

LIMIT THEOREMS FOR STOCHASTIC DIFFERENCE-DIFFERENTIAL EQUATIONS

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1. Introduction

There are extensive works on the limit theorems for sequences of stochastic ordinary differential equations written in the form:

$$(1.1) \quad \frac{d\varphi_t}{dt} = f_t^n(\varphi_t) + \bar{g}_t^n(\varphi_t),$$

where $f_t^n = f_t^n(x)$, $t \geq 0$ is a stochastic process and $\bar{g}_t^n = \bar{g}_t^n(x)$, $t \geq 0$ is a deterministic function, both of which take values in the space of vector fields. The case where $\{f_t^n\}_n$ satisfies certain mixing conditions has been studied by Khas'minskii [7], Kesten-Papanicolaou [6] and others. The limit process is characterized as a diffusion process governed by a stochastic differential equation based on a Brownian motion. Further, the approximation theorem of stochastic differential equation studied by Wong-Zakai [18], Ikeda-Watanabe [4] etc. is also formulated in this way. A unified method of treating these problems was proposed by Kunita [9].

On the other hand, a lot of attention has also been shown to the discrete time approximation of stochastic differential equations. Approximating sequence of equations is written as

$$(1.2) \quad \varphi_{k+1} = \varphi_k + f_k^n(\varphi_k)\xi_k^n + \bar{g}_k^n(\varphi_k), \quad k = 1, 2, \dots$$

where $\{\xi_k^n\}$ is an array of random variables with certain mixing conditions and $\{f_k^n, \bar{g}_k^n\}_n$ is an array of continuous maps of the state space into itself. See Kushner [13], H. Watanabe [17] and Fujiwara [2]. The limit process is either a diffusion process mentioned above or a diffusion process with jumps governed by a stochastic differential equation based on a Lévy process.

In this paper we will present a unified method which is applicable both to stochastic ordinary differential equation (1.1) and to stochastic difference

equation (1.2). Equations are written as stochastic difference-differential equations of the form:

$$(1.3) \quad \varphi_t^n = x_0 + \int_0^t \{f_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n)\} dA_u^n.$$

Here $\{f_u^n\}$ is a sequence of vector field valued stochastic processes and $\{\bar{g}_u^n\}$ is a sequence of deterministic vector field valued functions of u . $\{A_u^n\}$ is a sequence of deterministic nondecreasing càdlàg (right continuous with left hand limits) functions of u .

Assumptions required for equations (1.3) and techniques employed for the proof of the limit theorem, are closely related to those in [9], where stochastic ordinary differential equations are discussed. However there are some important differences. In this paper, we get a sharper estimate of the solution. This enabled us to obtain a better result than authors' previous works. Another different point in this paper is that we introduced Lévy measures so that the limit process can have jumps and can be characterized as a Markov process associated with Lévy's infinitesimal generator.

The organization of this paper is as follows. In the next section we state our main theorem (Theorem 2.1). Assumptions on equations are presented in an abstract manner. They appear to be technical and complicated. A reason why we presented the theorem in this form is that we want to apply the theorem to various types of limit theorems. The theorem will be proved at Section 3. Since the proof is long, it will be divided into four subsections.

In the last section we will apply the theorem to stochastic ordinary differential equations. Two cases will be discussed separately. The first is the case where certain uniform mixing (or ϕ -mixing) conditions are satisfied (Theorem 4.2). The second is the case where certain strong mixing conditions are satisfied (Theorem 4.6). Our typical results are Corollaries 4.5 and 4.8. There we discuss the weak convergence of solutions of equations

$$(1.4) \quad \frac{d\varphi_t}{dt} = \sqrt{n}f(\varphi_t, nt),$$

where $f(x, t) = (f^1(x, t), \dots, f^d(x, t))$ is a stationary process with mean 0. We show that the sequence of solutions $\{\varphi^n\}$ converges in law to a diffusion process if the following (a) and (b) are satisfied:

- (a): $E[\sup_{|\alpha| \leq N} |\partial_x^\alpha f(x, 0)|^2] < \infty$ for all $N > 0$ and $|\alpha| \leq 2$.
- (b): The uniform mixing rate $\phi(t)$ satisfies $\int_0^\infty \phi(s)^{1/2} ds < \infty$,

or if the following (a)' $_\delta$ and (b)' $_\gamma$ are satisfied for some $\delta \in (0, \infty)$ and $\gamma \in (0, \delta/2(2 + \delta)(1 + d))$.

- (a)' $_{\delta}$: $E[\sup_{|x| \leq N} |\partial_x^\alpha f(x, 0)|^{2+\delta}] < \infty$ for all $N > 0$ and $|\alpha| \leq 2$,
- (b)' $_{\gamma}$: The strong mixing rate $\alpha(t)$ satisfies $\int_0^\infty \alpha(s)^\gamma ds < \infty$.

Here $\partial_x^\alpha = (\partial/\partial x^1)^{\alpha_1} \cdots (\partial/\partial x^d)^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The diffusion and drift coefficients (characteristics) of the limit process are given by

$$(1.5) \quad a^{ij}(x) = E\left[\left(\int_0^\infty f^i(x, s) ds\right) f^j(x, 0)\right] + E\left[\left(\int_0^\infty f^j(x, s) ds\right) f^i(x, 0)\right],$$

$$(1.6) \quad b^i(x) = \sum_{j=1}^d E\left[\left(\int_0^\infty \frac{\partial f^i}{\partial x^j}(x, s) ds\right) f^j(x, 0)\right].$$

Similar limit theorems have been discussed by many authors. See Khas'minskii [7], Kesten-Papanicolaou [6], Kunita [8], [9], Kushner [13], and references therein. Conditions assumed in these works are much stronger than ours. Concerning moment conditions, $f(x, t)$ is assumed to be bounded in [7], [13] and is assumed to satisfy Condition (a)' $_{\delta}$ with $\delta > 2$ in [9]. It seems to us that our present condition (a) (existence of the second moment) would be the best possible one as far as the moment conditions are concerned, since our assertion can be regarded as a central limit theorem. Conditions (a)' $_{\delta}$ and (b)' $_{\gamma}$ with $\delta \in (0, \infty)$, $\gamma \in (0, \delta/2(2 + \delta)(1 + d))$ is also a relaxation of conditions in [6] and [8].

Another interesting application of our main theorem will be to the sequence of stochastic difference equations (1.2). We show in Fujiwara [3] that the sequence of solutions of stochastic difference equations of the form (1.2) converges in law to a certain Markov process with jumps under mixing conditions for $\{\xi_k^n\}$ similar to (a) or (a)' $_{\delta}$. Futher, we show in Kunita [11] that the sequence of random measures of the form

$$(1.7) \quad B^n(t, E) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \{I_E(\sqrt{n} \xi_k^n) - \pi_n(E)\},$$

where I_E is the indicator function of the set E and π_n is the law of $\sqrt{n} \xi_1^n$, converges in law to a Brownian random measure $B(t, E)$ and the sequence of random measures

$$(1.8) \quad N^n(t, F) = \sum_{k=1}^{[nt]} I_F(\xi_k^n),$$

converges to a Poisson random measure $N(t, F)$. These limit theorems are applied for the limit theorem of solutions of stochastic difference equations (1.2).

2. A convergence theorem

We begin by introducing some function spaces. Let k be a nonnegative integer

and let $\mathbf{C}^k = \mathbf{C}^k(\mathbf{R}^d, \mathbf{R}^d)$ be the set of all \mathbf{C}^k -maps from \mathbf{R}^d into itself. For $f \in \mathbf{C}^k$, we define the norm $\|f\|_k^*$ by

$$(2.1) \quad \|f\|_k^* = \sup_{x \in \mathbf{R}^d} \left(\frac{|f(x)|}{1 + |x|} \right) + \sum_{1 \leq |\alpha| \leq k} \sup_{x \in \mathbf{R}^d} |\partial_x^\alpha f(x)|.$$

We denote by \mathbf{C}_{b*}^k the set of all $f \in \mathbf{C}^k$ such that $\|f\|_k^* < \infty$. The space \mathbf{C}_{b*}^k and the norm $\|f\|_0^*$ are often denoted by \mathbf{C}_{b*} and $\|f\|^*$. We also denote by $\|f\|$ ($\|f\|_k$) the supremum norm (the \mathbf{C}^k -norm, respectively).

Now let (Ω, \mathcal{F}, P) be a probability space with a family of filtrations $\{\mathcal{F}_t^n; t \geq 0\}_{n \in \mathbf{N}}$. Suppose that for each $n \in \mathbf{N}$ we are given an $\{\mathcal{F}_t^n\}$ -adapted càdlàg stochastic process $f_t^n = f_t^n(x), t \geq 0$ with values in \mathbf{C}_{b*}^2 and a deterministic càdlàg function $\bar{g}_t^n = \bar{g}_t^n(x), t \geq 0$ with values in \mathbf{C}_{b*}^1 . Suppose further that we are given a sequence of nondecreasing, deterministic and càdlàg functions $\{A_t^n\}$ such that $A_0^n = 0$. We often use the following abbreviations. For nondecreasing processes A and B , we denote by $A \ll B$ if $B - A$ is also nondecreasing. We set $\Delta A_t^n = A_t^n - A_{t-}^n$,

$$(2.2) \quad f_{u,\delta}^n(x) = f_u^n(x) I_{(\|f_t^n \Delta A_t^n\|^* \leq \delta)}, \quad \bar{f}_{u,M}^n(x) = f_u^n(x) I_{(\delta < \|f_t^n \Delta A_t^n\|^* \leq M)},$$

and

$$(2.3) \quad \bar{f}_{u,\delta}^n(x) = E[f_{u,\delta}^n(x)], \quad \tilde{f}_{u,\delta}^n(x) = f_{u,\delta}^n(x) - \bar{f}_{u,\delta}^n(x),$$

for $0 < \delta \leq M$, where it is assumed that the expectations $\bar{f}_{u,\delta}^n(x)$ exist for all n, δ, u, x and $\tilde{f}_{u,\delta}^n, u \geq 0$ is a \mathbf{C}_{b*}^2 -valued càdlàg function for any n and δ . Note that $f_{u,\delta}^n = f_u^n$ and $\bar{f}_{u,M}^n = 0$ hold if $\Delta A_u^n = 0$. We denote by $f_u^{n,(i)}$ the i -th component of f_u^n . We also denote by $\mathbf{R}^d \otimes \mathbf{R}^e (\mathbf{S}_d)$ the set of all $d \times e$ real matrices (the set of all $d \times d$ real, symmetric, nonnegative definite matrices, respectively).

Consider the sequence of stochastic difference-differential equations (1.3). For almost all $\omega \in \Omega$, it has a unique global solution $\varphi_t^n, t \geq 0$. It is a càdlàg process with values in \mathbf{R}^d . The purpose of this paper is to discuss the weak convergence of the sequence of càdlàg processes $\{\varphi_t^n\}_n$. Let $\mathbf{D}_d = \mathbf{D}([0, \infty), \mathbf{R}^d)$ be the space of all càdlàg functions from $[0, \infty)$ to \mathbf{R}^d , endowed with Skorohod's J_1 -topology. See Jacod-Shiryayev [5] for the Skorohod space. We denote by $\mathcal{B}(\mathbf{D}_d)$ its topological Borel field. Then the law of the càdlàg process $\varphi_t^n, t \geq 0$ can be defined on the space $(\mathbf{D}_d, \mathcal{B}(\mathbf{D}_d))$ as usual. The sequence of càdlàg processes $\{\varphi_t^n\}_n$ is said to converge in law if the corresponding sequence of laws on $(\mathbf{D}_d, \mathcal{B}(\mathbf{D}_d))$ converges weakly.

To establish the weak convergence of $\{\varphi_t^n\}_n$ we introduce the following system of conditions (A. I)~(A. IV) for $\{f_t^n(x), \bar{g}_t^n(x), A_t^n\}_n$.

(A.I): (1) For every compact set K in \mathbf{R}^d and positive constants M, T , there exists a sequence of nondecreasing càdlàg processes $\{D^n\}_n$ satisfying following properties (i) and either, (ii) or (ii)'

(i) For all $s \leq t \leq t' \leq T$,

$$(2.4) \quad \sum_{|\alpha| \leq 2, |\beta| \leq 1} \left\{ \int_{(s,t)} \sup_{x,y \in K} |E[\int_{(v,t')} \partial_y^\alpha \tilde{f}_{u,M}^n(y) dA_u^n | \mathcal{F}_v^n]| | \partial_x^\beta \tilde{f}_{v,M}^n(x) | dA_v^n \right. \\ \left. + \int_{(s,t)} \sup_{x \in K} | \partial_x^\beta f_{v,M}^n(x) |^2 \Delta A_v^n dA_v^n \right\} \leq D_t^n - D_s^n.$$

(ii) The sequence of compensators $\{D^{n,\beta}\}_n$ of $\{D^n\}_n$ is C -tight in the space \mathbf{D}_1 , that is, it is tight and any limit law is supported on the space of continuous paths $\mathbf{C}([0, \infty), \mathbf{R}^1)$. In addition, $\{D_t^{n,\beta}\}_n$ is uniformly integrable for each t .

(ii)' $\{D^n\}_n$ is C -tight in the space \mathbf{D}_1 and $\{D_t^n\}_n$ is uniformly integrable for each t .

(2) For every compact set K and for every pair of positive constants $\delta < M$, there exists a sequence of deterministic nondecreasing functions $\{\bar{D}^n\}_n$ satisfying the following properties.

(i)

$$(2.5) \quad \sum_{|\beta| \leq 1} \left\{ \int_{(0,t)} \sup_{x \in K} | \partial_x^\beta (\tilde{f}_{u,M}^n(x) + \bar{g}_u^n(x)) | dA_u^n \right. \\ \left. + \int_{(0,t)} E [\sup_{x \in K} | \partial_x^\beta f_{u,M}^{n,\delta}(x) |] dA_u^n \right\} \ll \bar{D}_t^n.$$

(ii) For every $T > 0$, $\sup_{n \in \mathbf{N}} \bar{D}_T^n$ and

$$(2.6) \quad \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{s < t \leq T, t-s \leq \theta} (\bar{D}_t^n - \bar{D}_s^n) = 0.$$

(A.II): (1) There exists a family of σ -finite measures $\nu_u(df)$, $u \geq 0$ on \mathbf{C}_b^* satisfying the following properties of (i) and (ii).

