_r^{(1)} \le \cdots \le \tilde{\lambda}_r^{(r)}$ be the eigenvalues of the power GCD matrix $((P_q)^\varepsilon)$ defined on the set P_q and the power GCD matrix $((T_r)^\varepsilon)$ defined on the set T_r respectively. Then it is known (see [**16**]) that the eigenvalues of the tensor product matrix $((P_q)^\varepsilon) \otimes ((T_r)^\varepsilon)$ are given by the set

$$\left\{\mu_q^{(i)} \cdot \tilde{\lambda}_r^{(j)}\right\}_{1\le j\le r}^{1\le i\le q}.$$

Notice that

$$\mu_q^{(1)} \cdot \tilde{\lambda}_r^{(1)} \le \cdots \le \mu_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{3.3}$$

Since for any $i_1, i_2 \in \mathbf{Z}$,

$$(1+bi_1)(1+bi_2) = 1 + bi_1 + bi_2 + b^2 i_1 i_2 \equiv 1 \pmod{b},$$

the arithmetic progression $\{1+bi\}_{i=0}^\infty$ is closed under the usual multiplication. So the tensor product set $P_q \odot T_r \subset \{1+bi\}_{i=0}^\infty$. For any integer $r \ge 2$, define an integer n_r by

$$n_r := \frac{p_{q-1} \cdot \pi_{r-1} - 1}{b} + 1.$$

Then $P_q \odot T_r \subseteq \{1+bi\}_{i=0}^{n_r-1}$. Thus the power GCD matrix $((P_q \odot T_r)^\varepsilon)$ defined on $P_q \odot T_r$ is a principal submatrix of the $n_r \times n_r$ power GCD matrix $((1+bi, 1+bj)^\varepsilon)$ defined on the set $\{1, 1+b, \ldots, 1+b(n_r-1)\}$. Let $\bar{\lambda}_{qr}^{(1)} \le \cdots \le \bar{\lambda}_{qr}^{(qr)}$ be the eigenvalues of $((P_q \odot T_r)^\varepsilon)$. Then by Cauchy's interlacing inequalities we have

$$\lambda_{n_r}^{(q)} \le \bar{\lambda}_{qr}^{(q)}. \tag{3.4}$$

But by (3.3)

$$\bar{\lambda}_{qr}^{(q)} \le \mu_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{3.5}$$

So it follows from (3.4) and (3.5) that

$$\lambda_{n_r}^{(q)} \le \mu_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{3.6}$$

On the other hand, in Theorem 2.5, if we choose $x = x_1 = 1$ and $x_i = \pi_{i-1}$ for $i \ge 2$, then by (3.2) the two conditions of Theorem 2.5 are satisfied. It then follows immediately from Theorem 2.5 that

$$\lim_{r\to\infty} \tilde{\lambda}_r^{(1)} = 0. \tag{3.7}$$

It follows from Proposition 1.1 that the subsequence $\{\lambda_{n_r}^{(q)}\}_{r=1}^\infty$ of the sequence $\{\lambda_n^{(q)}\}_{n=1}^\infty$ converges and

$$\lim_{r\to\infty} \lambda_{n_r}^{(q)} \ge 0. \tag{3.8}$$

Hence by (3.6)–(3.8), $\lim_{r\to\infty}\lambda_{n_r}^{(q)} = 0$. Finally, again by Proposition 1.1, the desired result $\lim_{n\to\infty}\lambda_n^{(q)} = 0$ follows immediately. □

We are now in a position to give the main result of this section.

