APPROXIMATIONS TO THE AREA OF AN n-DIMENSIONAL ELLIPSOID

D. H. LEHMER

1. Introduction. The fact that the perimeter $S(a, b)$ of an ellipse is not an elementary function of its semiaxes a, b has led to many suggested approximations of S in finite form. Many of these have been given as geometric constructions, but all may be reduced to algebraic formulas. Among the oldest and most familiar[1] are:

	Approximation	(λ, μ)
	$M = \pi(a + b)$	$(1, 0)$
	$R = \sqrt{2}\pi(a^2 + b^2)^{1/2}$	$(2, 0)$
(1)	$G = 2\pi(ab)^{1/2}$	$(1, 1)$
	$H_1 = 4\pi(ab)/(a + b) = G^2/M$	$(-1, 0)$
	$H_2 = \sqrt{8}\pi(ab)(a^2 + b^2)^{-1/2} = G^2/R$	$(-2, 0)$

and certain linear combinations of these such as

$$(M + R)/2, \ (3M - G)/2, \ldots .$$

It is clear that each of the simple approximations (1) seeks to replace the given ellipse by a circle of approximately the same circumference, whose radius r is an average of a and b of the type

$$r(a, b) = \{\tfrac{1}{2}(a^\lambda b^\mu + a^\mu b^\lambda)\}^q \qquad (q^{-1} = \lambda + \mu)$$

for the values of λ, μ indicated above. To compare approximations of this sort or their combinations one can expand each in powers of the eccentricity

$$\alpha = (1 - b^2 a^{-2})^{1/2} \qquad (a \geq b)$$

and test them against Legendre's exact expansion

$$S(a, b) = 2\pi a \left[1 - \frac{\alpha^2}{4} - \frac{3\alpha^4}{64} - \ldots - (2\nu - 1)^{-1}\binom{2\nu}{\nu}^2\left(\frac{\alpha}{4}\right)^{2\nu} - \ldots\right]. \tag{2}$$

The same questions have been raised by a number of authors about the surface area $S(a, b, c)$ of the ellipsoid with semiaxes a, b, c. Here familiar approximations are

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[1]The approximation M has been attributed to Bernoulli, but appears to be older. It is to be found in Kepler's astronomical notes. We have not attempted a complete search of the literature on this problem. However, a short bibliography is given at the end of the paper.

$$
\begin{array}{lrcll}
 & & & \text{Approximation} & (\lambda, \mu, \nu) \\
 & M & = & 4\pi[(a+b+c)/3]^2 & (1, 0, 0) \\
 & A & = & 4\pi(a^2+b^2+c^2)/3 & (2, 0, 0) \\
(3) & R & = & 4\pi[(b^2c^2+a^2c^2+a^2b^2)/3]^{1/2} & (2, 2, 0) \\
 & F & = & 4\pi(bc+ac+ab)/3 & (1, 1, 0) \\
 & G & = & 4\pi(abc)^{2/3} & (1, 1, 1)
\end{array}
$$

together with such linear combinations as Peano's

$$(4) \qquad (A+4F)/5$$

or Pólya's

$$(5) \qquad (64F-2A-27G)/35.$$

Each approximation (3) is to be interpreted as replacing the ellipsoid by a nearly equivalent sphere whose radius r is an average of a, b, c, of the type

$$r(a,b,c) = [\tfrac{1}{6}(a^\lambda b^\mu c^\nu + a^\lambda b^\nu c^\mu + a^\nu b^\lambda c^\mu + a^\nu b^\mu c^\lambda + a^\mu b^\lambda c^\nu + a^\mu b^\nu c^\lambda)]^q$$

with $q^{-1} = \lambda + \mu + \nu$, where (λ, μ, ν) are as indicated above. As in the case of the ellipse we may expand an approximation in what is now a double power series in the eccentricities

$$\alpha = (1 - b^2a^{-2})^{\frac{1}{2}}, \qquad \beta = (1 - c^2a^{-2})^{\frac{1}{2}},$$

and compare the expansions with that of S:

$$
\begin{aligned}
S(a,b,c) &= 4\pi ab\{1 - (\alpha^2+\beta^2)/6 - (3\alpha^4 + 2\alpha^2\beta^2 + 3\beta^4)/120 - \dots\} \\
(6) \qquad &= 4\pi ab \sum_{\nu=0}^{\infty} (1-4\nu^2)^{-1}(\alpha\beta)^\nu P_\nu[(\alpha^2+\beta^2)/2\alpha\beta],
\end{aligned}
$$

where $P_\nu(x)$ is Legendre's polynomial. In adopting such a comparison as a criterion of goodness we are tacitly assuming that α and β are small. An approximation which agrees with (6) as far as $\nu = k_1$ is said to be better than another which agrees only as far as $\nu = k_2$, where $k_2 < k_1$. In other words we are dealing with "nearly spherical" ellipsoids.

In general the $(n-1)$-dimensional area of the n-dimensional ellipsoid may be approximated by any one of a class (defined below) of appropriately chosen functions of the n semiaxes. In this paper we consider the problem of finding the best approximation of a given class, best in the above sense. An example of such a result, and apparently the only one of its kind in the literature, is the following theorem of Sir Thomas Muir.

THEOREM 1. *Of all expressions of the type*

$$2\pi[\tfrac{1}{2}(a^\lambda + b^\lambda)]^{1/\lambda},$$

that with $\lambda = 3/2$ *is the best approximation to the perimeter of the ellipse whose semiaxes are* a, b.