(i) For every $T > 0$

$$(2.7) \quad \int_0^T \int_{\{\|f\|^* \leq 1\}} \|f\|_1^{*2} \nu_u(df) du < \infty \text{ and } \int_0^T \nu_u(\|f\|^* > 1) du < \infty.$$

(ii) For every bounded continuous function h on \mathbf{R}^d , compact set K in \mathbf{R}^d , and $\delta < M$ such that $\delta, M \in C(\nu) =: \{r > 0; \int_0^T \nu_u(\{f; \|f\|^* = r\}) du = 0$ for any $T > 0\}$,

$$(2.8) \quad \lim_{n \rightarrow \infty} E[\sup_{x \in K} | E[\sum_{s < u \leq t} h(f_u^n(x)) \Delta A_u^n I_{(\delta < \|f_u^n \Delta A_u^n\|^* \leq M)} | \mathcal{F}_s^n] \\ - \int_s^t \int_{\mathbf{C}_b^*} h(f(x)) I_{(\delta < \|f\|^* \leq M)} \nu_u(df) du] = 0.$$

In addition, $1 \in C(\nu)$.

(2) There exists a càdlàg function $a_0(u) = a_0(x, u)$ with values in $C_{b*}^2(\mathbf{R}^d, \mathbf{S}_d)$ such that for every compact set K and $s \leq t$

$$(2.9) \quad \lim_{\delta \in C(\nu) \downarrow 0} \limsup_{n \rightarrow \infty} E[\sup_{x \in K} | E[\int_{(s,t)} \tilde{f}_{u,\delta}^{n,(i)}(x) \tilde{f}_{u,\delta}^{n,(j)}(x) \Delta A_u^n dA_u^n | \mathcal{F}_s^n] - \int_s^t a_0^{ij}(x, u) du |] = 0.$$

(3) There exist càdlàg functions $a_1(u) = a_1(x, u)$ with values in $C_{b*}^2(\mathbf{R}^d, \mathbf{R}^d \otimes \mathbf{R}^d)$ and $c(u) = c(x, u)$ with values in $C_{b*}^1(\mathbf{R}^d, \mathbf{R}^d)$ such that for every compact set K , $M > 0$, and $s \leq t$

$$(2.10) \quad \lim_{n \rightarrow \infty} E[\sup_{x \in K} | E[\int_{(s,t)} \int_{(v,t)} \tilde{f}_{u,M}^{n,(i)}(x) \tilde{f}_{v,M}^{n,(j)}(x) dA_u^n dA_v^n | \mathcal{F}_s^n] - \int_s^t a_1^{ij}(x, u) du |] = 0,$$

and

$$(2.11) \quad \lim_{n \rightarrow \infty} E[\sup_{x \in K} | E[\int_{(s,t)} \int_{(v,t)} \sum_{j=1}^d \left(\frac{\partial}{\partial x^j} \tilde{f}_{u,M}^{n,(i)} \right) (x) \tilde{f}_{v,M}^{n,(j)}(x) dA_u^n dA_v^n | \mathcal{F}_s^n] - \int_s^t c^i(x, u) du |] = 0,$$

respectively.

(4) There exists a càdlàg function $b(u) = b(x, u)$ with values in $C_{b*}^1(\mathbf{R}^d, \mathbf{R}^d)$ such that for every compact set K and $s \leq t$

$$(2.12) \quad \limsup_{n \rightarrow \infty} \sup_{x \in K} | \int_{(s,t)} \{ \tilde{f}_{u,1}^n(x) + \bar{g}_u^n(x) \} dA_u^n - \int_s^t b(x, u) du |] = 0.$$

(A.III): (1) For every compact set K , $s \leq t$, $M > 0$,

$$(2.13) \quad \limsup_{n \rightarrow \infty} \sum_{s \in [0,t], |\alpha| \leq 2} E[\sup_{x \in K} | E[\int_{(s,t)} \partial_x^\alpha \tilde{f}_{u,M}^n(x) dA_u^n | \mathcal{F}_s^n] |] = 0.$$

(2) For every compact set K , $s \leq t$, $M > 0$,

$$(2.14) \quad \lim_{n \rightarrow \infty} \sum_{|\alpha| \leq 2} E[\int_{(s,t)} \sup_{x,y \in K} | E[\int_{(v,t)} \partial_v^\alpha \tilde{f}_{u,M}^n(y) dA_v^n | \mathcal{F}_v^n] | \times | \tilde{f}_{v,M}^n(x) |^2 \Delta A_v^n dA_v^n] = 0.$$

(A.IV): For every $t > 0$,

$$(2.15) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P[\sup_{u \leq t} \| f_u^n \Delta A_u^n \| * > M] = 0.$$

Our main theorem in this paper is stated as follows.

THEOREM 2.1. Assume (A.I) ~ (A.IV). Then, the family of \mathbf{D}_d -valued random variables $\{\varphi^n\}_n$ satisfying (1.3) converges in law to the unique solution of the following stochastic differential equation:

$$(2.16) \quad \varphi_t = x_0 + \int_{(0,t]} \sigma(\varphi_{u-}, u) dB_u + \int_{(0,t]} (b + c)(\varphi_{u-}, u) du + \int_{(0,t]} \int_{\{\|f\| \leq 1\}} f(\varphi_{u-}) \tilde{N}(dudf) + \int_{(0,t]} \int_{\{\|f\| > 1\}} f(\varphi_{u-}) N(dudf),$$

where (i) $\sigma(u) = \sigma(x, u)$ is a càdlàg function with values in $\mathbf{C}_b^1(\mathbf{R}^d, \mathbf{R}^d \otimes \mathbf{R}^r)$ such that $\sigma(u)\sigma(u)^* = a_0(u) + \{a_1(u) + a_1(u)^*\}$ where a^* denotes the transpose of the matrix a ,

(ii) B_t is an r -dimensional standard Brownian motion, and

(iii) $N(dudf)$ is a Poisson random measure with the intensity measure $du\nu_u(df)$ and $\tilde{N}(dudf)$ denotes the compensated measure defined by $\tilde{N}(dudf) = N(dudf) - du\nu_u(df)$.

See Ikeda-Watanabe [4] for stochastic integrals based on Poisson random measures and stochastic differential equations of jump type such as (2.16).

We defer the proof of Theorem 2.1 to the next section.

3. Proof of Theorem 2.1

In this section, we will give a proof of Theorem 2.1, by applying the so-called martingale method. However, since it seems difficult to apply the method for $\{\varphi^n\}_n$ satisfying (1.3) itself, we divide the proof into several steps. First, we introduce a family of stochastic processes which are uniformly bounded and have uniformly bounded jumps, as follows. For given $L > 0$, let $r_L(x)$ be a nonnegative smooth function from \mathbf{R}^d to \mathbf{R}^1 such that $r_L(x) \equiv 1$ for $|x| \leq L$ and that $r_L(x) \equiv 0$ for $|x| \geq L + 1$. We fix such a function r_L for each L in the following discussion. For given $M \in C(\nu)$ and $L > 0$, define the localized and truncated process $\varphi^{n,M,L}$ of φ^n as the solution of the following equation:

$$(3.1) \quad \varphi_t^{n,M,L} = x_0 + \int_{(0,t]} \{f_{u,M,L}^n(\varphi_{u-}^{n,M,L}) + \bar{g}_{u,L}^n(\varphi_{u-}^{n,M,L})\} dA_u^n,$$

where we set $f_{u,M,L}^n(x) = r_L(x) f_{u,M}^n(x)$, $\bar{g}_{u,L}^n(x) = r_L(x) \bar{g}_u^n(x)$. Then for arbitrary $T > 0$ there exists a positive constant $C_{ML,T}$ such that $|\varphi_t^{n,M,L}| \leq C_{ML,T}$ and $|\Delta\varphi_t^{n,M,L}| \leq C_{ML,T}$ for all $n, t \leq T$, and $\omega \in \Omega$. Throughout section 3, we denote

by K the compact set $\{x; |x| \leq C_{M,L,T}\}$.

The first step of our proof is to show the weak convergence of $\{\varphi^{n,M,L}\}_n$ for fixed M and L . For this aim, we will apply the martingale method for $\{\varphi^{n,M,L}\}_n$. In fact, in subsection 3-1, the tightness is shown. See Proposition 3.2. Further, in section 3-2, it is shown that any weak limit is unique and coincides with the law of some stochastic differential equation. See Proposition 3.4. After establishing the first step (Proposition 3.10), we will proceed into the second step of removing the restriction of localization in (3.1). This problem is discussed in section 3-3. See Proposition 3.11. In final section 3-4, we will remove the restriction of truncation on jumps, and then we will complete our proof of Theorem 2.1. Though the outline of the proof follows that given in Fujiwara [2], it should be noted that the proof given below does not depend on the specific property of mixing arrays of random variables.

First of all we give a formula of the change of variables for the process φ_t^n , which will be often used in the proof of Theorem 2.1.

LEMMA 3.1. *Let $F(x), x \in \mathbf{R}^d$ be a C^1 -function and let $\phi_t = (\phi_t^1, \dots, \phi_t^d)$ be an \mathbf{R}^d -valued càdlàg process of bounded variation. Set $\Delta\phi_u^i = \phi_u^i - \phi_{u-}^i$. Then for any $s < t$,*

$$\begin{aligned} (3.2) \quad & F(\phi_t) - F(\phi_s) \\ &= \sum_{i=1}^d \int_{(s,t]} \frac{\partial F}{\partial x^i}(\phi_{u-}) d\phi_u^i + \sum_{s < u \leq t} \left(F(\phi_u) - F(\phi_{u-}) - \sum_{i=1}^d \frac{\partial F}{\partial x^i}(\phi_{u-}) \Delta\phi_u^i \right) \\ &= \sum_{i=1}^d \int_{(s,t]} \left\{ \int_0^1 \frac{\partial F}{\partial x^i}(\phi_{u-} + \theta\Delta\phi_u) d\theta \right\} d\phi_u^i. \end{aligned}$$

Here we note that

$$\int_0^1 \frac{\partial F}{\partial x^i}(\phi_{u-} + \theta\Delta\phi_u) d\theta = \frac{\partial F}{\partial x^i}(\phi_{u-}) \quad \text{if } \Delta\phi_u = 0.$$

The proof is easy and is left to the reader.

3-1. Tightness of $\{\varphi^{n,M,L}\}_n$

The aim of this subsection is to prove the following proposition.

PROPOSITION 3.2. *Let M and L be fixed. Then $\{\varphi^{n,M,L}\}_n$ of (3.1) is tight in \mathbf{D}_d .*

For simplicity, we discuss only on 1-dimensional case ($d = 1$). Also, in order to avoid the notational complexity, we consider the equation (3.1) as if $r_L \equiv 1$, and

we omit the superscripts and the subscripts M, L in $\varphi^{n,M,L}, f_{u,M}^n, \bar{f}_{u,M}^n, \tilde{f}_{u,M}^n, f_{u,M}^{n,\delta}$, and $\bar{f}_{u,M}^{n,\delta}$. Further, we will give the proof under the assumption (A.I)-(1)-(ii) because the proof goes similarly when we assume (A.I)-(1)-(ii)'. In order to show the tightness of $\{\varphi^n = \varphi^{n,M,L}\}_n$, we refer to a criterion originated by Aldous [1]. Following Kurtz [12] Theorem 2.7, it is sufficient to prove that for all $T > 0$,

$$(3.3) \quad \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T^n} E[|\varphi_{\tau+\theta}^n - \varphi_\tau^n|^2] = 0,$$

where \mathcal{T}_T^n denotes the set of all $\{\mathcal{F}_t^n\}$ -stopping times not greater than T . Note that $\{\varphi^n\}_n$ which we are now considering is uniformly bounded. To show (3.3), we prepare the following lemma.

LEMMA 3.3. *There exist a positive constant C , stochastic processes $M_{s,t}^n$, and non-decreasing processes R_t^n satisfying the following (i) ~ (iii):*

(i) *For all $s \leq t \leq T$ and $n \in \mathbf{N}$,*

$$(3.4) \quad |\varphi_t^n - \varphi_s^n|^2 \leq C\{D_t^n - D_s^n + E[D_t^n - D_s^n] + \bar{D}_t^n - \bar{D}_s^n\} + M_{s,t}^n + R_t^n.$$

(ii) *For all $\tau \in \mathcal{T}_T^n, \theta > 0$ and $n \in \mathbf{N}, E[M_{\tau,\tau+\theta}^n] = 0$ holds.*

(iii) *For all $t, \lim_{n \rightarrow \infty} E[R_t^n] = 0$ holds.*

Proof. Note that $f_u^n + \bar{g}_u^n = \tilde{f}_u^n + (\bar{f}_u^n + \bar{g}_u^n)$. By integration by parts, we have

$$\begin{aligned} |\varphi_t^n - \varphi_s^n|^2 &= 2 \int_{(s,t)} (\varphi_{u-}^n - \varphi_s^n) d\varphi_u^n + \sum_{s < u \leq t} |\Delta\varphi_u^n|^2 \\ &= 2 \int_{(s,t)} (\varphi_{u-}^n - \varphi_s^n) \tilde{f}_u^n(\varphi_{u-}^n) dA_u^n \\ &\quad + 2 \int_{(s,t)} (\varphi_{u-}^n - \varphi_s^n) \{ \bar{f}_u^n(\varphi_{u-}^n + \bar{g}_u^n(\varphi_{u-}^n)) \} dA_u^n + \sum_{s < u \leq t} |\Delta\varphi_u^n|^2 \\ &=: 2I_1^n + 2I_2^n + I_3^n. \end{aligned}$$

We first estimate I_i^n for $i = 2, 3$. For I_2^n , we have

$$(3.5) \quad |I_2^n| \leq 2C_{ML,T} \int_{(s,t)} \sup_{x \in K} |\tilde{f}_u^n(x) + \bar{g}_u^n(x)| dA_u^n \leq 2C_{ML,T}(\bar{D}_t^n - \bar{D}_s^n),$$

because $|\varphi_u^n| \leq C_{ML,T}$ for all n, u , and ω .