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THEOREM 3.3. *Let $a,\ b \geq 1$ and $e \geq 0$ be integers and $0 < \varepsilon \leq 1$. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((a+bi, a+bj)^\varepsilon)$ defined on the set $\{a+be, a+b(e+1), \ldots, a+b(e+n-1)\}$. Then for any given integer $q \geq 1$*

$$\lim_{n\to\infty} \lambda_n^{(q)} = 0.$$

Proof. For the arithmetic progression $\{a+bi\}_{i=e}^{\infty}$, consider its subsequence

$$\{a+b(e+(a+be)i)\}_{i=0}^{\infty} = \{(a+be)(1+bi)\}_{i=0}^{\infty}.$$

For any integer $m \geq 1$, let $\mu_m^{(1)} \leq \cdots \leq \mu_m^{(m)}$ be the eigenvalues of the $m \times m$ power GCD matrix $((W_m)^\varepsilon)$ defined on the set

$$W_m := \{a+be, (a+be)(1+b), \ldots, (a+be)(1+b(m-1))\}$$

and let $\tilde{\mu}_m^{(1)} \leq \cdots \leq \tilde{\mu}_m^{(m)}$ be the eigenvalues of the $m \times m$ power GCD matrix $((\tilde{W}_m)^\varepsilon)$ defined on the set

$$\tilde{W}_m := \{1, 1+b, \ldots, 1+b(m-1)\}.$$

Thus for $1 \leq i \leq m,\ \mu_m^{(i)} = (a+be)^\varepsilon \cdot \tilde{\mu}_m^{(i)}$. In particular,

$$\mu_m^{(q)} = (a+be)^\varepsilon \tilde{\mu}_m^{(q)}. \tag{3.9}$$

Now let

$$m_n := 1 + \left\lfloor \frac{n-1}{a+be} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. Choose n so that $m_n \geq q$.

By Cauchy's interlacing inequalities

$$\lambda_n^{(q)} \leq \mu_{m_n}^{(q)}, \tag{3.10}$$

and by (3.9) and (3.10),

$$\lambda_n^{(q)} \leq (a+be)^\varepsilon \tilde{\mu}_{m_n}^{(q)}. \tag{3.11}$$

By Lemma 3.2

$$\lim_{m\to\infty} \tilde{\mu}_m^{(q)} = 0,$$

so

$$\lim_{n\to\infty} \tilde{\mu}_{m_n}^{(q)} = 0.$$

By Proposition 1.1 and (3.11) we get

$$\lim_{n\to\infty} \lambda_n^{(q)} = 0. \qquad \square$$

Furthermore, applying again Cauchy's interlacing inequalities, it follows from Proposition 1.1 and Theorem 3.3 that the following result holds.

THEOREM 3.4. *Let a, $b \geq 1$ and $e \geq 0$ be integers and $0 < \varepsilon \leq 1$. Let $\{x_i\}_{i=1}^{\infty}$ be any given strictly increasing infinite sequence of positive integers which contains the arithmetic progression $\{a + bi\}_{i=e}^{\infty}$ as its subsequence. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((x_i, x_j)^{\varepsilon})$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. Then for any given integer $q \geq 1$*

$$\lim_{n\to\infty} \lambda_n^{(q)} = 0.$$

Finally we give the following immediate consequence as the conclusion of this section.

COROLLARY 3.5. *Let $0 < \varepsilon \leq 1$. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((i, j)^{\varepsilon})$ defined on the set $S_n = \{1, \ldots, n\}$. Then for any given integer $q \geq 1$*

$$\lim_{n\to\infty} \lambda_n^{(q)} = 0.$$

4. Lower bound for the smallest eigenvalue and limit behavior of the q-th largest eigenvalue. In this section, we assume always that $1 \leq x_1 < \cdots < x_n < \cdots$ is an arbitrary given infinite sequence of positive integers. Let $\varepsilon > 0$ be any given real number. Let $S_n = \{x_1, \ldots, x_n\}$. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the $n \times n$ power GCD matrix $((S_n)^{\varepsilon})$ defined on the set S_n. By Bourque and Ligh's result [**4**] the matrix $((S_n)^{\varepsilon})$ is positive definite and so $\lambda_n^{(i)} > 0$ for all $1 \leq i \leq n$. We will improve this result by giving a lower bound for the smallest eigenvalue $\lambda_n^{(1)}$. We need the following structure theorem.