We consider approximations of the type

$$(7)\qquad P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Pi_n[r(\lambda_1, \lambda_2, \ldots, \lambda_n)]^{n-1},$$

where

$$\Pi_n = 2\pi^{n/2}/\Gamma(n/2)$$

is the area of the n-dimensional unit sphere and

$$r(\lambda_1, \lambda_2, \ldots, \lambda_n) = \left(\frac{1}{n!}\sum a_1^{\lambda_1}a_2^{\lambda_2}\ldots a_n^{\lambda_n}\right)^q,$$

in which $q^{-1} = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ and the sum extends over all $n!$ permutations of the λ's. If exactly k of the λ's are different from zero we say that P is of class k and dimension n. Muir's Theorem 1 is concerned with the case $n = 2$, $k = 1$. In § **3** we show that in general the best approximation of dimension n and class k is attained by certain algebraic numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$. These are given explicitly for $k = 1$ and $k = 2$. The case $k = n$ gives the best approximation of dimension n. Sections **8** and **9** are devoted to this case for $n = 2$ and $n = 3$. In considering ellipsoids of dimension $n \geq 3$ one discovers an exceptional class which we have called "well poised ellipsoids", whose areas are particularly amenable to approximations. These are given separate consideration in §§ **7**, **10** and **14**. In §§ **11-14** we consider the simpler approximations obtained from certain integral values of the λ's and the best linear combinations of such approximations.

The methods of this paper have been applied also to the question of approximating the electrostatic capacity of the ellipsoid. The same general results have been obtained. Another type of special ellipsoid presents itself in this case. A short account of these results may appear later.

2. Generalities. We consider the n-dimensional ellipsoid E_n whose Cartesian equation is

$$(8)\qquad \sum_{i=1}^{n} x_i^2 a_i^{-2} = 1 \qquad (a_1 \geq a_2 \geq \ldots \geq a_n > 0).$$

To express the $n - 1$ dimensional area S of E_n in terms of the $n - 1$ eccentricities α_i defined by

$$\alpha_i^2 = 1 - a_n^2 a_i^{-2} \qquad (i = 1, 2, \ldots, n-1)$$

so that $1 > \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} \geq 0$, we may proceed as follows. The "top" half of S is given by

$$(9)\qquad \tfrac{1}{2}S = \int\left\{1 + \sum_{i=1}^{n-1}\left(\frac{\partial x_n}{\partial x_i}\right)^2\right\}^{\frac{1}{2}} dx_1 \ldots dx_{n-1},$$

where the integration extends over the projection of E_n onto the coordinate hyperplane $x_n = 0$. Substituting $x_i = a_i y_i$ and

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$$x_n = \{ a_n^2 - \sum_{i=1}^{n-1} a_i^2(1 - a_i^2)y_i^2 \}^{\frac{1}{2}}$$

into (9) we find

$$(10) \quad S = 2a_1a_2 \ldots a_{n-1}\int_{S_{n-1}} (1 - \sum_{i=1}^{n-1} a_i^2y_i^2)^{\frac{1}{2}}(1 - \sum_{i=1}^{n-1} y_i^2)^{-\frac{1}{2}} dV_{n-1}$$

the integration extending over the interior of the unit $n - 1$ dimensional sphere. The first factor of the integrand may be replaced by its binomial series and the result integrated termwise. As a typical integral one obtains

$$I = I(m_1, m_2, \ldots, m_{n-1}) = \int_{S_{n-1}} y_1^{2m_1}y_2^{2m_2} \ldots y^{2m_{n-1}}(1 - \sum_{i=1}^{n-1} y_i^2)^{-\frac{1}{2}} dV_{n-1}.$$

This may be evaluated by introducing spherical coordinates $(\rho, \phi_1, \phi_2, \ldots, \phi_{n-2})$, where

$$0 \leq \rho \leq 1, \; 0 \leq \phi_i \leq \pi, \; 0 \leq \phi_{n-2} \leq 2\pi \qquad (i = 1, 2, \ldots, n - 3)$$

in terms of which

$$y_i = \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{n-i-1} \cos \phi_{n-i} \qquad (\phi_{n-1} \equiv 0)$$

The integral now becomes the product of $n - 1$ single integrals. Using repeatedly the Beta function formula

$$\int_0^{\pi} \sin^{2t}\phi \cos^{2u}\phi d\phi = \mathrm{B}(t + \tfrac{1}{2}, u + \tfrac{1}{2})$$

we find

$$I(m_1, \ldots, m_{n-1}) = \Gamma(\tfrac{1}{2}) [\prod_{i=1}^{n-1} \Gamma(m_i + \tfrac{1}{2})]/\Gamma (m_1 + m_2 + \ldots + m_{n-1} + \tfrac{1}{2} n)$$

$$= \frac{\pi^{n/2}}{\Gamma(^n/_2)} \frac{\prod_{i=1}^{n-1} \prod_{j=1}^{m-1} (1 + 2j)}{\prod_{n=0}^{\tau-1} (n + 2h)} \qquad (\tau = m_1 + m_2 + \ldots + m_{n-1}).$$