For I_3^n , since $\Delta\varphi_u^n = \{\tilde{f}_u^n(\varphi_{u-}^n) + \bar{f}_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n)\} \Delta A_u^n$ we obtain by (A.I)

$$(3.6) \quad \begin{aligned} |I_3^n| &\leq 2 \int_{(s,t)} \sup_{x \in K} \{ |\tilde{f}_u^n(x)|^2 + |\bar{f}_u^n(x) + \bar{g}_u^n(x)|^2 \} \Delta A_u^n dA_u^n \\ &\leq 2 \{ D_t^n - D_s^n + (\bar{D}_t^n - \bar{D}_s^n) \sup_n \bar{D}_t^n \}. \end{aligned}$$

We next consider I_1^n . Applying the formula of the change of variables (Lemma 3.1) to the function $(x - \varphi_s^n)\tilde{f}_u^n(x)$, we have

$$(3.7) \quad (\varphi_{u-}^n - \varphi_s^n)\tilde{f}_u^n(\varphi_{u-}^n) = \int_{(s,u)} \int_0^1 \{\tilde{f}_u^n(\varphi_{v-}^n + \theta\Delta\varphi_v^n) + (\varphi_{v-}^n + \theta\Delta\varphi_v^n - \varphi_s^n)\partial\tilde{f}_u^n(\varphi_{v-}^n + \theta\Delta\varphi_v^n)\} d\theta d\varphi_v^n.$$

Therefore,

$$(3.8) \quad I_1^n = \int_{(s,t)} \int_{(s,u)} \int_0^1 \{\tilde{f}_u^n(\varphi_{v-}^n + \theta\Delta\varphi_v^n) + (\varphi_{v-}^n + \theta\Delta\varphi_v^n - \varphi_s^n)\partial\tilde{f}_u^n(\varphi_{v-}^n + \theta\Delta\varphi_v^n)\} d\theta \times \{f_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n dA_u^n.$$

For fixed s , set

$$(3.9) \quad h_{v,t}^n(x, z) = \int_{(v,t)} \int_0^1 \{\tilde{f}_u^n(x + \theta z) + (x + \theta z - \varphi_s^n)\partial\tilde{f}_u^n(x + \theta z)\} d\theta dA_u^n.$$

Then Fubini's theorem implies that

$$(3.10) \quad I_1^n = \int_{(s,t)} h_{v,t}^n(\varphi_{v-}^n, \Delta\varphi_v^n) \{f_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n.$$

Further, set

$$(3.11) \quad \check{h}_{v,t}^n(x, z) = E[h_{v,t}^n(x, z) | \mathcal{F}_v^n], \quad \hat{h}_{v,t}^n(x, z) = h_{v,t}^n(x, z) - \check{h}_{v,t}^n(x, z).$$

Then, by (3.11), we have

$$(3.12) \quad I_1^n = \int_{(s,t)} \check{h}_{v,t}^n(\varphi_{v-}^n, \Delta\varphi_v^n) \{f_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n + \int_{(s,t)} \hat{h}_{v,t}^n(\varphi_{v-}^n, \Delta\varphi_v^n) \tilde{f}_v^n(\varphi_{v-}^n) dA_v^n + \int_{(s,t)} \hat{h}_{v,t}^n(\varphi_{v-}^n, \Delta\varphi_v^n) (\bar{f}_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)) dA_v^n =: I_{11}^n + I_{12}^n + I_{13}^n.$$

Define $M_{s,t}^n = I_{11}^n$. We show that it satisfies (ii). For arbitrary $\tau \in \mathcal{T}_T^n$ and $\theta > 0$, it holds that

$$(3.13) \quad E\left[\int_{(\tau,\tau+\theta)} \check{h}_{v,\tau+\theta}^n(\varphi_{v-}^n, \Delta\varphi_v^n) \{f_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n\right] = E\left[\int_{(0,T)} I_{(\tau < v < \tau+\theta)} E[\check{h}_{v,t+\theta}^n(x, z) | \mathcal{F}_v^n] |_{t=\tau, x=\varphi_{v-}^n, z=\Delta\varphi_v^n} \times \{f_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n\right] = 0,$$

because $I_{(\tau < v < \tau+\theta)}$ is \mathcal{F}_v^n measurable and τ is so on the set $I_{(\tau < v < \tau+\theta)}$. For I_{12}^n , we

have

$$(3.14) \quad |I_{12}^n| \leq (1 + 3C_{ML,T}) \times \sum_{|\alpha| \leq 1} \int_{(s,t)} \sup_{x,y \in K} |E[\int_{(v,t)} \partial_y^\alpha \tilde{f}_u^n(y) dA_u^n | \mathcal{F}_v^n]| |\tilde{f}_v^n(x)| dA_v^n.$$

Hence, combining this with (3.5), and (3.6), it is easily seen that (A.I) implies that

$$(3.15) \quad |I_2^n| + |I_3^n| + |I_{12}^n| \leq C\{D_t^n - D_s^n + \bar{D}_t^n - \bar{D}_s^n\},$$

for some constant C depending only on M, L, T . For I_{13}^n , by (A.I)-(2), we have

$$(3.16) \quad |I_{13}^n| \leq C \sum_{|\alpha| \leq 1} \int_{(s,t)} E[\sup_{x \in K} |E[\int_{(v,t)} \partial_x^\alpha \tilde{f}_u^n(x) dA_u^n | \mathcal{F}_v^n]|] d\bar{D}_v^n.$$

Define a nondecreasing process R_t^n by the right hand side of (3.16). Then, it is easy to see that (A.I)-(2) and (A.III)-(1) imply (iii). Thus, we have completed the proof of Lemma 3.3. □

We continue the proof of Proposition 3.2. By Lemma 3.3, we have for all $\tau \in \mathcal{T}_T^n$ and $0 < \theta < 1$

$$(3.17) \quad E[|\varphi_{\tau+\theta}^n - \varphi_\tau^n|^2] \leq C\{E[D_{\tau+\theta}^{n,\theta} - D_\tau^{n,\theta}] + E[\bar{D}_{\tau+\theta}^n - \bar{D}_\tau^n]\} + E[R_{\tau+1}^n].$$

Since $\{D^{n,\theta}\}_n$ is C -tight, it is easy to see that

$$\lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T^n} P[D_{\tau+\theta}^{n,\theta} - D_\tau^{n,\theta} > \delta] = 0$$

for all $\delta > 0$. Combining this with the uniform integrability of $D_{\tau+1}^{n,\theta}$, we obtain

$$\lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T^n} E[D_{\tau+\theta}^{n,\theta} - D_\tau^{n,\theta}] = 0.$$

On the other hand, by (A.I)-(2)-(ii), the second term of the right hand side of (3.17) converges to 0 as $n \rightarrow \infty$. Further, by the property (iii) in Lemma 3.3, we see that (3.3) holds for $\{\varphi^{n,M,L}\}_n$. □

3-2. Identification of limit process

In the previous section, we have shown that $\{\varphi^{n,M,L}\}_n$ is tight in \mathbf{D}_d for each M and L , namely, any subsequence of the laws of $\{\varphi^{n,M,L}\}_n$ contains weakly convergent subsequence. The aim of this subsection is to show that the limit laws are identified with the law of the unique solution of a stochastic differential equation. What we would like to show is stated as follows.

PROPOSITION 3.4. *Let \mathbf{Q} be any weak limit of the laws of $\{\varphi^{n,M,L}\}_n$. Then, \mathbf{Q} is equal to the law of the solution $\varphi^{M,L}$ of the following stochastic differential equation:*

$$(3.18) \quad \begin{aligned} \varphi_t^{M,L} = & x_0 + \int_{(0,t)} \sigma_L(\varphi_{u-}^{M,L}, u) dB_u + \int_{(0,t)} (b_L + c_L)(\varphi_{u-}^{M,L}, u) du \\ & + \int_{(0,t)} \int_{\{\|f\|^* \leq 1\}} f_L(\varphi_{u-}^{M,L}) \tilde{N}(dudf) \\ & + \int_{(0,t)} \int_{\{1 < \|f\|^* \leq M\}} f_L(\varphi_{u-}^{M,L}) N(dudf), \end{aligned}$$

where $\sigma_L(x, u) = r_L(x)\sigma(x, u)$, $b_L(x, u) = r_L(x)b(x, u)$, $c_L(x, u) = r_L(x)^2c(x, u) + a_1(x, u) \times r_L(x) \nabla_x r_L(x)$, and $f_L(x) = r_L(x) f(x)$.

By Jacod’s theorem ([5], Theorem III. 2.26. p. 144), we know that the uniqueness of the solution of stochastic differential equation implies that of the martingale problem for the corresponding generator. Therefore, to show Proposition 3.4, it is sufficient to prove the following.

PROPOSITION 3.5. *Let \mathbf{Q} be any weak limit of $\{\varphi^{n,M,L}\}$. Define a linear operator $\mathcal{L}_u^{M,L}$ by*

$$(3.19) \quad \begin{aligned} \mathcal{L}_u^{M,L} F(x) = & \frac{1}{2} \sum_{i,j=1}^d a_L^{ij}(x, u) \frac{\partial^2 F}{\partial x^i \partial x^j}(x) + \sum_{i=1}^d (b_L^i(x, u) + c_L^i(x, u)) \frac{\partial F}{\partial x^i}(x) \\ & + \int_{\{\|f\|^* \leq M\}} \{F(x + f_L(x)) - F(x) - \sum_{i=1}^d f_L^i(x) I_{\{\|f\|^* \leq 1\}} \frac{\partial F}{\partial x^i}(x)\} \nu_u(df), \end{aligned}$$

where $a_L(x, u) = \sigma_L(x, u)\sigma_L(x, u)^*$. Then, \mathbf{Q} satisfies for all $s \leq t$ and $F \in \mathbf{C}_b^\infty(\mathbf{R}^d)$,

$$(3.20) \quad E^{\mathbf{Q}}[F(\varphi_t) - F(\varphi_s) - \int_s^t \mathcal{L}_u^{M,L} F(\varphi_u) du \mid \mathcal{D}_s] = 0,$$

where φ denotes a canonical element of \mathbf{D}_d and \mathcal{D}_s denotes the right continuous sub- σ -field generated by $\{\varphi_u; u \leq s\}$.

In the remainder of this section, we prove Proposition 3.5. We give the proof only in 1-dimensional case and we omit the superscripts and the subscripts M, L as in the section 3-1. For the limit measure \mathbf{Q} , set $J = \{t > 0; \mathbf{Q}(\Delta\varphi_t \neq 0) > 0\}$, which is at most countable. To prove (3.20), it is sufficient to show that

$$(3.21) \quad E^{\mathbf{Q}}[\{F(\varphi_t) - F(\varphi_s) - \int_s^t \mathcal{L}_u F(\varphi_u) du\} \Psi(\varphi_{u_1}, \varphi_{u_2}, \dots, \varphi_{u_m})] = 0$$

for all $s, t \in J^c, m \in \mathbf{N}, u_i \in J^c (i = 1, 2, \dots, m)$ such that $u_1 < u_2 < \dots < u_m \leq s$, and for bounded continuous function $\Psi : \mathbf{R}^m \rightarrow \mathbf{R}^1$. In the sequel, we may assume that the law of φ^n converges weakly to \mathcal{Q} .

Applying Lemma 3.1 repeatedly as in the proof of Lemma 3.3, we can obtain for $s \leq t$ and $F \in \mathbf{C}_b^\infty(\mathbf{R}^1)$

(3.22)

$$\begin{aligned}
 F(\varphi_t^n) - F(\varphi_s^n) &= \int_{(s,t]} F'(\varphi_{u-}^n) \tilde{f}_u^n(\varphi_{u-}^n) dA_u^n \\
 &\quad + \int_{(s,t]} F'(\varphi_{u-}^n) (\tilde{f}_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n)) dA_u^n \\
 &\quad + \sum_{s < u \leq t} \{F(\varphi_u^n) - F(\varphi_{u-}^n) - F'(\varphi_{u-}^n) \Delta\varphi_u^n\} \\
 &= \int_{(s,t]} \int_{(v,t]} (F' \tilde{f}_u^n)'(\varphi_{v-}^n) dA_u^n \tilde{f}_v^n(\varphi_{v-}^n) dA_v^n \\
 &\quad + \int_{(s,t]} F'(\varphi_{u-}^n) \{\tilde{f}_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n)\} dA_u^n \\
 &\quad + \sum_{s < u \leq t} \{F(\varphi_u^n) - F(\varphi_{u-}^n) - F'(\varphi_{u-}^n) \Delta\varphi_u^n\} \\
 &\quad + \int_{(s,t]} \int_{(v,t]} (F' \tilde{f}_u^n)'(\varphi_{v-}^n) dA_u^n \{\tilde{f}_v^n(\varphi_{v-}^n) + \bar{g}_v^n(\varphi_{v-}^n)\} dA_v^n \\
 &\quad + \int_{(s,t]} \sum_{s < v < u} \{F' \tilde{f}_u^n(\varphi_v^n) - F' \tilde{f}_u^n(\varphi_{v-}^n) \\
 &\quad \quad \quad - (F' \tilde{f}_u^n)'(\varphi_{v-}^n) \Delta\varphi_v^n\} dA_u^n \\
 &\quad + \int_{(s,t]} F'(\varphi_s^n) \tilde{f}_u^n(\varphi_s^n) dA_u^n \\
 &=: \sum_{i=1}^6 I_i^n(s, t).
 \end{aligned}$$

Set $J_i^n(s, t) = E[I_i^n(s, t) | \mathcal{F}_s^n], i = 1, \dots, 6$. We first show that the terms $J_i^n(s, t) (i = 4, 5, 6)$ are neglected when n tends to infinity. Since

$$\begin{aligned}
 |J_4^n(s, t)| &\leq \|F\|_2 \sum_{|\alpha| \leq 1} E[E[\int_{(s,t]} \sup_{x \in K} |E[\int_{(v,t]} \partial_x^\alpha \tilde{f}_u^n(x) dA_u^n | \mathcal{F}_v^n]|] \\
 &\quad \times \sup_{x \in K} |\tilde{f}_v^n(x) + \bar{g}_v^n(x)| dA_v^n | \mathcal{F}_s^n],
 \end{aligned}$$

(A.I)-(2) and (A.III)-(1) imply that for $i = 4$

$$(3.23) \quad \lim_{n \rightarrow \infty} E[|J_i^n(s, t)|] = 0.$$

Similarly, (A.III)-(1) implies that (3.23) holds for $i = 6$. For $i = 5$, since we have

$$|J_s^n(s, t)| \leq 2 \|F\|_3 \sum_{|\alpha| \leq 2} E \int_{(s,t)} \sup_{x,y \in K} |E \int_{(v,t)} \partial_x^\alpha \tilde{f}_u^n(y) dA_u^n | \mathcal{F}_v^n| \times \{ |\tilde{f}_v^n(x)|^2 + |\bar{g}_v^n(x) + \bar{f}_v^n(x)|^2 \} \Delta A_v^n dA_v^n | \mathcal{F}_s^n],$$

(A.I) and (A.III) imply (3.23) for $i = 5$.