LEMMA 4.1. *Let $\varepsilon > 0$. Let $\bar{S} = \{d_1, \ldots, d_m\}$ be a factor closed set (i.e. $\bar{S}$ contains every positive divisor of d for all $d \in \bar{S}$) containing S_n and let $A = (a_{ij})_{n\times m}$ be the $n \times m$ matrix defined by:*

$$a_{ij} = \begin{cases} \sqrt{J_{\varepsilon}(d_j)}, & \textit{if } d_j | x_i \\ 0, & \textit{otherwise}, \end{cases}$$

where $J_{\varepsilon} := \xi_{\varepsilon} * \mu$ is the generalized Jordan's totient function and ξ_{ε} is defined by $\xi_{\varepsilon}(x) = x^{\varepsilon}$ for any $x \in \mathbf{Z}$. Then

$$((S_n)^{\varepsilon}) = A \cdot A^t.$$

Proof. It follows immediately from [**10**, Lemma 2]. □

REMARK. If ε is a positive integer, then J_{ε} becomes the Jordan's totient function (see, for example, [**1**], [**20**] or [**23**]). In particular, J_1 is just Euler's totient function φ.

Now let K_n be the set of all $n \times n$ lower triangular matrices that satisfy: Every main diagonal entry is 1, and every off-diagonal entry is 0 or 1. Obviously K_n is a finite set of nonsingular matrices and so the set $L_n := \{Y \cdot Y^t : Y \in K_n\}$ is also a finite set of $n \times n$ positive definite matrices. Then we can define a positive constant c_n depending only on n as follows.

$$c_n := \min_{Z \in L_n} \{\mu_n^{(1)}(Z) : \mu_n^{(1)}(Z) \text{ is the smallest eigenvalue of } Z\}. \tag{4.1}$$

We can now give a lower bound for the smallest eigenvalue in terms of the constant c_n defined in (4.1) and the generalized Jordan's totient function.

THEOREM 4.2. *Let $\varepsilon > 0$ and $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power GCD matrix $((x_i, x_j)^{\varepsilon})$ defined on the set $\{x_1, \ldots, x_n\}$. Then*

$$\lambda_n^{(1)} \geq c_n \cdot \min_{1 \leq i \leq n} \{J_{\varepsilon}(x_i)\}.$$

Proof. Let A be the $n \times m$ matrix defined in Lemma 4.1. Let $S_n = \{x_1, \ldots, x_n\}$. Then Lemma 4.1 gives

$$((S_n)^{\varepsilon}) = A \cdot A^t. \tag{4.2}$$

Note that given any $m \times m$ permutation matrix P we have

$$((S_n)^{\varepsilon}) = A \cdot A^t = A \cdot PP^t \cdot A^t = (AP) \cdot (AP)^t.$$

Thus we can permute the columns of A. So we can assume without loss of any generality that

$$d_l = x_l, \; l = 1, \ldots, n.$$

We now partition A as follows:

$$A = (\overbrace{\;B\;}^{n} \mid \overbrace{\;C\;}^{m-n}).$$

Therefore

$$\begin{aligned} A \cdot A^t &= (B \mid C) \cdot \begin{pmatrix} B^t \\ \hline C^t \end{pmatrix} \\ &= B \cdot B^t + C \cdot C^t. \end{aligned} \tag{4.3}$$

We introduce the following notation:

NOTATION. For real symmetric matrices G_1, G_2 of the same order, we write $G_1 \succeq G_2 \Longleftrightarrow G_1 - G_2$ is positive semi-definite.

Therefore it follows from (4.2) and (4.3) that

$$((S_n)^{\varepsilon}) \succeq B \cdot B^t.$$

Let $\delta_n^{(1)} \leq \cdots \leq \delta_n^{(n)}$ be the eigenvalues of $B \cdot B^t$. Then it is known (see [**15**]) that $\lambda_n^{(i)} \geq \delta_n^{(i)}$ for $1 \leq i \leq n$.