We can now write down the expansion of S in powers of α_i. Clearly, this is a symmetric function of the α's. For brevity we write these symmetric functions as follows

$$\Sigma\alpha_i^2 = S_1, \quad \Sigma\alpha_i^4\alpha_j^2 = S_{21}, \ldots.$$

The expansion of (10) thus becomes

$$S = \Pi_n a_1 \ldots a_{n-1} \{1 - S_1/2n - (3S_2 + 2S_{11})/8n(n + 2)$$
$$- (15S_3 + 9S_{21} + 6S_{111})/16n(n + 2)(n + 4)$$
$$(11) \quad -5(105S_4 + 60S_{31} + 54S_{22} + 36S_{211} + 24S_{1111})/128n(n+2)(n+4)(n+6)$$
$$- \ldots + 2^{-m} \prod_{h=0}^{m-1} \frac{2h - 1}{n + 2h} \sum_{(\mu)} \left(\prod_{\nu=1}^{n-1} \prod_{k=0}^{\mu_\nu - 1} \frac{1 + 2k}{1 + k} \right) S_{\mu_1\mu_2 \ldots \mu_{n-1}} + \ldots \}$$

where the sum over μ extends to all partitions of m:

$$m = \mu_1 + \mu_2 + \ldots + \mu_{n-1}$$

into $n-1$ non-negative integral parts. This expansion can be given in a somewhat less explicit form similar to (6). Let $R_m(z_1, z_2, \ldots z_{n-1})$ be the polynomial which is the coefficient of u^m in the generating function

$$\{(1 - z_1 u)(1 - z_2 u) \ldots (1 - z_{n-1} u)\}^{-\frac{1}{2}}$$

Then (11) can be written

$$S = \Pi_n a_1 \ldots a_{n-1} \sum_{m=0}^{\infty} \left(\prod_{h=0}^{m-1} \frac{2h-1}{2h+n} \right) R_m(\alpha_1^2, \alpha_2^2, \ldots, \alpha_{n-1}^2).$$

3. The function $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$. We consider now our approximating function (7). For $n = 3$, $P(\lambda_1, \lambda_2, \lambda_3)$ is a slight generalization of the function used by Pólya who took $q = 1/2$ and $\lambda_i \geq 0$. In general P shares with the area S three of the four properties noted by Pólya, namely P is

(i) Homogeneous of degree $n - 1$,

(ii) Symmetric in the a's,

(iii) Precisely equal to S when all the a's are equal.

Since the λ's are not restricted to be non-negative, $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ may not be continuous at $a_n = 0$. However, since in this paper we are not concerned with such degenerate ellipsoids this possible lack of continuity is not important. From a "practical" point of view it would be desirable to have the λ's real. However, some or even all of the λ's may be complex and yet, if they occur in conjugate pairs, the function P has a real interpretation closely approximating the area S of the "nearly spherical ellipsoids". In deference to the practical minded reader we call approximations by non-real λ's "improper". Since the λ's enter symmetrically, it is convenient to introduce their elementary symmetric functions

$$p_1 = q^{-1} = \Sigma\lambda_i, \quad p_2 = \Sigma\lambda_i\lambda_j, \quad p_3 = \Sigma\lambda_i\lambda_j\lambda_k, \ldots.$$

To expand P in powers of the eccentricities α_i we write

$$(1 - \alpha_i^2)^{\frac{1}{2}} = \beta_i,$$

so that

$$a_i = a_n \beta_i^{-1} \qquad (i = 1, 2, \ldots, n; \beta_n = 1).$$

On account of the homogeneity of P in the a_i we have

$$P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \Pi_n a_n^{n-1} \left\{ \frac{1}{n!} \sum \beta_1^{-\lambda_1} \beta_2^{-\lambda_2} \ldots \beta_n^{-\lambda_n} \right\}^{(n-1)q}$$

$$= \Pi_n (a_1 a_2 \ldots a_{n-1})(\beta_1 \beta_2 \ldots \beta_n) \left\{ \frac{1}{n!} \sum \beta_1^{-\lambda_1} \ldots \beta_n^{-\lambda_n} \right\}^{(n-1)q}$$

Now the factor

$$\beta_1 \ldots \beta_n = \prod_{i=1}^{n-1} (1 - \alpha_i^2)^{\frac{1}{2}} = 1 + Q_1(\alpha_1^2, \alpha_2^2, \ldots \alpha_{n-1}^2)$$

where Q_1 is an $(n - 1)$ fold power series in $\alpha_1^2, \alpha_2^2, \ldots \alpha_{n-1}^2$, which, being symmetric in the α's and of course absolutely convergent inside the unit sphere, may be expanded in terms of the monomial symmetric functions of the α's, thus

$$Q_1 = -S_1/2 - S_2/8 + S_{11}/4 + \ldots .$$

Similarly the function

$$\frac{1}{n!} \sum \beta_1^{-\lambda_1} \ldots \beta_n^{-\lambda_n} = 1 + p_1 S_1/n + (p_1^2 + 2p_1 - 2p_2) S_2/8n + p_2 S_{11}/2n(n-1)$$
$$= 1 + Q_2(\alpha_1^2, \alpha_2^2, \ldots, \alpha_{n-1}^2)$$

may be expanded in terms of the symmetric function of the α's with coefficients which are polynomials in $p_1, p_2, \ldots$ with rational coefficients. The coefficients in the expansion of

$$(\Pi_n a_1 a_2 \ldots a_n)^{-1} P(\lambda_1, \lambda_2, \ldots, \lambda_n) = (1 + Q_1)(1 + Q_2)^{(n-1)q}$$

are rational functions of $p_1, p_2, \ldots$ (whose denominators are polynomials in p_1 alone) with integer coefficients. Finally we see from (11) that the relative error