Next, we consider $I_1^n(s, t)$. To show the next lemma is one of principal parts of the proof.

LEMMA 3.6. For $u_i (i = 1, \dots, m)$ and Ψ given in (3.21), set $\Psi_s^n = \Psi(\varphi_{u_1}^n, \dots, \varphi_{u_m}^n)$ and $\Psi_s = \Psi(\varphi_{u_1}, \dots, \varphi_{u_m})$. Then we obtain

$$(3.24) \quad \lim_{n \rightarrow \infty} E \left[\int_{(s,t)} \int_{(v,t)} F''(\varphi_{v-}^n) \tilde{f}_u^n(\varphi_{v-}^n) \bar{f}_v^n(\varphi_{v-}^n) dA_u^n dA_v^n \times \Psi_s^n \right] = E \left[\int_{(s,t)} F''(\varphi_v) a_1(\varphi_v, v) dv \times \Psi_s \right],$$

$$(3.25) \quad \lim_{n \rightarrow \infty} E \left[\int_{(s,t)} \int_{(v,t)} F'(\varphi_{v-}^n) (\tilde{f}_u^n)'(\varphi_{v-}^n) \bar{f}_v^n(\varphi_{v-}^n) dA_u^n dA_v^n \times \Psi_s^n \right] = E \left[\int_{(s,t)} F'(\varphi_v) c(\varphi_v, v) dv \times \Psi_s \right].$$

Proof. We will give a proof only for (3.24) because (3.25) can be proved similarly. For $s \leq v \leq t, x$ and n , set

$$(3.26) \quad g_{v,t}^n(x) = E \left[\int_{(v,t)} \tilde{f}_u^n(x) dA_u^n | \mathcal{F}_v^n \right] \bar{f}_v^n(x).$$

Then, note that (A.I)-(1) and (A.II)-(3) imply that for all $s \leq t \leq t' \leq T$ and $|\alpha| \leq 1$

$$(3.27) \quad \int_{(s,t)} \sup_{x \in K} |\partial_x^\alpha g_{v,t}^n(x)| dA_v^n \leq D_t^n - D_s^n,$$

and

$$(3.28) \quad \lim_{n \rightarrow \infty} E \left[\sup_{x \in K} |E \left[\int_{(s,t)} g_{v,t}^n(x) dA_v^n | \mathcal{F}_s^n \right] - \int_s^t a_1(x, v) dv| \right] = 0.$$

Now, since φ^n converges in law to φ in \mathbf{D}_d , by Proposition 2 in Slomiński [15], we know that there exist a sequence of positive constants $\{\rho^i\}_i$ which converges to 0 as $i \rightarrow \infty$ and a sequence of $\{\mathcal{F}_t^n\}$ -stopping times $\{\sigma_k^{i,n}\}_{i,n,k}$ satisfying the following properties (i) ~ (iii):

(i) For each i and n

$$(3.29) \quad 0 = \sigma_0^{i,n} < \sigma_1^{i,n} < \dots, \quad \lim_{k \rightarrow \infty} \sigma_k^{i,n} = \infty.$$

(ii)

$$(3.30) \quad \lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} P[\min_k \{\sigma_{k+1}^{i,n} - \sigma_k^{i,n}; \sigma_k^{i,n} \leq T\} \leq \rho^i] = 0.$$

(iii)

$$(3.31) \quad \lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} P[\max_k \{ \sup_{v \in \{\sigma_k^{i,n}, \sigma_{k+1}^{i,n}\}} |\varphi_v^n - \varphi_{\sigma_k^{i,n}}^{n,i,n}|; \sigma_k^{i,n} \leq T\} > \varepsilon] = 0,$$

for all $\varepsilon > 0$. Hence, by (3.30) and (3.31), for arbitrary $\eta, \varepsilon > 0$, there exist i_0 and n_0 such that for all $i \geq i_0$ and $n \geq n_0$ we have

$$(3.32) \quad P[\min_k \{\sigma_{k+1}^{i,n} - \sigma_k^{i,n}; \sigma_k^{i,n} \leq T\} \leq \rho^i] < \eta,$$

and

$$(3.33) \quad P[\max_k \{ \sup_{v \in \{\sigma_k^{i,n}, \sigma_{k+1}^{i,n}\}} |\varphi_v^n - \varphi_{\sigma_k^{i,n}}^{n,i,n}|; \sigma_k^{i,n} \leq T\} > \varepsilon] < \eta.$$

In the discussion below, we fix η, ε , and i satisfying (3.32) and (3.33). For them, set

$$(3.34) \quad \Omega^n = \left\{ \omega; \min_k \{\sigma_{k+1}^{i,n} - \sigma_k^{i,n}; \sigma_k^{i,n} \leq T\} > \rho^i \right. \\ \left. \text{and } \max_k \{ \sup_{v \in \{\sigma_k^{i,n}, \sigma_{k+1}^{i,n}\}} |\varphi_v^n - \varphi_{\sigma_k^{i,n}}^{n,i,n}|; \sigma_k^{i,n} \leq T\} \leq \varepsilon \right\}.$$

Set $\tau_k^{i,n} = (\sigma_k^{i,n} \vee s) \wedge t$. Then, we have

$$(3.35) \quad |E[\{ \int_{(s,t)} g_{v,t}^n(\varphi_{v-}^n) dA_v^n - \int_{(s,t)} a_1(\varphi_{v-}^n, v) dv \} \Psi_s^n]| \\ \leq |E[\{ \int_{(s,t)} g_{v,t}^n(\varphi_{v-}^n) dA_v^n - \sum_{k=0}^{T/\rho^i} \int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^{n,i,n}) dA_v^n \} \Psi_s^n]| \\ + |E[\{ \sum_{k=0}^{T/\rho^i} \int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^{n,i,n}) dA_v^n - \sum_{k=0}^{T/\rho^i} \int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} a_1(\varphi_{\tau_k^{i,n}}^{n,i,n}, v) dv \} \Psi_s^n]| \\ + |E[\{ \sum_{k=0}^{T/\rho^i} \int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} a_1(\varphi_{\tau_k^{i,n}}^{n,i,n}, v) dv - \int_{(s,t)} a_1(\varphi_{v-}^n, v) dv \} \Psi_s^n]| \\ =: J_1^n + J_2^n + J_3^n.$$

For J_1^n , (A.I)-(1) implies that

$$(3.36) \quad J_1^n \leq \sum_{k=0}^{T/\rho^i} E[\int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} |g_{v,t}^n(\varphi_{v-}^n) - g_{v,t}^n(\varphi_{\tau_k^{i,n}}^{n,i,n})| dA_v^n] \Psi \\ \leq \sum_{k=0}^{T/\rho^i} E[\int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} \sup_{x \in K} |\partial_x g_{v,t}^n(x)| |\varphi_{v-}^n - \varphi_{\tau_k^{i,n}}^{n,i,n}| dA_v^n] \Psi \\ \leq \sum_{k=0}^{T/\rho^i} E[\int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} |\varphi_{v-}^n - \varphi_{\tau_k^{i,n}}^{n,i,n}| dD_v^{n,\rho}] \Psi$$

$$\begin{aligned} &\leq \sum_{k=0}^{T/\rho^i} E \left[\int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} |\varphi_{v-}^n - \varphi_{\tau_k^{i,n}}^n| dD_v^{n,\rho}; \Omega^n \right] \|\Psi\| \\ &\quad + \sum_{k=0}^{T/\rho^i} E \left[\int_{(\tau_k^{i,n}, \tau_{k+1}^{i,n})} |\varphi_{v-}^n - \varphi_{\tau_k^{i,n}}^n| dD_v^{n,\rho}; (\Omega^n)^c \right] \|\Psi\| \\ &=: J_{11}^n + J_{12}^n. \end{aligned}$$

Since

$$\sup_{v \in (\tau_k^{i,n}, \tau_{k+1}^{i,n})} |\varphi_{v-}^n - \varphi_{\tau_k^{i,n}}^n| \leq \varepsilon$$

on Ω^n and $D_t^{n,\rho}$ is a nondecreasing process, we have

$$(3.37) \quad J_{11}^n \leq \varepsilon \sup_n E[D_t^{n,\rho}] \|\Psi\|.$$

Note that (A.I)-(1) implies that $\sup_n E[D_t^{n,\rho}] < \infty$ for each t . On the other hand, by (3.32), (3.33) and (3.34), we have

$$\begin{aligned} J_{12}^n &\leq 2C_{ML,T} E[D_t^{n,\rho}; (\Omega^n)^c] \\ &\leq 2C_{ML,T} \{ E[D_t^{n,\rho}; D_t^{n,\rho} > 1/\sqrt{\eta}] + (1/\sqrt{\eta})P((\Omega^n)^c) \} \\ &\leq 2C_{ML,T} \{ E[D_t^{n,\rho}; D_t^{n,\rho} > 1/\sqrt{\eta}] + 2\sqrt{\eta} \}. \end{aligned}$$

Combining this with (3.37), we can see that $\limsup_{n \rightarrow \infty} J_1^n = 0$ because $\{D_t^{n,\rho}\}_n$ is uniformly integrable and $\eta, \varepsilon > 0$ are arbitrary.

For J_2^n we have

$$\begin{aligned} (3.38) \quad J_2^n &\leq \sum_{k=0}^{T/\rho^i} | E[\{ \int_{(\tau_k^{i,n}, t)} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n - \int_{(\tau_k^{i,n}, t)} a_1(\varphi_{\tau_k^{i,n}}^n, v) dv \} \Psi_s^n] | \\ &\quad + \sum_{k=0}^{T/\rho^i} | E[\{ \int_{(\tau_{k+1}^{i,n}, t)} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n - \int_{(\tau_{k+1}^{i,n}, t)} a_1(\varphi_{\tau_k^{i,n}}^n, v) dv \} \Psi_s^n] | \\ &=: J_{21}^n + J_{22}^n. \end{aligned}$$

Further, for $m \in \mathbb{N}$, set $\tau_k^{i,n,m} = t \wedge ([m\tau_k^{i,n}] + 1)/m$. Then we have by (3.27)

$$\begin{aligned} (3.39) \quad J_{21}^n &\leq \sum_{k=0}^{T/\rho^i} | E[\int_{(\tau_k^{i,n}, \tau_k^{i,n,m})} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n \Psi_s^n] | \\ &\quad + \sum_{k=0}^{T/\rho^i} | E[\{ \int_{(\tau_k^{i,n}, t)} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n - \int_{(\tau_k^{i,n,m}, t)} a_1(\varphi_{\tau_k^{i,n}}^n, v) dv \} \Psi_s^n] | \\ &\quad + \sum_{k=0}^{T/\rho^i} | E[\int_{(\tau_k^{i,n}, \tau_k^{i,n,m})} a_1(\varphi_{\tau_k^{i,n}}^n, v) dA_v^n \Psi_s^n] | \\ &\leq \sum_{k=0}^{T/\rho^i} E[D_{\tau_k^{i,n,m}}^{n,\rho} - D_{\tau_k^{i,n}}^{n,\rho}] \|\Psi\| + J_{21}^{n,m} + \|a_1\| \|\Psi\| \frac{1}{m} (\frac{T}{\rho^i} + 1), \end{aligned}$$

where we set

$$J_{21}^{n,m} = \sum_{k=0}^{T/\rho^i} |E[\{ \int_{(\tau_k^{i,n,m},t)} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n - \int_{(\tau_k^{i,n,m},t)} a_1(\varphi_{\tau_k^{i,n}}^n, v) dv \} \Psi_s^n]|.$$

Since (A.I)-(1) implies that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{k \leq T/\rho^i} E[D_{\tau_k^{i,n,m}}^{n,\rho} - D_{\tau_k^{i,n}}^{n,\rho}] = 0,$$

we have

$$(3.40) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=0}^{T/\rho^i} E[D_{\tau_k^{i,n,m}}^{n,\rho} - D_{\tau_k^{i,n}}^{n,\rho}] = 0.$$

We next show that for each $m \in \mathbb{N}$

$$(3.41) \quad \limsup_{n \rightarrow \infty} J_{21}^{n,m} = 0.$$

To see this, note that

$$(3.42) \quad \begin{aligned} J_{21}^{n,m} &\leq \sum_{k=0}^{T/\rho^i} |E[E[\int_{(\tau_k^{i,n,m},t)} g_{v,t}^n(\varphi_{\tau_k^{i,n}}^n) dA_v^n - \int_{(\tau_k^{i,n,m},t)} a_1(\varphi_{\tau_k^{i,n}}^n, v) dv \mid \mathcal{F}_{\tau_k^{i,n,m}}^n] \Psi_s^n]| \\ &\leq \sum_{k=0}^{T/\rho^i} \sum_{l=0}^{\lfloor mt \rfloor} E[\sup_{x \in K} |E[\int_{(((l+1)/m)\vee s, t)} g_{v,t}^n(x) dA_v^n \\ &\quad - \int_{(((l+1)/m)\vee s, t)} a_1(x, v) dv \mid \mathcal{F}_{(((l+1)/m)\vee s, t)}^n]|)] \Psi \|. \end{aligned}$$

Therefore, by (3.28), we obtain (3.41). Combining this with (3.40), we get $\lim_{n \rightarrow \infty} J_{21}^n = 0$, because we can take m as large as we want in (3.39). If we note that $\varphi_{\tau_k^{i,n}}^n$ is $\mathcal{F}_{\tau_k^{i,n}}^n$ -measurable for all k , we can see that the similar argument on J_{21}^n implies that $\lim_{n \rightarrow \infty} J_{22}^n = 0$. Hence, we have shown that $\lim_{n \rightarrow \infty} J_2^n = 0$.