Consider now the $n \times n$ matrix $B = (b_{ij})$. We have

$$b_{ij} = \begin{cases} \sqrt{J_{\varepsilon}(x_j)}, & \text{if } x_j | x_i \\ 0, & \text{otherwise.} \end{cases}$$

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In particular, B is lower triangular, and the main diagonal entries are $\sqrt{J_\varepsilon(x_i)}$, $i = 1, \ldots, n$. We can factor now

$$B = \tilde{B} \cdot D,$$

where $D = \operatorname{diag}(\sqrt{J_\varepsilon(x_1)}, \ldots, \sqrt{J_\varepsilon(x_n)})$ and $\tilde{B} = (\tilde{b}_{ij})$ is defined by

$$\tilde{b}_{ij} = \begin{cases} 1, & \text{if } x_j | x_i \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\tilde{B} \in K_n$. So we have

$$B \cdot B^t = \tilde{B}D \cdot D\tilde{B}^t = \tilde{B} \cdot D^2 \cdot \tilde{B}^t$$

and

$$G := (B \cdot B^t)^{-1} = (\tilde{B}^t)^{-1} \cdot D^{-2} \cdot \tilde{B}^{-1}.$$

We now use the spectral norm which is denoted by $\|\cdot\|$, and is well known to be the largest singular value of the matrix (see [**15**]). Let $\rho(\cdot)$ denote the spectral radius. Since G is positive definite, we have

$$\rho(G) = \|G\| = \|(\tilde{B}^t)^{-1} \cdot D^{-2} \cdot \tilde{B}^{-1}\| \leq \alpha \cdot \|(\tilde{B}^t)^{-1}\| \cdot \|\tilde{B}^{-1}\|,$$

where

$$\alpha := \|D^{-2}\| = \max_{1 \leq i \leq n} \left\{ \frac{1}{J_\varepsilon(x_i)} \right\} = \frac{1}{\min_{1 \leq i \leq n}\{J_\varepsilon(x_i)\}}.$$

It is also known that for any matrix T,

$$\|T \cdot T^t\| = \|T\| \cdot \|T^t\| = \|T\|^2.$$

So we have

$$\rho(G) = \|G\| \leq \alpha \cdot \|((\tilde{B}^t)^{-1}\tilde{B}^{-1})\| = \alpha \cdot \|(\tilde{B} \cdot \tilde{B}^t)^{-1}\|. \tag{4.4}$$

Since $\tilde{B} \cdot \tilde{B}^t$ is positive definite, we have

$$\|(\tilde{B} \cdot \tilde{B}^t)^{-1}\| = \rho((\tilde{B} \cdot \tilde{B}^t)^{-1}) = \frac{1}{\mu_n^{(1)}(\tilde{B} \cdot \tilde{B}^t)},$$

where $\mu_n^{(1)}(\tilde{B} \cdot \tilde{B}^t)$ denotes the smallest eigenvalue of $\tilde{B} \cdot \tilde{B}^t$. Therefore it follows from (4.4) that

$$\rho(G) \leq \frac{\alpha}{\mu_n^{(1)}(\tilde{B} \cdot \tilde{B}^t)} \leq \frac{\alpha}{c_n} = \frac{1}{c_n \cdot \min_{1 \leq i \leq n}\{J_\varepsilon(x_i)\}}.$$

Since $\rho(G) = \frac{1}{\delta_n^{(1)}}$ and $\lambda_n^{(1)} \geq \delta_n^{(1)}$, we conclude that

$$\lambda_n^{(1)} \geq \frac{1}{\rho(G)} \geq c_n \cdot \min_{1 \leq i \leq n}\{J_\varepsilon(x_i)\}$$

as required. □

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COROLLARY 4.3. *Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ GCD matrix (S_n) defined on the set $S_n = \{x_1, \ldots, x_n\}$. Then*

$$\lambda_n^{(1)} \geq c_n \cdot \min_{1 \leq i \leq n} \{\varphi(x_i)\}.$$

Proof. Since $J_1(x) = \varphi(x)$ for any positive integer x, the result follows immediately from Theorem 4.2. □

REMARK. Note that Bourque and Ligh's theorem [**4**] just states that $\lambda_n^{(1)} > 0$, so Theorem 4.2 gives a better lower bound for $\lambda_n^{(1)}$. Note also that Bourque and Ligh's theorem generalizes a result due to Beslin and Ligh [**3**] which deals with the special case $\varepsilon = 1$.