$$\frac{P - S}{S} = c_1 S_1 + c_2 S_2 + c_{11} S_{11} + \ldots \tag{12}$$

has coefficients c of the same type. The problem of determining λ's so that a specified number of these c's vanish is thus a purely algebraic one, and the best approximation $P(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0 \ldots, 0)$ of class r and dimension n is attained with λ's which are algebraic numbers. The expansion (12) may be developed quite in general, that is for a general n and arbitrary λ_i, and begins as follows:

$$\frac{P - S}{S} = \psi_1(f_4 + f_6)/8n^2(n+2) + (3p_1\psi_1 - \psi_2 + \psi_3)g_6/96n^3(n+2) + O(\alpha^8) \tag{13}$$

where the ψ's are independent of the α's and are given by:

$$\psi_1 = (n-1)(n+2)p_1 - 2n(n+2)p_2/p_1 - 2(n+1),$$
$$\psi_2 = (n-1)(n+2)p_1^2 - 6(n+1)p_1 - 8(n+1)(n+2)/(n+4),$$
$$\psi_3 = 6n^2(n+2)p_3/[(n-2)p_1)],$$

and the functions f_4, f_6, g_6 depend only on the α's and are given by:

$$f_4 = (n-1)S_2 - 2S_{11},$$
$$f_6 = (n-1)S_3 - 2S_{21},$$
$$g_6 = (n-1)(n-2)S_3 - 3(n-2)S_{21} + 12S_{111}.$$

If, in (13), a certain choice of λ's results in the vanishing of all terms involving symmetric functions of the α_i of weights $< d$ while the terms of weight d do not vanish, the approximation P is said to be of order d. Inspection of (13) yields

THEOREM 2. *All approximations* (7) *are of order* $d \geqslant 4$.

4. Approximations of Class 1. As a generalization of Theorem 1 we have

THEOREM 3. *Of all approximations of the type*

$$P(\lambda) = \Pi_n\{(a_1^\lambda + a_2^\lambda + \ldots + a_n^\lambda)/n\}^{(n-1)/\lambda} \tag{14}$$

that with

$$\lambda = 2(n+1)/[(n-1)(n+2)] \tag{15}$$

gives the best approximation to the area of the n-dimensional ellipsoid.

Proof. In this case we are considering approximations of class 1, so that

$$p_1 = \lambda,\ p_2 = p_3 = \ldots = 0.$$

Hence in (13)

$$\psi_1 = (n-1)(n+2)\,\lambda - 2(n+1).$$

Since, as we shall see later, $f_4 + f_6 > 0$, we must set $\psi_1 = 0$ to obtain the best approximation of type (14). This gives the theorem.

The approximation (14) is in general of order 6. In fact we have only to note that in this case $\psi_1 = 0$ and $\psi_3 = 0$ in (13). This gives

$$\begin{aligned} \frac{P-S}{S} &= -[(n-1)(n+2)\lambda^2 - 6(n+1)\lambda \\ &\quad - 8(n+1)(n+2)/(n+4)]g_6/[96n^3(n+2)] + \ldots \\ &= (n+1)(n^2+4n+5)g_6/[12(n-1)n^2(n+2)^2(n+4)] \end{aligned} \tag{16}$$

in view of (15). In case $n = 2$, the function g_6 vanishes identically. We shall see in § 8 that the approximation

$$P(3/2) = 2\pi[\tfrac{1}{2}(a_1^{3/2} + a_2^{3/2})]^{2/3}$$

is of order 8.

In case $n = 3$, $\lambda = 4/5$ and the relative error of the best approximation

$$P(4/5) = 4\pi[\tfrac{1}{3}(a_1^{4/5} + a_2^{4/5} + a_3^{4/5})]^{5/2} \tag{17}$$

of class 1 for the common ellipsoid is, by (14),

$$-13g_6/4725 = -13(\alpha_1^2 + \alpha_2^2)(\alpha_1^2 - 2\alpha_2^2)(2\alpha_1^2 - \alpha_2^2)/4725 + \ldots . \tag{18}$$

In case $n = 4$, the best approximation of class 1 is

$$P(5/9) = 2\pi^2[\tfrac{1}{4}(a_1^{5/9} + a_2^{5/9} + a_3^{5/9} + a_4^{5/9})]^{27/5} \tag{19}$$

and its relative error is by (16)

$$-185g_6/27648 = 185(\alpha_1^2 + \alpha_2^2 - \alpha_3^2)(\alpha_1^2 - \alpha_2^2 + \alpha_3^2)(-\alpha_1^2 + \alpha_2^2 + \alpha_3^2)/4608. \tag{20}$$