For J_3^n , by the similar argument on J_1^n , we can show that $\lim_{n \rightarrow \infty} J_3^n = 0$. Thus, we have completed the proof of Lemma 3.6. □

We continue the proof of Proposition 3.5. For $I_2^n(s, t)$ and $I_3^n(s, t)$ in (3.22), the following lemma holds.

LEMMA 3.7. *Let Ψ_s^n and Ψ_s be the functions given in Lemma 3.6. Then we have*

$$(3.43) \quad \begin{aligned} \lim_{n \rightarrow \infty} E[I_2^n(s, t) \Psi_s^n] &= E^Q[\int_s^t F'(\varphi_u) b(\varphi_u) du \Psi_s] \\ &\quad + E^Q[\int_s^t F'(\varphi_u) f(\varphi_u) I_{(1 < \|f\|^* \leq M)} \nu_u(df) du \Psi_s]. \end{aligned}$$

Proof. By (A.II)-(1) and (A.II)-(4), we can obtain the conclusion similarly

as in the proof of Lemma 3.6. □

Finally, we consider $I_3^n(s, t)$, for which we get the following lemma.

LEMMA 3.8. *Let Ψ_s^n and Ψ_s be the same functions as in Lemma 3.6. Then we have*

$$\begin{aligned}
 (3.44) \quad & \lim_{n \rightarrow \infty} E[I_3^n(s, t) \Psi_s^n] \\
 &= E^Q[\int_s^t a_0(\varphi_u, u) F''(\varphi_u) du \Psi_s] \\
 &+ E^Q[\int_s^t \int_{\{\|f\|^* \leq M\}} \{F(\varphi_u + f(\varphi_u)) - F(\varphi_u) - f(\varphi_u) F'(\varphi_u)\} \\
 &\hspace{15em} \times \nu_u(df) du \Psi_s].
 \end{aligned}$$

To prove this lemma, we give a preparatory lemma. Set

$$\Phi_u^n = \int_0^1 \int_0^1 a F''(\varphi_u^n + \alpha\beta \Delta\varphi_u^n) d\alpha d\beta.$$

Then, note that the mean value theorem gives

$$(3.45) \quad I_3^n(s, t) = \sum_{s < u \leq t} \Phi_u^n (\Delta\varphi_u^n)^2 = \int_{(s,t)} \Phi_u^n \{f_u^n(\varphi_u^n) + \bar{g}_u^n(\varphi_u^n)\}^2 \Delta A_u^n dA_u^n.$$

Since it is difficult to calculate the limit of the above term directly, we interpolate it by some terms which make the limit clear as follows.

LEMMA 3.9. (i) *For every $\delta > 0$,*

$$\begin{aligned}
 (3.46) \quad & \limsup_{n \rightarrow \infty} |E[\int_{(s,t)} \Phi_u^n \{f_u^n(\varphi_u^n) + \bar{g}_u^n(\varphi_u^n)\}^2 \\
 &\hspace{10em} - \tilde{f}_{u,\delta}^n(\varphi_u^n)^2 - f_u^{n,\delta}(\varphi_u^n)^2\} \Delta A_u^n dA_u^n \Psi_s^n]| = 0.
 \end{aligned}$$

(ii)

$$(3.47) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} |E[\int_{(s,t)} \{\Phi_u^n - \frac{1}{2} F''(\varphi_u^n)\} \tilde{f}_{u,\delta}^n(\varphi_u^n)^2 \Delta A_u^n dA_u^n \Psi_s^n]| = 0.$$

(iii) *For every $\delta > 0$,*

$$\begin{aligned}
 (3.48) \quad & \limsup_{n \rightarrow \infty} |E[\int_{(s,t)} \{\Phi_u^n - \int_0^1 \int_0^1 \alpha F''(\varphi_u^n + \alpha\beta f_u^{n,\delta}(\varphi_u^n)) \Delta A_u^n\} \\
 &\hspace{15em} \times f_u^{n,\delta}(\varphi_u^n)^2 \Delta A_u^n dA_u^n \Psi_s^n]| = 0.
 \end{aligned}$$

Proof. (i) Let $0 < \delta \leq M$. Then $f_u^n + \bar{g}_u^n = \tilde{f}_{u,\delta}^n + f_u^{n,\delta} + (\tilde{f}_{u,\delta}^n + \bar{g}_u^n)$ holds. Therefore

$$\begin{aligned} & (f_u^n + \bar{g}_u^n)^2 - (\tilde{f}_{u,\delta}^n)^2 - (f_u^{n,\delta})^2 \\ &= (\tilde{f}_{u,\delta}^n + \bar{g}_u^n)^2 + 2f_u^{n,\delta}(\tilde{f}_{u,\delta}^n + \tilde{f}_{u,\delta}^n + \bar{g}_u^n) + 2\tilde{f}_{u,\delta}^n(\tilde{f}_{u,\delta}^n + \bar{g}_u^n). \end{aligned}$$

Since $|\tilde{f}_{u,\delta}^n + \bar{g}_u^n| \leq \Delta \bar{D}_u^n$, we have

$$\begin{aligned} (3.49) \quad & |E[\int_{(s,t)} \Phi_u^n (f_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n))^2 - \tilde{f}_{u,\delta}^n(\varphi_{u-}^n)^2 - f_u^{n,\delta}(\varphi_{u-}^n)^2] \Delta A_u^n dA_u^n \Psi_s^n]| \\ & \leq \|F\|_2 \|\Psi\| \{ (\sup_{s \leq u \leq t} \Delta \bar{D}_u^n) (\bar{D}_t^n - \bar{D}_s^n) \\ & \quad + 2 \sum_{s < u \leq t} E[\sup_{x \in K} |f_u^{n,\delta}(x) \Delta A_u^n|] (\sup_{s < u \leq t} E[\sup_{x \in K} |f_{u,\delta}^n(x) \Delta A_u^n|] + \sup_{s < u \leq t} |\Delta \bar{D}_u^n|) \\ & \quad + 4 \sup_{s < u \leq t} E[\sup_{x \in K} |f_{u,\delta}^n(x) \Delta A_u^n|] (\bar{D}_t^n - \bar{D}_s^n) \}. \end{aligned}$$

By (A.I)-(2), we have

$$\sum_{s < u \leq t} E[\sup_{x \in K} |f_{u,\delta}^n(x) \Delta A_u^n|] \leq \sup_n \bar{D}_t^n < \infty,$$

and $\sup_{s < u \leq t} |\Delta \bar{D}_u^n| \rightarrow 0$. Therefore the first term of the right hand side of (3.49) converges to 0 as $n \rightarrow \infty$. Further, (A.I)-(1) implies that the second term converges to 0 as $n \rightarrow \infty$. Therefore we get (3.46).

(ii) If we note that $\sup_n \sup_{u,x} |\tilde{f}_{u,\delta}^n(x) \Delta A_u^n| \leq 2(1 + C_{ML,T})\delta$, we can see that

$$\begin{aligned} & |E[\int_{(s,t)} \{ \Phi_u^n - \frac{1}{2} F''(\varphi_{u-}^n) \} \tilde{f}_{u,\delta}^n(\varphi_{u-}^n)^2 \Delta A_u^n dA_u^n \Psi_s^n]| \\ & \leq \|F\|_3 \|\Psi\| \{ E[\int_{(s,t)} \sup_{x \in K} |\tilde{f}_u^n(x)| \tilde{f}_{u,\delta}^n(x)^2 (\Delta A_u^n)^2 dA_u^n] \\ & \quad + \sup_{s < u \leq t} \sup_{x \in K} |\bar{g}_u^n(x) + \tilde{f}_u^n(x)| \Delta A_u^n \times E[\int_{(s,t)} \sup_{x \in K} \tilde{f}_{u,\delta}^n(x)^2 \Delta A_u^n dA_u^n] \} \\ & \leq \|F\|_3 \|\Psi\| \{ 2\delta(1 + C_{ML,T}) + \sup_{s < u \leq t} \Delta \bar{D}_u^n \} E[D_t^{n,p}]. \end{aligned}$$

Therefore, (A.I) implies (3.47) by the same way as in the proof of (3.46).

(iii) Note that we have

$$\begin{aligned} & |E[\int_{(s,t)} \{ \Phi_u^n - \int_0^1 \int_0^1 \alpha F''(\varphi_{u-}^n + \alpha \beta f_u^{n,\delta}(\varphi_{u-}^n)) \Delta A_u^n \} d\alpha d\beta] f_u^{n,\delta}(\varphi_{u-}^n)^2 \\ & \quad \times \Delta A_u^n dA_u^n \Psi_s^n]| \end{aligned}$$

$$\begin{aligned} &\leq \|F\|_3 \|\Psi\| \{ E[\int_{(s,t)} \sup_{x \in K} (|\bar{g}_u^n(x) + \tilde{f}_{u,\delta}^n(x) | \Delta A_u^n \\ &\quad + |\tilde{f}_{u,\delta}^n(x) | \Delta A_u^n) f_u^{n,\delta}(x)^2 \Delta A_u^n dA_u^n] \\ &\leq \|F\|_3 \|\Psi\| \{ \sup_{s < u \leq t} \Delta \bar{D}_u^n + \sup_{s < u \leq t} E[\sup_{x \in K} |f_{u,\delta}^n(x) \Delta A_u^n|] \} E[D_t^{n,p}]. \end{aligned}$$

Hence, by the same argument as before, we have (3.48). □

We continue the proof of Proposition 3.5. As the discussion in the proof of Lemma 3.6, we can show that (A.II)-(1) and (A.II)-(2) imply that

$$(3.50) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} | E[\int_{(s,t)} F''(\varphi_{u-}^n) \tilde{f}_{u,\delta}^n(\varphi_{u-}^n)^2 \Delta A_u^n dA_u^n \Psi_s^n] - E^Q[\int_s^t F''(\varphi_u) a_0(\varphi_u) du \Psi_s] | = 0,$$

and

$$(3.51) \quad \begin{aligned} &\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} | E[\int_{(s,t)} \int_0^1 \int_0^1 \alpha F''(\varphi_{u-}^n + \alpha \beta f_{u,\delta}^{n,\delta}(\varphi_{u-}^n) \Delta A_u^n) d\alpha d\beta \\ &\quad \times f_u^{n,\delta}(\varphi_{u-}^n)^2 \Delta A_u^n dA_u^n \Psi_s^n] \\ &- E^Q[\int_s^t \int_{\{||f||^* \leq 1\}} \int_0^1 \int_0^1 \alpha F''(\varphi_u + \alpha \beta f(\varphi_u)) d\alpha d\beta f(\varphi_u)^2 \nu_u(df) du \Psi_s] | \\ &= 0. \end{aligned}$$

Since the proof is similar to that of (3.24) and (3.25), we will only consider (3.51). For any fixed $\delta > 0$, set

$$(3.52) \quad g_u^n(x) = \int_0^1 \int_0^1 \alpha F''(x + \alpha \beta f_{u,\delta}^{n,\delta}(x) \Delta A_u^n) d\alpha d\beta f_{u,\delta}^{n,\delta}(x)^2 \Delta A_u^n.$$

Then, as (3.27), we have for all $s \leq t$ and $|\alpha| \leq 1$

$$\int_{(0,t)} \sup_{x \in K} | \partial_x^\alpha g_u^n(x) | dA_u^n \ll C \times D_t^n.$$

In fact, since $| f_u^{n,\delta}(x) \Delta A_u^n | = | f_{u,M}^{n,\delta}(x) \Delta A_u^n | \leq M(1 + C_{ML,T})$, it follows from (2.4) that

$$\begin{aligned} &\int_{(0,t)} \sup_{x \in K} | \partial_x^\alpha g_u^n(x) | dA_u^n \\ &\ll \|F\|_3 \sum_{u \leq t} \sup_{x \in K} \{ | \partial_x^\alpha f_u^{n,\delta}(x) \Delta A_u^n | | f_u^{n,\delta}(x) \Delta A_u^n |^2 + | \partial_x^\alpha (f_u^{n,\delta}(x) \Delta A_u^n)^2 | \} \end{aligned}$$

$$\begin{aligned} &\ll \|F\|_3 \sum_{u \leq t} \sup_{x \in K} \{ M(1 + C_{ML,T}) | \partial_x^\alpha f_u^{n,\delta}(x) \Delta A_u^n | | f_u^{n,\delta}(x) \Delta A_u^n | \\ &\hspace{20em} + | \partial_x^\alpha (f_u^{n,\delta}(x) \Delta A_u^n)^2 | \} \\ &\ll \|F\|_3 \sum_{u \leq t} \sup_{x \in K} \{ (M(1 + C_{ML,T}) + 1) | \partial_x^\alpha (f_u^{n,\delta}(x) \Delta A_u^n)^2 | \} \\ &\ll \{ M(1 + C_{ML,T}) + 1 \} \|F\|_3 D_t^n. \end{aligned}$$

Then we can prove (3.51) similarly as in the proof of Lemma 3.6. Thus, we have completed the proof of Proposition 3.5. □

Therefore, combining Proposition 3.2 and Proposition 3.4, we obtain the following.