COROLLARY 4.4. *Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers so that $x_1 = 1$. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power GCD matrix $((x_i, x_j)^{\varepsilon})$ defined on the set $\{x_1, \ldots, x_n\}$. Then for $\varepsilon > 0$ we have*

$$\lambda_n^{(1)} \geq c_n.$$

Proof. Since $J_\varepsilon(1) = 1$ and $J_\varepsilon(x) \geq 1$ for any integer $x \geq 2$, the result follows immediately from Theorem 4.2. □

LEMMA 4.5. *Let $x > 1$ be a positive integer.*

(i) *If $\varepsilon > 1$, then*

$$J_\varepsilon(x) \geq \frac{x^{\varepsilon}}{\zeta(\varepsilon)}.$$

(ii) *For $\varepsilon = 1$, we have*

$$J_1(x) \geq \frac{x \cdot e^{-\gamma}}{\log x}\left(1 - \frac{C}{\log x}\right),$$

where $C > 0$ is a constant and γ is Euler's constant.

(iii) *For $0 < \varepsilon < 1$, we have*

$$J_\varepsilon(x) = x^{\varepsilon(1-\delta)} \cdot g(x),$$

where $0 < \delta < 1$ is a constant and $g(x)$ is a function depending only on ε and δ satisfying that $g(x) \to \infty$ as $x \to \infty$.

(iv) *If $\varepsilon > 0$, then $\lim_{x \to \infty} J_\varepsilon(x) = \infty$.*

Proof. First for any real number ε, J_ε is multiplicative. For a prime p and a positive integer l,

$$J_\varepsilon(p^l) = (\xi_\varepsilon * \mu)(p^l) = p^{l\varepsilon}\left(1 - \frac{1}{p^{\varepsilon}}\right).$$

So we have

$$J_\varepsilon(x) = x^{\varepsilon} \prod_{p|x}\left(1 - \frac{1}{p^{\varepsilon}}\right). \tag{4.5}$$

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It follows immediately that for $\varepsilon > 0$

$$J_\varepsilon(x) \geq x^\varepsilon \prod_{p \in I}\left(1 - \frac{1}{p^\varepsilon}\right), \tag{4.6}$$

where I denotes the set of all positive prime numbers.

If $\varepsilon > 1$, then Euler's formula (see, for example, [**1**] or [**17**]) says that

$$\zeta(\varepsilon) = \prod_{p \in I}\left(1 - \frac{1}{p^\varepsilon}\right)^{-1},$$

where ζ means the usual Riemann's zeta function. So we have

$$\prod_{p \in I}\left(1 - \frac{1}{p^\varepsilon}\right) = \frac{1}{\zeta(\varepsilon)}. \tag{4.7}$$

Therefore (i) follows from (4.6) and (4.7).

If $\varepsilon = 1$, by (4.5) we have

$$J_1(x) \geq x \prod_{p \leq x}\left(1 - \frac{1}{p}\right). \tag{4.8}$$

But a theorem of Mertens (see [**1**], or [**23**], or [**24**]) gives

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where γ is Euler's constant. So there exists a constant $C > 0$ such that

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) \geq \frac{e^{-\gamma}}{\log x}\left(1 - \frac{C}{\log x}\right). \tag{4.9}$$

Now (ii) follows immediately from (4.8) and (4.9).

Let now $0 < \varepsilon < 1$. Write

$$f(x) = \frac{x^{\varepsilon(1-\delta)}}{J_\varepsilon(x)}.$$

Then $f(x)$ is multiplicative. We claim that $f(x) \to 0$ when $x \to \infty$. By Theorem 316 of [7], it is sufficient to prove that $f(p^m) \to 0$ when $p^m \to \infty$. But

$$\frac{1}{f(p^m)} = p^{m\varepsilon\delta}\left(1 - \frac{1}{p^\varepsilon}\right) \to \infty$$

as $p^m \to \infty$. Therefore the claim is proved. Now let $g = \frac{1}{f}$. Then $g(x) \to \infty$ when $x \to \infty$ and $J_\varepsilon(x) = x^{\varepsilon(1-\delta)} \cdot g(x)$ as desired. Thus (iii) is proved.