5. Approximations of Class 2. The next problem is that of determining two parameters λ and μ so that the approximation

$$(21)\qquad P(\lambda, \mu) = \Pi_n \left(\frac{1}{n(n-1)} \sum_{i \neq j} a_i^{\lambda} a_j^{\mu} \right)^{(n-1)/(\lambda+\mu)}$$

is the best possible. With two parameters at our disposal we may arrange to make both ψ_1 and ψ_2 of (13) vanish. Since $p_3 = 0$, ψ_3 vanishes also so that we have an approximation of order ≥ 8. Setting $\psi_2 = 0$ we find

$$(n-1)(n+2)(n+4)\, p_1 = 3(n+1)(n+4) \pm \Delta^{\frac{1}{2}}$$

where

$$\Delta = (n+1)(n+4)(8n^3 + 33n^2 + 45n + 4).$$

Setting $\psi_1 = 0$, gives

$$n(n+2)(n+4)\, p_2 = 2\, p_1(n+1)(n+4) + 4(n+1)(n+2).$$

Since $\Delta > 0$, it follows that (λ, μ) are either both real or conjugate complex. Their discriminant however is proportional to

$$n(n-1)^2(n+2)^2(n+4)(p_2^2 - 4p_2) = -2(n+1)(A \pm B\Delta^{\frac{1}{2}}),$$

where

$$A = 4n^4 + 7n^3 - 21n^2 - 64n - 16, \qquad B = n - 4.$$

Moreover

$$A^2 - B^2\Delta = 16n(n-1)^2(n+2)^2(n^3 + n^2 - 7n - 16) > 0$$

for $n \geq 4$. Therefore we have

THEOREM 4. *The best approximation of type* (21) *is improper for* $n > 3$.

For $n = 3$ one of the two possible pairs (λ, μ) is real and the other complex. All four satisfy

$$3675\, x^4 - 8820\, x^3 + 1904\, x^2 + 2240\, x + 3328 = 0.$$

The real pair is approximately

$$(22)\qquad \begin{aligned} \lambda &= 1.70002966918802203200, \\ \mu &= 1.43018127413249642468. \end{aligned}$$

The other pair gives not as good an approximation, and besides is improper.

For $n = 2$, the function ψ_1 vanishes for

$$(23)\qquad 8p_2 = 2p_1^2 - 3p_1.$$

This condition alone assures an approximation of order 8 since g_6 vanishes identically. To get the best approximation of class 2 a further study of the case $n = 2$ is required (see § 8).

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6. Approximations of Class 3. If, for $n > 2$, we seek the best choice of the three parameters λ, μ, ν, then (13) tells us that we can obtain approximations of order ≥ 8 by choosing $p_1 \neq 0$ arbitrarily and then determining p_2 and p_3 by

$$(24) \qquad 2n(n+2)p_2 = (n-1)(n+2)p_1^2 - 2(n+1)p_1$$

and

$$(25) \qquad 6n^2(n+2)(n+4)p_3 = (n-2)[(n-1)(n+2)(n+4)p_1^3 - 6(n+1)(n+4)p_1^2 - 8(n-1)(n+2)p_1]$$

A further expansion of the error $(P - S)/S$ would be necessary to obtain, in the general case, the best choice of p_1. This is done for $n = 3$ in § **9.**

7. Special Ellipsoids. Before proceeding to consider further details for $n = 2$ and $n = 3$ we call attention to a special class of ellipsoids whose approximations by functions of the type we have been considering, or indeed by any combination of them, are particularly close.

In the preceding discussion of the error expansion (13) we have tacitly assumed that the functions f_4 f_6, and g_6 (which depend only on the shape of the ellipsoid) are different from zero. This assumption, which is of course true of ellipsoids in general, is strictly true for the functions f_4 and f_6. In fact these functions may be written:

$$f_4 = \Sigma(\alpha_i^2 - \alpha_j^2)^2,$$

$$f_6 = \Sigma(\alpha_i^2 + \alpha_j^2)(\alpha_i^2 - \alpha_j^2)^2,$$

where the sums extend over all integers $i < j \leq n$, and we have taken α_n to be zero. Hence for real, non-spherical ellipsoids these functions are strictly positive. Therefore the condition $\psi_1 = 0$ is necessary and sufficient for the approximation P to be of order ≥ 6.

In the case of g_6, however, there are real non-spherical ellipsoids for which $g_6 = 0$ and for these ellipsoids the vanishing of the coefficient of g_6 in (13) is sufficient but not necessary for the approximation to be of order ≥ 8. A conspicuous class of such special ellipsoids consists of those whose Cartesian equations

$$A_1x_1^2 + A_2x_2^2 + \ldots + A_nx_n^2 = 1$$

are such that

$$(26) \qquad A_1 + A_n = A_2 + A_{n-1} = \ldots = A_n + A_1.$$

As far as the author is aware such ellipsoids have not received attention. In what follows such an ellipsoid will be called "well poised". In terms of the semiaxes a_i an ellipsoid is well poised in case the harmonic mean of a_i^2 and a_{n+1-i}^2 is the same for all i. In terms of its eccentricities α_i an ellipsoid is well poised in case the arithmetic mean of α_i^2 and α_{n+1-i}^2 is the same for all i. For

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$n = 2$, we have no condition, and all ellipses are well poised. For $n = 3$ there is a one parameter family of ellipsoids which are well poised. These are characterized by

$$a_1^2 = 2a_2^2.$$

For $n = 4$, the well poised ellipsoids are characterized by

$$a_1^2 = a_2^2 + a_3^2.$$

By (18) and (20) we have

THEOREM 5. *If an ellipsoid is well poised, its function g_6 vanishes.*

Proof. The function $4g_6$ may be written

$$4g_6 = 4n^2S_3 - 12nS_1S_2 + 8S_1^3. \tag{27}$$

Now we can write

$$a_i^2 + a_{n+1-i}^2 = K$$

and raising both sides to the first, second and third powers and, summing over i from l to n, we obtain:

$$\begin{aligned} 2S_1 &= nK \\ 2S_2 &= nK^2 - 2\sigma, \\ 2S_3 &= nK^3 - 3K\sigma, \end{aligned} \tag{28}$$

where

$$\sigma = \sum_{i=1}^{n} a_i^2 a_{n+1-i}^2.$$

Substituting from (28) into (27) we find that g_6 vanishes identically in K and σ. This proves the theorem.