PROPOSITION 3.10. *For each $M \in C(\nu)$ and $L > 0$, $\varphi^{n,M,L}$ of (3.1) converges in law to $\varphi^{M,L}$ which is the solution of (3.18).*

3-3. In this subsection, we will remove the restriction of localization in the equation (3.1). Though the procedure seems routine or similar to the discussion in section 3-5 in Fujiwara [2], we will give it for the completeness.

For each $n \in \mathbf{N}$ and $M \in C(\nu)$, we define the truncated process $\varphi^{n,M}$ of φ^n by

$$(3.53) \quad \varphi_t^{n,M} = x_0 + \int_{(0,t)} \{ f_{u,M}^n(\varphi_{u-}^{n,M}) + \bar{g}_u^n(\varphi_{u-}^{n,M}) \} dA_u^n.$$

The aim of this subsection is to show the following proposition.

PROPOSITION 3.11. *For each $M \in C(\nu)$, the family of \mathbf{D}_d -valued random variables $\{\varphi^{n,M}\}_n$ converges in law to the unique solution φ^M of the stochastic differential equation:*

$$(3.54) \quad \begin{aligned} \varphi_t^M &= x_0 + \int_{(0,t)} \sigma(\varphi_{u-}^M, u) dB_u + \int_{(0,t)} (b + c)(\varphi_{u-}^M, u) du \\ &\quad + \int_{(0,t)} \int_{\{\|f\|^* \leq 1\}} f(\varphi_{u-}^M) \tilde{N}(dudf) \\ &\quad + \int_{(0,t)} \int_{\{\|f\|^* \leq M\}} f(\varphi_{u-}^M) N(dudf). \end{aligned}$$

Proof. For given $L > 0$, define $S_L(\varphi) = \inf \{t \geq 0; |\varphi_t| \geq L \text{ or } |\varphi_{t-}| \geq L\}$ for $\varphi \in \mathbf{D}_d$. Then, it is a $\{\mathcal{D}_t\}$ -stopping time and lower semicontinuous from \mathbf{D}_d to $[0, \infty]$. Set $\mathcal{D}_{S_L} = \{A \in \mathcal{D}; A \cap \{S_L \leq t\} \in \mathcal{D}_t\}$ as usual. Then, since it is clear that $S_L(\varphi^{n,M,L}) = S_L(\varphi^{n,M})$ and that $\varphi_t^{n,M,L} = \varphi_t^{n,M}$ for $t \leq S_L(\varphi^{n,M})$, we have

$$(3.55) \quad P_n^{M,L} = P_n^M \quad \text{on} \quad \mathcal{D}_{S_L},$$

for each n , where $P_n^{M,L}$ and P_n^M denote the laws of $\varphi^{n,M,L}$ and, $\varphi^{n,M}$ respectively. Similarly, we have

$$(3.56) \quad P^{M,L} = P^M \quad \text{on} \quad \mathcal{D}_{S_L},$$

where $P^{M,L}$ and P^M denote the laws of $\varphi^{M,L}$ of (3.18) and φ^M of (3.54), respectively. Combining (3.55) and (3.56) with Proposition 3.10, we obtain the conclusion by a similar argument in Lemma 11.1.1 in Stroock-Varadhan [16]. □

3-4. In this subsection, we will remove the restriction of truncation for jumps in the equation (3.53) and complete the proof of Theorem 2.1. First, we prepare the following lemma.

LEMMA 3.12. *The family $\{\varphi^M\}_M$ of (3.54) converges in law to the solution of (2.16) as $M \rightarrow \infty$.*

Proof. Let $N(dudf)$ be the Poisson random measure in (2.16), and for $M > 0$ define a stopping time τ_M by $\tau_M = \inf\{t; N((0, t], \|f\|^* > M) > 0\}$. Then, it is clear that $\varphi_t^M = \varphi_t$ if $t < \tau_M$, where $\varphi^M(\varphi)$ is the solution of (3.54) ((2.16) respectively). Hence for arbitrary t and bounded \mathcal{D}_t -measurable continuous function G , we have

$$\begin{aligned} E[G(\varphi^M)] &= E[G(\varphi^M); t < \tau_M] + E[G(\varphi^M); t \geq \tau_M] \\ &= E[G(\varphi); t < \tau_M] + E[G(\varphi^M); t \geq \tau_M] \\ &= E[G(\varphi)] - E[G(\varphi); t \geq \tau_M] + E[G(\varphi^M); t \geq \tau_M]. \end{aligned}$$

Therefore we have

$$\begin{aligned} |E[G(\varphi^M)] - E[G(\varphi)]| &\leq 2 \|G\| P[\tau_M \leq t] \\ &\leq 2 \|G\| P[N((0, t], \|f\|^* > M) > 0] \\ &= 2 \|G\| \{1 - \exp(-\int_0^t \nu_u(\|f\|^* > M) du)\} \rightarrow 0 \end{aligned}$$

as $M \rightarrow \infty$, because $\int_0^t \nu_u(\|f\|^* = \infty) du = 0$. Since t is arbitrary, we can conclude that φ^M converges in law to φ of (2.16). □

Finally, we show that for every bounded continuous function G on \mathbf{D}_d

$$(3.57) \quad \lim_{n \rightarrow \infty} E[G(\varphi^n)] = E^q[G(\varphi)].$$

For arbitrary $t > 0$, suppose that G is \mathcal{D}_t -measurable. Then, it holds that

$$\begin{aligned} E[G(\varphi^n)] &= E[G(\varphi^n); \sup_{u \leq t} \|f_u^n \Delta A_u^n\|^* \leq M] + E[G(\varphi^n); \sup_{u \leq t} \|f_u^n \Delta A_u^n\|^* > M] \\ &= E[G(\varphi^{n,M})] - E[G(\varphi^{n,M}); \sup_{u \leq t} \|f_u^n \Delta A_u^n\|^* > M] \\ &\quad + E[G(\varphi^n); \sup_{u \leq t} \|f_u^n \Delta A_u^n\|^* > M]. \end{aligned}$$

Hence, we have

$$\begin{aligned} |E[G(\varphi^n)] - E^0[G(\varphi)]| &\leq |E[G(\varphi^n)] - E[G(\varphi^{n,M})]| + |E[G(\varphi^{n,M})] - E[G(\varphi^M)]| \\ &\quad + |E[G(\varphi^M)] - E^0[G(\varphi)]| \\ &\leq 2 \|G\| P[\sup_{u \leq t} \|f_u^n \Delta A_u^n\|^* > M] + |E[G(\varphi^{n,M})] - E[G(\varphi^M)]| \\ &\quad + |E[G(\varphi^M)] - E^0[G(\varphi)]|. \end{aligned}$$

Therefore, by (A.IV), Proposition 3.11, and Lemma 3.12, we get (3.57). Further, since $t > 0$ is arbitrary, (3.57) holds for every bounded continuous function G .

By the above discussion, we have proved Theorem 2.1. □

4. Case of diffusions

As applications of Theorem 2.1, we will consider a sequence of stochastic ordinary differential equations:

$$(4.1) \quad \frac{d\varphi_t}{dt} = f_t^n(\varphi_t),$$

where $\{f_t^n\}$ is a sequence of $\{\mathcal{F}_t^n\}$ -adapted càdlàg process with values in \mathbf{C}_{b*}^2 . It is assumed that $\bar{f}_t^n(x) = E[f_t^n(x)]$ exists for all t, x and $\bar{f}_t^n = \bar{f}_t^n(x)$ is a \mathbf{C}_{b*}^2 -valued càdlàg function for any n . We set $\tilde{f}_t^n(x) = f_t^n(x) - \bar{f}_t^n(x)$ as before. We denote by φ_t^n the solution starting at a fixed point x_0 at time 0. Since conditions for Theorem 2.1 can be simplified in this case, we state it for a reference. We will follow the notations in section 2.

(A.I)’: For every $N, T > 0$, there exists a sequence of nondecreasing continuous processes $\{D^n\}_n$ satisfying the following properties (i) and (ii).

- (i) For all $s \leq t \leq t' \leq T$,

$$(4.2) \quad \sum_{|\alpha| \leq 2, |\beta| \leq 1} \left\{ \int_s^{t'} \sup_{|x|, |y| \leq N} |E[\int_v^{t'} \partial_y^\alpha \tilde{f}_u^n(y) du | \mathcal{F}_v^n]| | \partial_x^\beta \tilde{f}_v^n(x) | dv + \int_s^t \sup_{|x| \leq N} | \partial_x^\beta \tilde{f}_u^n(x) | du \right\} \leq D_t^n - D_s^n.$$

(ii) $\{D^n\}_n$ is tight in the space $\mathbf{C}([0, \infty), \mathbf{R}^1)$, and $\{D_t^n\}_n$ is uniformly integrable for all t .

(A.II)': There exist a càdlàg function $a_1(t) \equiv a_1(x, t)$ with values in $\mathbf{C}_{b*}^2(\mathbf{R}^d, \mathbf{R}^d \otimes \mathbf{R}^d)$ and càdlàg functions $b(t) \equiv b(x, t), c(t) \equiv c(x, t)$ with values in $\mathbf{C}_{b*}^1(\mathbf{R}^d, \mathbf{R}^d)$ such that for all $N > 0$ and $s \leq t$

$$(4.3) \quad \lim_{n \rightarrow \infty} E[\sup_{|x| \leq N} |E[\int_s^t \int_v^t \tilde{f}_u^{n,(i)}(x) \tilde{f}_v^{n,(j)}(x) dudv | \mathcal{F}_s^n] - \int_s^t a_1^{ij}(x, u) du|] = 0,$$

$$(4.4) \quad \lim_{n \rightarrow \infty} E[\sup_{|x| \leq N} |E[\int_s^t \int_v^t \sum_{j=1}^d \left(\frac{\partial}{\partial x^j} \tilde{f}_u^{n,(i)}(x)\right) \tilde{f}_v^{n,(j)}(x) dudv | \mathcal{F}_s^n] - \int_s^t c^i(x, u) du|] = 0,$$

and

$$(4.5) \quad \lim_{n \rightarrow \infty} \sup_{|x| \leq N} | \int_s^t \tilde{f}_u^n(x) du - \int_s^t b(x, u) du | = 0.$$

(A.III)': For all $N > 0$ and $t > 0$,

$$(4.6) \quad \lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \sum_{|\alpha| \leq 2} E[\sup_{|x| \leq N} |E[\int_s^t \partial_x^\alpha \tilde{f}_u^n(x) du | \mathcal{F}_s^n]|] = 0.$$

Then, by Theorem 2.1, we obtain the following diffusion approximation theorem. It can be regarded as an improvement of the previous work by Kunita [9].

THEOREM 4.1. Assume (A.I)', (A.II)', and (A.III)'. Then, the family of continuous stochastic processes $\{\varphi^n\}_n$ satisfying (4.1) converges in law to the unique solution of the following stochastic differential equation:

$$(4.7) \quad \varphi_t = x_0 + \int_0^t \sigma(\varphi_u, u) dB_u + \int_0^t (b + c)(\varphi_u, u) du,$$

where $\sigma(u) = \sigma(x, u)$ is a càdlàg function with values in $\mathbf{C}_{b*}^1(\mathbf{R}^d, \mathbf{R}^d \otimes \mathbf{R}^r)$ such that $\sigma(u)\sigma(u)^* = a_1(u) + a_1(u)^*$ and B_t is an r -dimensional standard Brownian motion.

The solution φ_t of (4.7) is often called a diffusion process with characteristics $(a_1(t) + a_1(t)^*, b(t) + c(t))$.

We shall apply the above theorem to the case where the driving processes $\{f_t^n(x)\}_n$ satisfy suitable mixing conditions. Define the filtrations $\{\mathcal{F}_{s,t}^n : s \leq u \leq t\}$ by

$$(4.8) \quad \mathcal{F}_{s,t}^n = \sigma[f_u^n; s \leq u \leq t].$$

The strong mixing rate $\alpha^n(t)$ and the uniform mixing rate $\phi^n(t)$ of the family of the σ -fields $\{\mathcal{F}_{s,t}^n\}_{s \leq t}$ are defined by

$$(4.9) \quad \alpha^n(t) = \sup_{s \geq 0} \sup_{A \in \mathcal{F}_{s+t,\infty}^n, B \in \mathcal{F}_{0,s}^n} |P(A \cap B) - P(A)P(B)|,$$

$$(4.10) \quad \phi^n(t) = \sup_{s \geq 0} \sup_{A \in \mathcal{F}_{s+t,\infty}^n, B \in \mathcal{F}_{0,s}^n} |P(A|B) - P(A)|.$$

We will first consider the case where a uniform mixing condition is satisfied. We introduce the following conditions.

(C.1): For all positive numbers N, T and $|\beta| \leq 1$,

$$\sup_n \sup_{t \leq T} \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_t^n(x)| < \infty.$$

(C.2): The sequence of the uniform mixing rates $\{\phi^n(t)\}$ satisfies

$$\lim_{n \rightarrow \infty} \int_0^T \phi^n(s)^{1/2} ds = 0$$

for all T . Further the following family of random variables are uniformly integrable for any N, T and $|\alpha| \leq 2$

$$\left\{ \left(\int_0^T \phi^n(s)^{1/2} ds \right) \sup_{|x| \leq N} |\partial_x^\alpha \tilde{f}_t^n(x)|^2; \quad t \in [0, T], n = 1, 2, \dots \right\}.$$

(C.3): There exists a sequence $\{h_n\}$ of positive numbers converging to 0 such that the limits

$$(4.11) \quad b(x, t) = \lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \tilde{f}_u^n(x) du,$$

$$(4.12) \quad a^{ij}(x, t) = \lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \int_t^u E [\tilde{f}_u^{n,(i)}(x) \tilde{f}_v^{n,(j)}(x)] dudv,$$

$$(4.13) \quad c^i(x, t) = \lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \int_t^u E \left[\sum_{j=1}^d \frac{\partial \tilde{f}_u^{n,(i)}}{\partial x^j}(x) \tilde{f}_v^{n,(j)}(x) \right] dudv$$

exist uniformly on compact sets of $\mathbf{R}^d \times [0, \infty)$.