The statement of (iv) follows immediately from (i)–(iii). □

Corollary 4.6. *Let $\{x_i\}_{i=1}^\infty$ be an arbitrary given strictly increasing infinite sequence of positive integers so that $x_1 > 1$. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power*

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GCD matrix $((x_i, x_j)^\varepsilon)$ *defined on the set* $\{x_1, \ldots, x_n\}$. *Then for* $\varepsilon > 1$ *we have*

$$\lambda_n^{(1)} \geq c_n \cdot \frac{x_1^\varepsilon}{\zeta(\varepsilon)}.$$

For $\varepsilon = 1$, *we have*

$$\lambda_n^{(1)} \geq c_n \cdot \min_{1\leq i\leq n} \left\{ \frac{x_i \cdot e^{-\gamma}}{\log x_i} \left(1 - \frac{C}{\log x_i}\right) \right\}, \tag{4.10}$$

where $C > 0$ *is a constant and* γ *is Euler's constant;*

For $0 < \varepsilon < 1$, *we have*

$$\lambda_n^{(1)} \geq c_n \cdot \min_{1\leq i\leq n} \left\{ x_{n-i}^{\varepsilon(1-\delta)} \cdot g(x_{n-i}) \right\},$$

where $0 < \delta < 1$ *is a constant and* $g(x)$ *is the function defined in Lemma* 4.5.

Proof. It follows from Theorem 4.2 and Lemma 4.5. □

REMARK. If x is sufficiently large, then $\frac{x}{\log x}(1 - \frac{C}{\log x})$ is strictly increasing we deduce that if x_1 is sufficiently large, then (4.10) becomes

$$\lambda_n^{(1)} \geq c_n \cdot \frac{x_1 \cdot e^{-\gamma}}{\log x_1} \left(1 - \frac{C}{\log x_1}\right).$$

THEOREM 4.7. *Let* $q \geq 1$ *be an arbitrary given integer and* $\{x_i\}_{i=1}^\infty$ *an arbitrary given strictly increasing infinite sequence of positive integers. Let* $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ *be the eigenvalues of the* $n \times n$ *power GCD matrix* $((x_i, x_j)^\varepsilon)$ *defined on the set* $\{x_1, \ldots, x_n\}$. *Then, if* $n > q$, *the following four statements hold.*

(i) *For* $\varepsilon > 1$,

$$\lambda_n^{(n-q+1)} \geq c_q \cdot \frac{x_{n-q+1}^\varepsilon}{\zeta(\varepsilon)}.$$

(ii) *For* $\varepsilon = 1$,

$$\lambda_n^{(n-q+1)} \geq c_q \cdot \min_{0\leq i\leq q-1} \left\{ \frac{x_{n-i} \cdot e^{-\gamma}}{\log x_{n-i}} \left(1 - \frac{C}{\log x_{n-i}}\right) \right\}, \tag{4.11}$$

where $C > 0$ *is a constant and* γ *is Euler's constant.*

(iii) *For* $0 < \varepsilon < 1$,

$$\lambda_n^{(n-q+1)} \geq c_q \cdot \min_{0\leq i\leq q-1} \left\{ x_i^{\varepsilon(1-\delta)} \cdot g(x_i) \right\},$$

where $0 < \delta < 1$ *is a constant and* $g(x)$ *is the function defined in Lemma* 4.5.

(iv) *For* $\varepsilon > 0$, $\lim_{n\to\infty} \lambda_n^{(n-q+1)} = \infty$.

Proof. Let $R_q = ((x_{n-i}, x_{n-j})^\varepsilon)$ be the $q \times q$ power GCD matrix defined on the set $\{x_{n-q+1}, \ldots, x_n\}$. Let $\mu_q^{(1)} \leq \cdots \leq \mu_q^{(q)}$ be the eigenvalues of R_q. Then Theorem 4.2 applied to R_q gives

$$\mu_q^{(1)} \geq c_q \cdot \min_{0\leq i\leq q-1} \{J_\varepsilon(x_{n-i})\},$$

where c_q is a constant depending only on q and is defined by (4.1). By Cauchy's interlacing inequalities we have

$$\lambda_n^{(n-q+1)} \geq \mu_q^{(1)}.$$

So we have

$$\lambda_n^{(n-q+1)} \geq c_q \cdot \min_{0\leq i\leq q-1}\{J_\varepsilon(x_{n-i})\}. \tag{4.12}$$

By Lemma 4.5 applied to $J_\varepsilon(x_{n-i})$ for $0 \leq i \leq q-1$, the statements of (i)–(iii) follow immediately from (4.12).