In the case of well poised ellipsoids a separate discussion of best possible approximations and their errors is required. We proceed to consider further details for $n = 2$ and $n = 3$.

8. The case $n = 2$. In the case of the ellipse there remain two questions (a) What is the order and error of Muir's approximation? and (b) what is the best of all approximations of type (7) when $n = 2$? As a matter of fact Muir showed that his approximation

$$P(3/2) = 2\pi[\tfrac{1}{2}(a^{3/2} + b^{3/2})]^{2/3} \tag{29}$$

is of order 8. The expansion of $P(3/2)$ in powers of $\alpha^2 = 1 - b^2a^{-2}$ yields

$$P(3/2) = 2\pi a[1 - \alpha^2/4 - 3\alpha^4/2^6 - 5\alpha^6/2^8 - 11\alpha^8/2^{10} - 7\alpha^{10}/2^{10} - \dots].$$

Comparing this with (2) we find the relative error

$$(P - S)/S = -\alpha^8/2^{14} - \alpha^{10}/2^{13} - \dots. \tag{30}$$

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It is difficult to appreciate the smallness of this error. Applied to the earth's orbit for example the very simple formula (29) differs from the true perimeter of the ellipse by less than a wave length of visible light.

The best approximation to the perimeter is of class 2

$$P(\lambda, \mu) = 2\pi[\tfrac{1}{2}(a^\lambda b^\mu + a^\mu b^\lambda)]^{1/p_1}. \tag{31}$$

Making use of (23) all symmetric functions of λ and μ may be expressed as polynomials in $p_1 = \lambda + \mu$. Actually carrying out the expansion of $P(\lambda, \mu)$ in powers of α^2 we find

$$P(\lambda, \mu) = 2\pi a\{\tfrac{1}{2}[(1 - \alpha^2)^{\lambda/2} + (1 - \alpha^2)^{\mu/2}]\}^{1/p_1}$$
$$= 2\pi a\left(1 - \frac{\alpha^2}{4} - \frac{3\alpha^4}{2^6} - \frac{5\alpha^6}{2^8} - \frac{(6p_1 + 79)\alpha^8}{2^{13}} - \frac{(42p_1 + 161)\alpha^{10}}{2^{15}} - \dots\right).$$

Comparing this with the actual expansion (2) we find

$$2^{16}[P(\lambda, \mu) - S] = -2\pi a(12p_1 - 17)(4\alpha^8 + 7\alpha^{10}) + O(\alpha^{12}).$$

To obtain the best choice of λ, μ we must choose $p_1 = 17/12$. By (23) this implies $p_2 = -17/576$. Hence λ, μ are the roots of

$$576x^2 - 816x - 17 = 0.$$

That is

$$\lambda = [17 + 3(34)^{\frac{1}{2}}]/24 = 1.4372023201889958\,92,$$
$$\mu = [17 - 3(34)^{\frac{1}{2}}]/24 = -0.0205356535223292\,25.$$

With this best choice the approximation (31) is actually of order 12. In fact one finds the relative error is only

$$[P(\lambda, \mu) - S]/S = 2^{-19}\alpha^{12} + O(\alpha^{14}).$$

This error is smaller than (30) by a factor of approximately $-\alpha^4/32$.

9. Best approximations for $n = 3$. We consider here the general three dimensional ellipsoid and approximations of class 3. According to (24) and (25) for $n = 3$ we would choose p_2 and p_3 in terms of $p_1 = \lambda + \mu + \nu$ so that

$$15p_2 = 5p_1^2 - 4p_1, \tag{32}$$

$$945p_3 = 35p_1^3 - 84p_1^2 - 80p_1. \tag{33}$$

Expanding $P(\lambda, \mu, \nu)$ in terms of α^2 and β^2 under the assumptions (32) and (33) and comparing the results with the exact expansion (6) we find that

$$[P(\lambda, \mu, \nu) - S]/S = -(7p_1 - 23)(\alpha^4 - \alpha^2\beta^2 + \beta^4)^2/28350.$$

Hence we should choose $p_1 = 23/7$ and in view of (32) and (33)

$$p_2 = 667/245 \text{ and } p_3 = 391/5145.$$

Therefore λ, μ, ν are roots of the cubic equation

$$5145x^3 - 16905x^2 + 14007x - 391 = 0,$$

so that approximately

$$\begin{aligned} \lambda &= 1.7813912991422280387, \\ \mu &= 1.4754082089850733373, \\ \nu &= 0.2891477758693195 45. \end{aligned}$$

A recomputation of the relative error, using the above values of p_1, p_2 and p_3 shows that this best approximation is of order 10 and that

$$(P - S)/S = (\alpha^6 + \beta^6)(\alpha^2 - 2\beta^2)(2\alpha^2 - \beta^2)/72765 + O(\alpha^{12}).$$

When applied to the earth's surface this gives an error of about 30 square inches.