THEOREM 4.2. *Assume that $\{f_t^n\}_n$ in equation (4.1) satisfies (C.1) ~ (C.3). Then $\{\varphi^n\}_n$ converges in law to a diffusion process with characteristics $(a(t) + a(t)^*, b(t) + c(t))$.*

Before the proof we show a lemma. We fix N, T and set

$$(4.14) \quad K^n(v) = \sum_{|\alpha| \leq 2, |\beta| \leq 1} \left\{ \sup_{|x| \leq N, |y| \leq N} \int_v^T |E[\partial_y^\alpha \tilde{f}_u^n(y) | \mathcal{F}_v^n]| \, du \, |\partial_x^\beta \tilde{f}_v^n(x)| + \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_v^n(x)| \right\}.$$

LEMMA 4.3. *The family of random variables $\{K^n(v); n = 1, 2, \dots, v \in [0, T]\}$ is uniformly integrable for any T .*

Proof. By a uniform mixing inequality, we have

$$(4.15) \quad |E[\partial_y^\alpha \tilde{f}_u^n(y) | \mathcal{F}_v^n]| \leq 2\phi^n(u - v)^{\frac{1}{2}} \{ E[|\partial_y^\alpha \tilde{f}_u^n(y)|^2 | \mathcal{F}_v^n]^{\frac{1}{2}} + E[|\partial_y^\alpha \tilde{f}_u^n(y)|^2]^{\frac{1}{2}} \}.$$

See [9], Proof of Lemma 5.6.2. Consequently $K^n(v)$ is dominated by the sum of the following four terms.

$$(4.16) \quad \int_0^{T-v} \phi^n(u)^{\frac{1}{2}} \sup_{|y| \leq N} E[|\partial_y^\alpha \tilde{f}_{u+v}^n(y)|^2 | \mathcal{F}_v^n] \, du,$$

$$(4.17) \quad \int_0^{T-v} \phi^n(u)^{\frac{1}{2}} \sup_{|y| \leq N} E[|\partial_x^\alpha \tilde{f}_{u+v}^n(y)|^2] \, du,$$

$$(4.18) \quad \left(\int_0^{T-v} \phi^n(u)^{\frac{1}{2}} \, du \right) \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_v^n(x)|^2,$$

$$(4.19) \quad \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_v^n(x)|.$$

Clearly (4.17) and (4.19) are bounded by a positive constant not depending on v and n . Further (4.18) is uniformly integrable by (C.2). We will consider (4.16). By (C.2), there exists a positive convex increasing function $G(\lambda), \lambda > 0$ such that $\lim_{\lambda \rightarrow \infty} G(\lambda)/\lambda = \infty$ and

$$(4.20) \quad \sup_n \sup_{t \leq T} E[G(c_n \sup_{|y| \leq N} |\partial_y^\alpha \tilde{f}_t^n(y)|^2)] \equiv K < \infty,$$

where $c_n = \int_0^T \phi^n(t)^{1/2} dt$. See Meyer [14]. By Jensen's inequality, we have

$$(4.21) \quad \begin{aligned} E[G(\int_0^{T-v} \phi^n(u)^{\frac{1}{2}} \sup_{|y| \leq N} E[|\partial_y^\alpha \tilde{f}_{u+v}^n(y)|^2 | \mathcal{F}_v^n] \, du)] \\ \leq c_n^{-1} \int_0^T \phi^n(u)^{\frac{1}{2}} E[G(c_n \sup_{|y| \leq N} E[|\partial_y^\alpha \tilde{f}_{u+v}^n(y)|^2 | \mathcal{F}_v^n])] \, du \\ \leq c_n^{-1} \int_0^T \phi^n(u)^{\frac{1}{2}} E[G(c_n E[\sup_{|y| \leq N} |\partial_y^\alpha \tilde{f}_{u+v}^n(y)|^2 | \mathcal{F}_v^n])] \, du. \end{aligned}$$

By Jensen's inequality concerning conditional expectation, the above is bounded by

$$c_n^{-1} \int_0^T \phi^n(u)^{\frac{1}{2}} E[G(c_n \sup_{|y| \leq N} |\partial_y^\alpha \tilde{f}_{u+y}^n(y)|^2)] du \leq K < \infty.$$

Consequently, (4.16) is uniformly integrable. □

Proof of Theorem 4.2. We take $D_t^n = \int_0^t K^n(v) dv$. Then, (A.I)'-(i) is obviously satisfied. Since $\{K^n(v); n = 1, 2, \dots, v \in [0, T]\}$ is uniformly integrable, it is easily verified that $\{\int_0^t K^n(v) dv\}_n$ is also uniformly integrable for each t . We will prove that the latter sequence is tight. The property

$$(4.22) \quad \limsup_{c \rightarrow \infty} \sup_n P[\int_0^t K^n(v) dv > c] = 0$$

is obvious from the uniform integrability of $\{\int_0^t K^n(v) dv\}_n$. Let $0 < \theta < T$ be a constant and σ be a stopping time such that $\sigma + \theta \leq T$. Then we have

$$\begin{aligned} \int_\sigma^{\sigma+\theta} K^n(v) dv &= \int_\sigma^{\sigma+\theta} K^n(v) I_{[0,c]}(K^n(v)) dv + \int_\sigma^{\sigma+\theta} K^n(v) I_{(c,\infty)}(K^n(v)) dv \\ &\leq c\theta + \int_0^T K^n(v) I_{(c,\infty)}(K^n(v)) dv \end{aligned}$$

for any $c > 0$. Therefore

$$(4.23) \quad E[\int_\sigma^{\sigma+\theta} K^n(v) dv] \leq c\theta + \int_0^T E[K^n(v); K^n(v) > c] dv.$$

For any $\varepsilon > 0$, there exists $c > 0$ such that $E[K^n(v); K^n(v) > c] < \varepsilon T^{-1}$ holds for all n and v , since $\{K^n(v)\}_{n,v}$ is uniformly integrable. Next choose $\theta_0 > 0$ such that $c\theta_0 < \varepsilon$. Then we have for any $|\theta| < \theta_0$

$$(4.24) \quad \sup_n \sup_{\sigma \in \mathcal{T}_T} E[\int_\sigma^{\sigma+\theta} K^n(v) dv] < 2\varepsilon,$$

where \mathcal{T}_T is the set of all stopping times less than T . Therefore $\{\int_0^t K^n(v) dv\}_n$ satisfies Aldous's condition. Then it is tight. Thus we have proved (A.I)'.

We will next prove (A.II)'. Set

$$K^n(u, v, x) = \tilde{f}_u^{n,(i)}(x) \tilde{f}_v^{n,(j)}(x), \quad \bar{K}^n(u, v, x) = E[K^n(u, v, x)].$$

We can show similarly as in [9], Chapter 5 that (4.12) implies

$$(4.25) \quad \lim_{n \rightarrow \infty} \int_s^t \int_s^u \bar{K}^n(u, v, x) dudv = \int_s^t a^{ij}(x, r) dr,$$

for any $s < t$. Hence it is sufficient to prove

$$(4.26) \quad \lim_{n \rightarrow \infty} E[\sup_{|x| \leq N} |\int_s^t \int_s^u E[\bar{K}^n(u, v, x) | \mathcal{F}_s^n] dudv|] = 0,$$

where $\tilde{K}^n = K^n - \bar{K}^n$. Let $c_n = \int_0^T \phi^n(s)^{1/2} ds$ as before. Since

$$\left\{ c_n \sup_{|x| \leq N} |f_u^n(x)|^2; n = 1, 2, \dots, u \in [0, T] \right\}$$

is uniformly integrable, for any $\varepsilon > 0$ there exists $a > 0$ such that for all n, u

$$(4.27) \quad E[c_n \sup_{|x| \leq N} |f_u^n(x)|^2; c_n \sup_{|x| \leq N} |f_u^n(x)|^2 > a] < \varepsilon.$$

Now consider the truncated process $f_{u,a}^n = f_u^n I_{(\sup_{|x| \leq N} |f_u^n(x)| < (a/c_n)^{1/2})}$. Set $\tilde{f}_{u,a}^n = E[f_{u,a}^n]$, $\tilde{f}_{u,a}^n = f_{u,a}^n - \tilde{f}_{u,a}^n$ and $K_a^n(u, v, x) = \tilde{f}_{u,a}^{n,(i)}(x) \tilde{f}_{v,a}^{n,(j)}(x)$. Define \tilde{K}_a^n and \tilde{K}^n as above. Then $\tilde{K}^n = (\tilde{K}_a^n - \bar{K}^n) + (K^n - K_a^n) + \tilde{K}_a^n$ holds. We have

$$\begin{aligned} K^n(u, v, x) - K_a^n(u, v, x) &= \tilde{f}_{u,a}^n(x) (\tilde{f}_v^n(x) - \tilde{f}_{v,a}^n(x)) + (\tilde{f}_u^n(x) - \tilde{f}_{u,a}^n(x)) \tilde{f}_{v,a}^n(x) \\ &\quad + (\tilde{f}_u^n(x) - \tilde{f}_{u,a}^n(x)) (\tilde{f}_v^n(x) - \tilde{f}_{v,a}^n(x)) \\ &=: L_1^n + L_2^n + L_3^n. \end{aligned}$$

Apply (4.15) and (4.27) to each term of the right hand side. Then

$$E\left[\int_s^t \int_s^u \sup_{|x| \leq N} |E[L_i^n | \mathcal{F}_s^n]| dudv\right] \leq 4T(C\varepsilon)^{1/2}, \quad i = 1, 2,$$

where $C = \sup_{n,u} \{c_n E[\sup_{|x| \leq N} |f_u^n(x)|^2]\}$, and

$$E\left[\int_s^t \int_s^u \sup_{|x| \leq N} |E[L_3^n | \mathcal{F}_s^n]| dudv\right] \leq 4T\varepsilon.$$

Consequently we obtain

$$(4.28) \quad \sup_{|x| \leq N} |\tilde{K}_a^n - \bar{K}^n| \leq E[\sup_{|x| \leq N} |K^n - K_a^n|] \leq 4T\varepsilon^{1/2}(2C^{1/2} + \varepsilon^{1/2}).$$

Further by a uniform mixing inequality ([9], Lemma 5.6.3),

$$(4.29) \quad \begin{aligned} & \left| \int_s^t \int_s^u E[\tilde{K}_a^n(u, v, x) | \mathcal{F}_s^n] dudv \right| \\ & \leq 4 \int_s^t \int_s^u \phi^n(u-v)^{\frac{1}{2}} \phi^n(v-s)^{\frac{1}{2}} \|\tilde{f}_{u,a}^{n,(i)}(x)\|_{(\infty)} \|\tilde{f}_{v,a}^{n,(j)}(x)\|_{(\infty)} dudv, \end{aligned}$$

where $\|X\|_{(p)}$ ($1 \leq p \leq \infty$) is the L^p norm of the random variable X with respect to the measure P . The right hand side of the above is dominated by

$$\left(\int_0^T \phi^n(s)^{\frac{1}{2}} ds\right)^2 \sup_u \|\tilde{f}_{u,a}^{n,(i)}(x)\|_{(\infty)} \sup_v \|\tilde{f}_{v,a}^{n,(j)}(x)\|_{(\infty)}.$$

Since $\|\tilde{f}_{u,a}^{n,(i)}\|_{(\infty)} \leq 2(a/c_n)^{1/2}$, the above is bounded by $4ac_n$. It converges to 0 a.s. as $n \rightarrow \infty$. This fact together with (4.28) proves that (4.26) holds. Therefore (4.3)

in Condition (A.II)' is verified. Other properties in (A.II)' and (A.III)' can be proved similarly. □

We will apply Theorem 4.2 to two problems. The first is the approximation problem of stochastic differential equation.

Let $\nu^n(t) = (\nu_1^n(t), \dots, \nu_m^n(t))$, $n = 1, 2, \dots$ be a sequence of \mathbf{R}^m -valued stochastic processes with the uniform mixing rate $\phi^n(t)$. We assume the following $(\nu.1) \sim (\nu.3)$.

- ($\nu.1$): $E[\nu_k^n(t)] = 0$ for any n, k, t .
- ($\nu.2$): $\lim_{n \rightarrow \infty} \int_0^T \phi^n(s)^{1/2} ds = 0$. Further, $\{(\int_0^T \phi^n(s)^{1/2} ds) | \nu_k^n(t) |^2, n = 1, 2, \dots, t \in [0, T]\}$ is uniformly integrable for any T .
- ($\nu.3$): There exists a sequence of positive numbers $\{h_n\}$ converging to 0 such that the following limit exists uniformly on compact sets of $[0, \infty)$:

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \int_t^u E[\nu_k^n(u) \nu_l^n(v)] dudv = v_{kl}(t).$$

Now define

$$(4.30) \quad X^n(t) = \int_0^t \nu^n(s) ds.$$

It can be regarded as a solution of the equation $d\varphi_t/dt = \nu^n(t)$. Then Theorem 4.2 tells us that $\{X^n(t)\}$ converges in law to a Brownian $X(t) = (X_1(t), \dots, X_m(t))$ with mean 0 and covariance $\int_0^t v_{kl}(u) du$.

Now let $f_k, k = 0, \dots, m$ be deterministic functions of \mathbf{C}_b^2 . Consider a sequence of stochastic ordinary differential equations:

$$(4.31) \quad \frac{d\varphi_t}{dt} = f_0(\varphi_t, t) + \sum_{k=1}^m f_k(\varphi_t, t) \nu_k^n(t).$$

Let φ_t^n be the solution starting at x_0 at time 0. Then we obtain the following from Theorem 4.2 immediately.