The statement of (iv) follows immediately from the lower bounds of (i)–(iii). □

REMARKS.

1. If $\varepsilon = 1$ and n is sufficiently large, then (4.11) becomes

$$\lambda_n^{(n-q+1)} \geq c_q \cdot \frac{x_{n-q+1} \cdot e^{-\gamma}}{\log x_{n-q+1}}\left(1 - \frac{C}{\log x_{n-q+1}}\right).$$

2. Mertens' theorem gives an asymptotic formula for the following product

$$\prod_{p\leq x}\left(1 - \frac{1}{p^\varepsilon}\right) \tag{4.13}$$

if $\varepsilon = 1$. We believe that there exists a similar but more complicated asymptotic formula for the product (4.13) for the case $0 < \varepsilon < 1$. Such asymptotic formula should give a more explicit lower bound for $J_\varepsilon(x)$ and hence for $\lambda_n^{(1)}$ and for $\lambda_n^{(n-q+1)}$ if $0 < \varepsilon < 1$.

5. Concluding remarks and questions. Let $\{x_i\}_{i=1}^\infty$ be an arbitrary strictly increasing infinite sequence of positive integers. For an integer $n \geq 1$, let $S_n = \{x_1, \ldots, x_n\}$. Let $0 < \varepsilon \leq 1$ and $q \geq 1$ a given integer. Let $\lambda_n^{(1)} \leq \cdots \leq \lambda_n^{(n)}$ be the eigenvalues of the power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set S_n. It follows from Theorem 2.5 that if for every $i \neq j$, $(x_i, x_j) = x_1$ and $\sum_{i=1}^\infty \frac{1}{x_i} = \infty$, then $\lim_{n\to\infty}\lambda_n^{(1)} = 0$. Then by Cauchy's interlacing inequalities and Proposition 1.1 we have that for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers which contains a subsequence $\{x'_i\}_{i=1}^\infty$ satisfying that for every $i \neq j$, $(x'_i, x'_j) = x'_1$ and $\sum_{i=1}^\infty \frac{1}{x'_i} = \infty$, $\lim_{n\to\infty}\lambda_n^{(1)} = 0$. On the other hand, by Theorem 3.4 we know that for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers containing the arithmetic progression $\{a + bi\}_{i=e}^\infty$ as its subsequence, $\lim_{n\to\infty}\lambda_n^{(q)} = 0$. First we would like to understand for what sequences $\{x_i\}_{i=1}^\infty$, $\lim_{n\to\infty}\lambda_n^{(1)} = 0$. Namely, we have the following question:

QUESTION 5.1. Characterize all strictly increasing infinite sequences $\{x_i\}_{i=1}^\infty$ of positive integers so that $\lim_{n\to\infty}\lambda_n^{(1)} = 0$, where $\lambda_n^{(1)}$ is the smallest eigenvalue of the power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set $\{x_1, \ldots, x_n\}$ and ε is a positive real number.

Consequently we propose a further problem.

QUESTION 5.2. Given any integer $q \geq 1$, characterize all strictly increasing infinite sequences $\{x_i\}_{i=1}^\infty$ of positive integers so that $\lim_{n\to\infty}\lambda_n^{(q)} = 0$, where $\lambda_n^{(q)}$ is the q-th smallest eigenvalue of the power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set $\{x_1, \ldots, x_n\}$ and ε is a positive real number.

Finally, we suggest a conjecture as the conclusion of this paper.

Conjecture 5.3. *Let $\varepsilon > 1$ and $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power GCD matrix $((x_i, x_j)^\varepsilon)$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. Then $\lim_{n\to\infty}\lambda_n^{(1)} > 0$.*

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