10. Well poised ellipsoids for $n = 3$. In the previous section we have discussed 3-dimensional ellipsoids in general. In the special case of well poised ellipsoids the condition (33) is not necessary since $g_6 = 0$. For these ellipsoids

$$\alpha^2 = 2\beta^2.$$

Under these assumptions $P(\lambda, \mu, \nu)$ becomes a function of α alone. In this case the expansion of the error proceeds as follows (assuming that $\psi_1 = 0$):

$$(34)\quad [P(\lambda, \mu, \nu) - S]/S = [4725p_3/p_1 - (175p_1^2 - 364p_1 - 584)]\alpha^8/403200 + \dots .$$

Thus for approximations of class 1, (17) is still best but the error in this case is only

$$[P(4/5) - S]/S = 53\alpha^8/2800 + \dots .$$

For approximations of class 2, the values (22) no longer are best. Instead (34) shows that we should choose p_1 so that

$$175p_1^2 - 364p_1 - 584 = 0.$$

The condition $\psi_1 = 0$ then gives

$$525p_2 = 224p_1 + 584.$$

This in turn implies that λ, μ should be the two real roots of the quartic equation

$$153125x^4 - 318500x^3 - 34440x^2 + 81760x + 247616 = 0.$$

Approximately, these roots are:

$$(35)\qquad \begin{aligned} \lambda &= 1.6941890650239756 3155, \\ \mu &= 1.4478915369275326 8049, \end{aligned}$$

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and they differ slightly from the pair (22). The error involved in this approximation is of order 12 and is in fact

$$(P - S)/S = [15573634069 + 613132377\sqrt{3759}]\alpha^{12}\, 2^9\, 3^4\, 5^{11}\, 7^3\, 13.$$

For approximations of class 3 according to (34) we should choose p_3 such that

$$4725p_3 = 175p_1^3 - 364p_1^2 - 584p_1. \tag{36}$$

In this case the error takes the form

$$(P - S)/S = -[7007p_1^3 + 1710709p_1^2 - 14403103p_1 + 12494475]\alpha^{12}/2^9 3^5 5^5 7^2 11\cdot 13.$$

Hence for the best approximation possible we should choose for p_1 a root of

$$7007x^3 + 1710709x^2 - 14403103x + 12494475 = 0.$$

These roots are approximately:

$$r_1 = 0.98262974495222543686,$$
$$r_2 = 7.19198324487317717019,$$
$$r_3 = -252.31747013268253999959.$$

Each root leads to a choice of p_1, p_2, p_3 and hence to a set of (λ, μ, ν). These latter are all real only when $p_1 = r_2$. Then we get the approximate values:

$$\begin{aligned} \lambda &= 3.70372956924493413068, \\ \mu &= 2.54278064910480668412, \\ \nu &= 0.94547302652343088710, \end{aligned} \tag{37}$$

which represent algebraic numbers of degree 9. With these λ, μ, ν the function $P(\lambda, \mu, \nu)$ is an approximation of order ≥ 14 for all well poised ellipsoids. There are reasons to believe that the order is actually 16.

11. Simple approximations and their combinations. Thus far our insistance upon the most accurate approximation has led us in most cases to irrational λ's and to approximations P whose applications are somewhat laborious. It is clear that if the λ's are simply integers the resulting loss of accuracy is to a certain degree made up by a gain in practical simplicity of application. By taking linear combinations of two or more such simple approximations one may regain lost accuracy. One may determine the best linear combination of a given set of P's by using the general formula (13). It is clear that the same criterion of goodness, the same functions f_4, f_6 and g_6, and the same special cases of well poised ellipsoids will occur as before in dealing with such questions. To illustrate these remarks we consider the four simple approximations:

$$
(38)\quad
\begin{aligned}
A &= P(2, 0, 0, \ldots, 0) = \Pi_n\left\{\frac{1}{n}\Sigma a_i^2\right\}^{(n-1)/2}\\
B &= P(2, 2, \ldots, 2, 0) = \Pi_n a_1a_2\ldots a_n\left\{\frac{1}{n}\Sigma a_i^{-2}\right\}^{\frac{1}{2}}\\
F &= P(1, 1, \ldots, 1, 0) = \Pi_n a_1a_2\ldots a_n n^{-1}\Sigma a_i^{-1},\\
G &= P(1, 1, \ldots, 1, 1) = \Pi_n\{a_1a_2\ldots a_n\}^{(n-1)/n}.
\end{aligned}
$$

The first and last of these are the areas of the hyperspheres whose radii are respectively the root mean square and the geometric mean of the semiaxes of the ellipsoid, the approximations B and F are the areas of the hyperspheres whose equatorial sections are the arithmetic mean and the root mean square respectively of the n principal sections of the ellipsoid.

According to (13) we have as relative errors of A, B, F and G:

$$
\begin{aligned}
(A - S)/S &= (n^2-3)(f_4+f_6)/[4n^2(n+2)]+(n^2+4n-3)g_6/[12n^3(n+4)],\\
(B - S)/S &= (f_4+f_6)/[4n^2(n+2)] - g_6/[4n^3(n+4)],\\
(G - S)/S &= -(n+1)(f_4+f_6)/[4n^2(n+2)] + (n+1)g_6/[12n^3(n+4)],\\
(F - S)/S &= -(f_4+f_6)/[8n(n+2)] + g_6/[16n^2(n+4)].
\end{aligned}
$$

For $n > 2$ it is seen that B is better than F, F better than G, and G better than A. The best combinations of A, B, F, G, two at a time arranged (for $n > 4$) in order of increasing accuracy[2] follows.