COROLLARY 4.4. *Assume that $\{\nu_k^n(t)\}_n$ satisfies $(\nu.1) \sim (\nu.3)$. Then the sequence $\{\varphi_t^n\}_n$ converges in law. The limit φ_t satisfies the stochastic differential equation:*

$$(4.32) \quad \varphi_t = x_0 + \int_0^t \{f_0(\varphi_u, u) + c(\varphi_u, u)\} du + \sum_{k=1}^m \int_0^t f_k(\varphi_u, u) dX_k(u),$$

where $c^i(x, t)$ is given by

$$(4.33) \quad c^i(x, t) = \sum_{j=1}^d \sum_{k,l=1}^m f_k^j(x, t) \frac{\partial}{\partial x^j} f_l^i(x, t) v_{kl}(t).$$

Now let $f(t) \equiv f(x, t)$ and $g(t) \equiv g(x, t), x \in \mathbf{R}^d, t \in [0, \infty)$ be $\mathbf{C}_{b^*}^2$ -

valued càdlàg stochastic processes. We assume that $f(t)$ is of mean 0, i.e. $E[f(x, t)] = 0$ for all x and t . Consider a sequence of stochastic ordinary differential equations

$$(4.34) \quad \frac{d\phi_t}{dt} = \frac{1}{\sqrt{n}} f(\phi_t, t) + \frac{1}{n} g(\phi_t, t).$$

For each n , it has a unique global solution ϕ_t^n starting at any given point x_0 at time 0. Let n tend to infinity. Then ϕ_t^n converges to the trivial stochastic process $\phi_t^\infty \equiv x_0$. However if we consider the process $\varphi_t^n \equiv \phi_{nt}^n$, it satisfies

$$(4.35) \quad \frac{d\varphi_t^n}{dt} = \sqrt{n} f(\varphi_t^n, nt) + g(\varphi_t^n, nt).$$

Then $\{\varphi^n\}_n$ does not converges to a trivial process. We shall show its weak convergence applying Theorem 4.2. A similar problem has been discussed by many authors. See e.g. Khas'minskii [7], Kesten-Papanicolaou [6], Kunita [8], [9]. The following Corollary may be considered as an improvement of Theorem 5.6.1 in [9].

COROLLARY 4.5. *Assume that the driving processes $f(t) = f(x, t)$ and $g(t) = g(x, t)$ are stationary processes satisfying the following (a) and (b):*

- (a): *For every $N > 0$ and $|\alpha| \leq 2$, $E[\sup_{|x| \leq N} |\partial_x^\alpha f(x, 0)|^2] < \infty$ and $E[\sup_{|x| \leq N} |\partial_x^\alpha g(x, 0)|^2] < \infty$.*
- (b): $\int_0^\infty \phi(s)^{1/2} ds < \infty$.

Then $\{\varphi^n\}$ converges in law to a stationary diffusion process with characteristics (a, b + c) which is given by

$$(4.36) \quad a^{ij}(x) = E\left[\left(\int_0^\infty f^i(x, u) du\right) f^j(x, 0)\right] + E\left[\left(\int_0^\infty f^j(x, u) du\right) f^i(x, 0)\right],$$

$$(4.37) \quad c^i(x) = \sum_{j=1}^d E\left[\left(\int_0^\infty \frac{\partial f^i}{\partial x^j}(x, u) du\right) f^j(x, 0)\right],$$

$$(4.38) \quad b^i(x) = E[g^i(x, 0)].$$

Proof. Set

$$f_t^n(x) = \sqrt{n} f(x, nt) + g(x, nt), \quad \mathcal{F}_t^n = \mathcal{F}_{nt}, \quad \phi^n(t) = \phi(nt).$$

Then (a) and (b) imply (C.1) and (C.2) immediately. Further since $f(x, t)$ is stationary, we have

$$\sqrt{n} E\left[\int_t^{t+1/\sqrt{n}} \int_t^u \tilde{f}_u^{n,(i)}(x) \tilde{f}_v^{n,(j)}(x) dudv\right]$$

$$\begin{aligned} &= \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} E\left[\left(\int_v^{\sqrt{n}} f^i(x, u) du\right) f^j(x, v)\right] dv + o(n, x, t) \\ &= E\left[\left(\int_0^\infty f^i(x, u) du\right) f^j(x, 0)\right] \\ &\quad - \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} E\left[\left(\int_{\sqrt{n}}^\infty f^i(x, u) du\right) f^j(x, v)\right] dv + o(n, x, t) \\ &= E\left[\left(\int_0^\infty f^i(x, u) du\right) f^j(x, 0)\right] + o(n, x, t), \end{aligned}$$

where $o(n, x, t)$ converges to 0 uniformly in (x, t) on compact sets as $n \rightarrow \infty$. We can show similarly the existence of b and c in Condition (C.3). □

We will next consider equation (4.1) in the case where a strong mixing condition is satisfied. We introduce a condition associated with two positive numbers δ, γ :

(C.2)' $_{\delta, \gamma}$: The sequence of the strong mixing rates $\{\alpha^n(t)\}$ satisfies

$$\lim_{n \rightarrow \infty} \int_0^T \alpha^n(s)^\gamma ds = 0$$

for any T . Further, for any N, T and $|\alpha| \leq 2$, the sequence

$$\left\{ \left(\int_0^T \alpha^n(t)^\gamma dt \right) \sup_{t \leq T} \left\| \sup_{|x| \leq N} \left| \partial_x^\alpha \tilde{f}_t^n(x) \right\| \right\|_{(2+\delta)}^2 \right\}_n$$

is bounded.

THEOREM 4.6. *Assume Conditions (C.1), (C.2)' $_{\delta, \gamma}$ and (C.3) for some $\delta \in (0, \infty)$ and $\gamma \in (0, \delta/2(2 + \delta)(1 + d))$. Then $\{\varphi^n\}_n$ converges in law to a diffusion process with characteristics $(a(t) + a(t)^*, b(t) + c(t))$.*

Proof. We shall again apply Theorem 4.1. We shall prove that the sequence of the driving processes $\{f^n\}_n$ satisfies Condition (A.I)'. Consider $K^n(v)$ of (4.14). It is sufficient to prove that $\{K^n(v); n = 1, 2, \dots, v \in [0, T]\}$ is uniformly integrable since the fact implies the tightness and the uniform integrability of $\{\int_0^t K^n(v) dv\}_n$ as in the proof of Theorem 4.2. In the sequel we will prove

$$(4.39) \quad \sup_{n, v} E[K^n(v)^{1+\varepsilon_0}] < \infty,$$

for sufficiently small positive number ε_0 , which implies the uniform integrability of $\{K^n(v)\}$ immediately.

The process $K^n(v)$ is bounded by the sum of the following terms.

$$(4.40) \quad K_1^n(v) = \sup_{|\nu| \leq N} \int_0^{T-\nu} |E[\partial_\nu^\alpha \tilde{f}_{u+\nu}^n(y) | \mathcal{F}_\nu^n]| du \cdot \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_\nu^n(x)|,$$

$$(4.41) \quad K_2^n(v) = \sup_{|x| \leq N} |\partial_x^\beta \tilde{f}_\nu^n(x)|,$$

where $|\alpha| \leq 2$ and $|\beta| \leq 1$. The property (4.39) is obvious for $K_2^n(v)$ by (C.1). We will prove (4.39) for $K_1^n(v)$ only. We consider the case $\alpha = \beta = 0$. Since

$$(4.42) \quad K_1^n(v) \leq \int_0^{T-\nu} \sup_{|\nu| \leq N} |E[\tilde{f}_{u+\nu}^n(y) | \mathcal{F}_\nu^n]| \sup_{|x| \leq N} |\tilde{f}_\nu^n(x)| du,$$

we have by Hölder's inequality,

$$\|K_1^n(v)\|_{(1+\varepsilon_0)} \leq \int_0^{T-\nu} \left\| \sup_{|\nu| \leq N} |E[\tilde{f}_{u+\nu}^n(y) | \mathcal{F}_\nu^n]| \right\|_{(p(1+\varepsilon_0))} \left\| \sup_{|x| \leq N} |\tilde{f}_\nu^n(x)| \right\|_{(q(1+\varepsilon_0))} du,$$

where p, q are numbers greater than 1 such that $p^{-1} + q^{-1} = 1$. Next let p' and q' be positive numbers such that $\{p(1 + \varepsilon_0)\}^{-1} = p'^{-1} + q'^{-1}$. Then by a mixing inequality ([8], Lemma 3.10.2), there exists a positive constant $C = C(p', q', d, N)$ such that

$$(4.43) \quad \left\| \sup_{|\nu| \leq N} |E[\tilde{f}_{u+\nu}^n(y) | \mathcal{F}_\nu^n]| \right\|_{(p(1+\varepsilon_0))} \leq C \alpha^n(u)^{\frac{1}{p'(d+1)}} \left\| \sup_{|\nu| \leq N} |\tilde{f}_{u+\nu}^n(y)| \right\|_{(q')}.$$

Consequently we obtain

$$(4.44) \quad \|K_1^n(v)\|_{(1+\varepsilon_0)} \leq C \left(\int_0^T \alpha^n(u)^{\frac{1}{p'(d+1)}} du \right) \sup_{\nu \leq u \leq T} \left\| \sup_{|\nu| \leq N} |\tilde{f}_\nu^n(y)| \right\|_{(q')} \\ \times \left\| \sup_{|x| \leq N} |\tilde{f}_\nu^n(x)| \right\|_{(q(1+\varepsilon_0))}.$$

Set $q = (2 + \delta)/(1 + \varepsilon_0)$ and $q' = 2 + \delta$. Then we have $(p'(d + 1))^{-1} = (\delta - 2\varepsilon_0)/(1 + \varepsilon_0)(2 + \delta)(d + 1)$. If $\varepsilon_0 > 0$ is sufficiently small, the quantity is not less than γ of Condition (C.2)' $_{\delta, \gamma}$. Then (4.44) is bounded. Therefore (4.39) is satisfied for $K_1^n(v)$.

Next, property (4.3) of Condition (A.II)' can be verified similarly as in the proof of Theorem 4.2. Indeed, set $c_n = \int_0^T \alpha^n(s)^\gamma ds$ and define the truncated processes $f_{u,a}^n$ similarly as in the proof of Theorem 4.2. Then we can prove that for any $\varepsilon > 0$ there exists $a > 0$ such that

$$E\left[\int_s^T \int_s^u E\left[\sup_{|x| \leq N} |K^n(u, v, x) - K_a^n(u, v, x)| | \mathcal{F}_s^n \right] dudv \right] \leq C\varepsilon^{\delta/2(2+\delta)},$$

where C is a constant independent of n and a . Further, instead of the uniform mixing inequality (4.29), we have inequality:

$$(4.45) \quad \left\| \int_s^t \int_s^u \sup_{|x| \leq N} |E[\tilde{K}_a^n(u, v, x) | \mathcal{F}_s^n]| dudv \right\|_{(1+\varepsilon_0)}$$

$$\begin{aligned} &\leq C \int_s^t \int_s^u \alpha^n(u-v)^r \alpha^n(v-s)^r \sup_{|x| \leq N} \|\tilde{f}_{u,a}^{n,(t)}(x)\|_{(\infty)} \\ &\quad \times \sup_{|x| \leq N} \|\tilde{f}_{v,a}^{n,(j)}(x)\|_{(\infty)} du dv \end{aligned}$$

where $r = \{2(1 + \epsilon_0)(1 + d)\}^{-1}$. See Lemma 3.10.3 in [8]. Since $r \geq \gamma$ holds for sufficiently small ϵ_0 , the right hand side converges to 0 as $n \rightarrow \infty$. These two facts imply (4.3). □

We again consider equation (4.31) in the case where $\{\nu_k^n(t)\}_n$ satisfies a strong mixing condition. We introduce:

($\nu.2$)’ $_{\delta,\gamma}$: $\lim_{n \rightarrow \infty} \int_0^T \alpha^n(s)^r ds = 0$. Further the sequence

$$\left\{ \left(\int_0^T \alpha^n(s)^r ds \right) \sup_{t \leq T} \|\nu_k^n(t)\|_{(2+\delta)}^2 \right\}_n$$

is bounded for any T .

COROLLARY 4.7. Assume that $\{\nu_k^n(t)\}_n$ satisfies ($\nu.1$), ($\nu.2$)’ $_{\delta,\gamma}$ and ($\nu.3$) for some $\delta \in (0, \infty)$ and $\gamma \in (0, \delta/2(2 + \delta))$. Then the assertion of Corollary 4.4 is valid.

Indeed inequalities (4.43) and (4.45) are valid even if we replace $p'(d + 1)$ by p' and $\{(1 + \epsilon_0)(1 + d)\}^{-1}$ by $(1 + \epsilon_0)^{-1}$, respectively. See [8]. Then the corollary follows.

Finally we consider equation (4.35).

COROLLARY 4.8. Assume that the driving processes $f(t)$ and $g(t)$ are stationary processes satisfying the following (a)’ $_{\delta}$ and (b)’ $_{\gamma}$ for some $\delta \in (0, \infty)$ and $\gamma \in (0, \delta/2(2 + \delta)(1 + d))$:

- (a)’ $_{\delta}$: For every $N > 0$ and $|\alpha| \leq 2$, $\|\sup_{|x| \leq N} |\partial_x^\alpha f(x, 0)|\|_{(2+\delta)} < \infty$ and $\|\sup_{|x| \leq N} |\partial_x^\alpha g(x, 0)|\|_{(2+\delta)} < \infty$.
- (b)’ $_{\gamma}$: $\int_0^\infty \alpha(t)^r dt < \infty$.

Then $\{\varphi^n\}$ converges in law to a diffusion process with characteristics given by Corollary 4.5.

Suppose that if $f(t)$ and $g(t)$ are represented by

$$f(t) = \sum_{k=1}^m f_k(x) \nu_k(t), \quad g(t) = \sum_{k=1}^{m'} g_k(x) \mu_k(t) + g_0(x),$$

where $(\nu_k(t), \mu_k(t))$ is a stationary process with the strong mixing rate $\alpha(t)$, then $(a)_\delta'$ and $(b)_\gamma'$ with $\delta \in (0, \infty)$ and $\gamma \in (0, \delta/2(2 + \delta))$ imply the weak convergence of $\{\varphi^n\}$.

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