	Approximation	Relative Error
(a)	$[2(n+1)F-nG]/(n+2)$	$(n+1)g_6/[24n^2(n+4)]$
(b)	$[(n+1)A+(n^2-3)G]/[(n-1)(n+2)]$	$(n+1)(n+3)g_6/[6n^3(n+2)(n+4)]$
(c)	$[nA+2(n^2-3)F]/[(2n-3)(n+2)]$	$(n+1)(5n-6)g_6/[24n^2(n+2)(n+4)(2n-3)]$
(d)	$[(n^2-3)B-A]/[(n-2)(n+2)]$	$(n^2+n-3)g_6/[3n^3(n-2)(n+2)(n+4)]$
(e)	$[(n+1)B+G]/(n+2)$	$-(n+1)g_6/[6n^3(n+2)(n+4)]$
(f)	$(nB+2F)/(n+2)$	$-g_6/[8n^2(n+2)(n+4)]$

The best combinations three at a time are:

$$
(39)\quad
\begin{aligned}
&[n(n+1)B+8(n+1)F-3nG]/(n+2)(n+4)],\\
&[8(n-1)(n+1)(n+3)F-n(5n^2+8n-15)G-n(n+1)A]/[3(n-1)(n+2)(n+4)],\\
&[(n-1)(n+1)(n+3)B-(n+1)A-2(n^2+n-3)G]/[n-1)(n+2)(n+4)],\\
&[3nA+n(5n^2+8n-15)B+16(n^2+n-3)F]/[5n-6)(n+2)(n+4)].
\end{aligned}
$$

[2]For $n = 3$, the order of increasing accuracy is (b), (d), (a), (c), (e), (f). For $n = 4$, the order is (b), (a), (d), (c), (e), (f).

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These approximations are all of order ≥ 8. Their errors are not obtainable from (13). We proceed to discuss the cases $n = 2$ and $n = 3$ more fully.

11. Combinations for the ellipse. When $n = 2$ we have somewhat different results. In the first place B and A coincide and become what we have called R in (1). Also F becomes M in (1). The relative errors for each of the functions R, M, and G are:

$$(R - S)/S = (\alpha^4 + \alpha^6)/2^6 + 211\,\alpha^8/2^{14} + \ldots ,$$
$$(M - S)/S = -(\alpha^4 + \alpha^6)/2^6 - 221\,\alpha^8/2^{14} + \ldots ,$$
$$(G - S)/S = -3\{(\alpha^4 + \alpha^6)/2^6 + 223\,\alpha^8/2^{14} + \ldots\}.$$

Hence R is slightly better than M and both are very nearly three times as good as G. The best combinations two at a time are:

$$(3M - G)/2, \qquad (R + M)/2, \qquad (3R + G)/4,$$

whose relative errors are respectively (neglecting terms of higher order)

$$3\alpha^8/2^{14}, \qquad - 5\alpha^8/2^{14}, \qquad 9\alpha^8/2^{14}.$$

The best of these is only one third as good as Muir's approximation (29).

The best approximation obtainable from all three is

$$(3R + 18M - 5G)/16,$$

whose relative error is only $- 77\alpha^{10}/2^{16}$.

13. Combinations for $n = 3$. For the ordinary ellipsoid the combinations three at a time (39) become the following:

APPROXIMATION	RELATIVE ERROR
$(A + 18B + 16G)/35$	$f_{10}/23760$
$(- 2A + 64F - 27G)/35$	$f_{10}/41580$
$(2A + 24B + 9G)/35$	$f_{10}/45360$
$(12B + 32F - 9G)/35$	$f_{10}/49896$

where the function of f_{10} is defined by

$$f_{10} = (\alpha^2 + \beta^2)(\alpha^4 - \alpha^2\beta^2 + \beta^4)(2\alpha^2 - \beta^2)(\alpha^2 - 2\beta^2) = f_4 f_6.$$

The second of these is Pólya's (5). The best combination of all four is

$$-(10A - 72B - 512F + 189G)/385,$$

with a relative error of

$$- f_{12}/2432430 + \ldots,$$

where

$$f_{12} = 7(\alpha^{12} + \beta^{12}) - 21\alpha^2\beta^2(\alpha^8 + \beta^8) + 15\alpha^4\beta^4(\alpha^4 + \beta^4) + 5\alpha^6\beta^6.$$

14. The case of well poised ellipsoids ($n = 3$). The results of § **13** are based on the assumption that $g_6 \neq 0$. For well poised ellipsoids the following results may be tabulated. It will be recalled that $\alpha^2 = 2\beta^2$. The principal parts of the relative errors are given. The table is arranged in order of increasing accuracy.

Approximation	Relative Error
A	$\alpha^4/20$
G	$-\alpha^4/30$
F	$-\alpha^4/80$
B	$\alpha^4/120$
$(2A + 3G)/5$	$\alpha^8/336$
$(6G - A)/5$	$\alpha^8/448$
$(A + 4F)/5$	$3\alpha^8/1792$
$(4B + G)/5$	$-\alpha^8/672$
$(8F - 3G)/5$	$\alpha^8/2688$
$(2F + 3B)/5$	$-\alpha^8/3584$
$(2A + 24B + 9G)/35$	$179\alpha^{12}/5765760$
$(A + 18B + 16F)/35$	$207\alpha^{12}/10250240$
$(64F + 2A - 27G)/35$	$-19\alpha^{12}/1537536$
$(12B + 32F - 9G)/35$	$-431\alpha^{12}/46126080$
$(6840B - 862A + 45824F - 16767G)/35035$	$-4649\alpha^{16}/9758228480$

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University of California